Student Solutions Guide
for
Discrete Mathematics

Second Edition

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Preface

This book should serve as a resource for students using *Discrete Mathematics*. It contains two components intended to supplement the textbook.

First, we provide a review for each chapter of the textbook. In these, the main definitions and results within each section are summarized. Since these summaries average approximately one page per section, they should serve as a useful study aid for students.

Second, we include the answers to the odd-numbered exercises from each section and all of the exercises from the review sections. These answers generally expand upon those listed in Appendix C of the textbook. However, for some exercises, the answers given here may still require further expansion to obtain the answers requested.

I wish to thank my many students who contributed to reducing the number of errors in this work.

Kevin Ferland

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Chapter 1

Review Sheets

1.0 Chapter 0

In base s, a nonnegative integer n is represented in the form

$$n = a_k a_{k-1} \cdots a_1 a_0, \qquad \text{(in base } s\text{)}$$

where the digits $a_k, a_{k-1}, \ldots, a_1, a_0$ represent elements from $\{0, 1, \ldots, s-1\}$. The corresponding value of n is determined by the equation

$$n = a_k s^k + a_{k-1} s^{k-1} + \cdots + a_1 s^1 + a_0 s^0.$$

Base s	Name
10	decimal
2	binary
8	octal
16	hexadecimal

Table 1.1: Common Number Systems

For any integer $s \geq 2$, the base s representation of a number is obtained by a sequence of divisions by s, each generating a remainder from $0, 1, \ldots, s-1$. When the quotient 0 is obtained, the base s representation is given by listing the sequence of remainders in reverse order.

1.1 Chapter 1

Section 1.1

Definition. A **statement** is a sentence that is either true or false, but not both.

Form	Translation
$\neg p$	**not** p (negation of p)
$p \wedge q$	p **and** q
$p \vee q$	p **or** q
$p \to q$	**if** p **then** q (p implies q)

Table 1: Basic Statement Forms

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \to q$
F	F	T	F	F	T
F	T	T	F	T	T
T	F	F	F	T	F
T	T	F	T	T	T

Table 2: Truth Table Defining $\neg, \wedge, \vee$, and $\to$

Definition. (a) The **exclusive or** operation $\oplus$ is defined by
$p \oplus q \;=\; (p \vee q) \wedge \neg(p \wedge q)$.

(b) The **if and only if** operation $\leftrightarrow$ is defined by
$p \leftrightarrow q \;=\; (p \to q) \wedge (q \to p)$.
Note that **iff** is also used to denote $\leftrightarrow$.

p	q	$p \oplus q$	$p \leftrightarrow q$
F	F	F	T
F	T	T	F
T	F	T	F
T	T	F	T

Table 3: Truth Table Defining $\oplus$ and $\leftrightarrow$

Definition. (a) A **tautology** is a statement form that is always true. We denote a tautology by $\underline{t}$.

(b) A **contradiction** is a statement form that is always false. We denote a contradiction by $\underline{f}$.

A statement whose form is a tautology or contradiction is also said to be a tautology or contradiction, respectively.

Definition. Two statement forms p and q are **logically equivalent**, written $p \equiv q$, if and only if the statement form $p \leftrightarrow q$ is a tautology. We write $p \not\equiv q$ when p and q are not logically equivalent.

Example. $\neg(p \to q) \equiv p \wedge \neg q$.

Example. $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$.

Example. $p \to q \equiv \neg p \vee q$.

Definition. Given the statement form $p \to q$,

(a) its **converse** is $q \to p$.

(b) its **contrapositive** is $\neg q \to \neg p$.

(c) its **inverse** is $\neg p \to \neg q$.

Example. An if-then statement is not logically equivalent to its converse but is logically equivalent to its contrapositive.

Theorem (Basic Logical Equivalences)**.**
Let p, q and r be statement variables. Then, the following logical equivalences hold:

(a)	$\neg\neg p$	$\equiv$	p	*Double Negative*
(b)	$(p \wedge q) \wedge r$	$\equiv$	$p \wedge (q \wedge r)$	*Associativity*
	$(p \vee q) \vee r$	$\equiv$	$p \vee (q \vee r)$	
(c)	$p \wedge q$	$\equiv$	$q \wedge p$	*Commutativity*
	$p \vee q$	$\equiv$	$q \vee p$	
(d)	$p \wedge (q \vee r)$	$\equiv$	$(p \wedge q) \vee (p \wedge r)$	*Distributivity*
	$p \vee (q \wedge r)$	$\equiv$	$(p \vee q) \wedge (p \vee r)$	
(e)	$\neg(p \wedge q)$	$\equiv$	$\neg p \vee \neg q$	*De Morgan's Laws*
	$\neg(p \vee q)$	$\equiv$	$\neg p \wedge \neg q$	
(f)	*If $p \to q$, then* $[p \wedge q$	$\equiv$	$p]$	*Absorption Rules*
	If $p \to q$, then $[p \vee q$	$\equiv$	$q]$	

Theorem (Interactions with Tautologies and Contradictions)**.** *Let p be a statement variable. Then, the following logical equivalences hold:*

(a)	$\neg \underline{t}$	$\equiv$	$\underline{f}$
	$\neg \underline{f}$	$\equiv$	$\underline{t}$
(b)	$p \wedge \underline{t}$	$\equiv$	p
	$p \vee \underline{t}$	$\equiv$	$\underline{t}$
(c)	$p \wedge \underline{f}$	$\equiv$	$\underline{f}$
	$p \vee \underline{f}$	$\equiv$	p
(d)	$p \wedge \neg p$	$\equiv$	$\underline{f}$
	$p \vee \neg p$	$\equiv$	$\underline{t}$
(e)	$\underline{t} \to p$	$\equiv$	p
	$p \to \underline{t}$	$\equiv$	$\underline{t}$
(f)	$p \to \underline{f}$	$\equiv$	$\neg p$
	$\underline{f} \to p$	$\equiv$	$\underline{t}$

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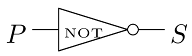
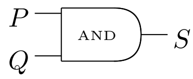
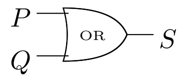
<tr><td>Symbol</td><td>
</td><td>
</td><td>
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Table 4: Basic Gates

Section 1.2

Definition (Set Equality (Informal Version))**.** Two sets A and B are said to be equal, written $A = B$, if and only if A and B contain exactly the same elements.

Definition. We define the sets of

integers	$\mathbb{Z}$		
real numbers	$\mathbb{R}$		
natural numbers	$\mathbb{N}$	$=$	$\{n : n \in \mathbb{Z}$ and $n \geq 0\}$,
positive integers	$\mathbb{Z}^+$	$=$	$\{n : n \in \mathbb{Z}$ and $n > 0\}$,
negative integers	$\mathbb{Z}^-$	$=$	$\{n : n \in \mathbb{Z}$ and $n < 0\}$,
positive real numbers	$\mathbb{R}^+$	$=$	$\{x : x \in \mathbb{R}$ and $x > 0\}$, and
negative real numbers	$\mathbb{R}^-$	$=$	$\{x : x \in \mathbb{R}$ and $x < 0\}$.

Definition (Subsets (Informal Version))**.** Let A and B be sets.

(a) We say that A is a **subset** of B, denoted $A \subseteq B$, if and only if every element of A is also an element of B.

(b) When it is not the case that $A \subseteq B$, we write $A \not\subseteq B$.

(c) If $A \subseteq B$ and B contains at least one element that A does not, then we say that A is a **proper subset** of B and write $A \subset B$.

Definition (Interval Notation)**.** Given real numbers a and b, define the **intervals**

$$
\begin{aligned}
(a,b) &= \{x : a < x < b\},\\
[a,b] &= \{x : a \leq x \leq b\},\\
[a,b) &= \{x : a \leq x < b\},\\
(a,b] &= \{x : a < x \leq b\},\\
(a,\infty) &= \{x : a < x\},\\
[a,\infty) &= \{x : a \leq x\},\\
(-\infty,b) &= \{x : x < b\}, \text{ and}\\
(-\infty,b] &= \{x : x \leq b\}.
\end{aligned}
$$

Definition. The **empty set**, denoted $\emptyset$, is the unique set that contains no elements.

Theorem ($\emptyset$ is Smallest)**.** *Given any set A, we have $\emptyset \subseteq A$.*

Definition (Cardinality (Informal Version))**.** The **cardinality** of a set A, denoted $|A|$, is the number of elements in A.

Definition (Finiteness (Informal Version))**.** A set A is said to be **finite** if and only if $|A|$ is a natural number. A set that is not finite is said to be **infinite**.

Section 1.3

Definition (Universal Statements)**.** The statement

$$\forall\, x \in \mathcal{U},\ p(x)$$

is defined to be true if and only if, for every value of $x \in \mathcal{U}$, the statement $p(x)$ holds. Consequently, it is false if and only if there is some $x \in \mathcal{U}$ for which $p(x)$ does not hold. The quantifier $\forall$ is read as "for every", "for all", or "for any".

Definition (Existential Statements)**.** The statement

$$\exists\, x \in \mathcal{U} \text{ such that } p(x)$$

is defined to be true if and only if, there exists some $x \in \mathcal{U}$ such that the statement $p(x)$ holds. Consequently, it is false if and only if, for every $x \in \mathcal{U}$, $p(x)$ does not hold. The quantifier $\exists$ is read as "there exists", "there is", or "there are".

Definition (Properties of Real Functions)**.** A real function f is said to be

(a) **constant** if $\exists\, c \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R}, f(x) = c$.

(b) **increasing** if $\forall\, x, y \in \mathbb{R},$ if $x < y,$ then $f(x) < f(y)$.

(c) **decreasing** if $\forall\, x, y \in \mathbb{R},$ if $x < y,$ then $f(x) > f(y)$.

(d) **nondecreasing** if $\forall\, x, y \in \mathbb{R},$ if $x \le y,$ then $f(x) \le f(y)$.

(e) **nonincreasing** if $\forall\, x, y \in \mathbb{R},$ if $x \le y,$ then $f(x) \ge f(y)$.

(f) **periodic** if $\exists\, p \in \mathbb{R}^+$ such that $\forall\, x \in \mathbb{R}, f(x + p) = f(x)$.

(g) **bounded above** if $\exists\, M \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R}, f(x) \le M$.

(h) **bounded below** if $\exists\, L \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R}, f(x) \ge L$.

We say that f is **bounded** if f is both bounded above and bounded below.

Definition (Operations on Real Functions)**.** Given a real number c and real functions f and g, we define

(a) the **constant multiple** cf by $\forall\, x \in \mathbb{R}, \quad (cf)(x) = c \cdot f(x)$.

(b) the **product** $f \cdot g$ by $\forall\, x \in \mathbb{R}, \quad (f \cdot g)(x) = f(x) \cdot g(x)$.

(c) the **sum** $f + g$ by $\forall\, x \in \mathbb{R}, \quad (f + g)(x) = f(x) + g(x)$.

(d) the **composite** $f \circ g$ by $\forall\, x \in \mathbb{R}, \quad (f \circ g)(x) = f(g(x))$.

Proposition (Negating $\forall$ and $\exists$)**.**

(a) $\neg\,[\forall\, x \in \mathcal{U}, p(x)] \quad \equiv \quad \exists\, x \in \mathcal{U}$ *such that* $\neg p(x)$

(b) $\neg\,[\exists\, x \in \mathcal{U}$ *such that* $p(x)] \quad \equiv \quad \forall\, x \in \mathcal{U}, \neg p(x)$

Section 1.4

Definition (Set Equality and Subsets (Formal Version))**.** Given sets A and B whose elements come from some universal set $\mathcal{U}$,

(a) we say that A **equals** B, written $A = B$, if and only if
$\forall\, x \in \mathcal{U}, \quad x \in A \leftrightarrow x \in B$.

(b) we say that A is a **subset** of B, written $A \subseteq B$, if and only if
$\forall\, x \in \mathcal{U}, \quad x \in A \rightarrow x \in B$.

Definition (Basic Set Operations)**.** Given sets A and B (subsets of some universal set $\mathcal{U}$),

(a) the **complement** of A, denoted A^c, is defined by
$\forall\, x \in \mathcal{U}, \quad x \in A^c \leftrightarrow x \notin A$ (i.e. $\neg(x \in A)$).

(b) the **intersection** of A and B, denoted $A \cap B$, is defined by
$\forall\, x \in \mathcal{U}, \quad x \in A \cap B \leftrightarrow x \in A$ and $x \in B$.

(c) the **union** of A and B, denoted $A \cup B$, is defined by
$\forall\, x \in \mathcal{U}, \quad x \in A \cup B \leftrightarrow x \in A$ or $x \in B$.

(d) the **difference** of A **minus** B, denoted $A \setminus B$, is defined by
$\forall\, x \in \mathcal{U}, \quad x \in A \setminus B \leftrightarrow x \in A$ and $x \notin B$.

(e) the **symmetric difference** of A and B, denoted $A \mathbin{\triangle} B$, is defined by
$\forall\, x \in \mathcal{U}, \quad x \in A \mathbin{\triangle} B \leftrightarrow x \in A \ \oplus \ x \in B$.

Definition. Given sets A and B,

(a) they are said to be **disjoint** if and only if $A \cap B = \emptyset$.

(b) the union $A \cup B$ is said to be a **disjoint union** if and only if A and B are disjoint.

Definition (General Products)**.**

(a) Given sets $A_1, A_2, \ldots, A_n$, the **n-fold product** $A_1 \times A_2 \times \cdots \times A_n$ is given by $A_1 \times A_2 \times \cdots \times A_n =$
$\{(x_1, x_2, \ldots, x_n) \ : \ x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n\}$.

(b) The elements $(x_1, x_2, \ldots, x_n)$ of $A_1 \times A_2 \times \cdots \times A_n$ are called **ordered n-tuples** (**ordered pairs** when $n = 2$).

(c) The n-fold product $A \times A \times \cdots \times A$ is denoted A^n.

Definition. Given a set A, the **power set** of A, denoted $\mathcal{P}(A)$, is the set of subsets of A, $\mathcal{P}(A) = \{B \ : \ B \subseteq A\}$. That is, $\forall\, B, \quad B \in \mathcal{P}(A) \leftrightarrow B \subseteq A$.

Theorem (Basic Set Identities)**.** *Let A, B and C be sets (in some universe $\mathcal{U}$). Then, the following identities hold:*

(a)	$(A^c)^c$	$=$	$A.$	*Double Complement*
(b)	$(A \cap B) \cap C$	$=$	$A \cap (B \cap C).$	*Associativity*
	$(A \cup B) \cup C$	$=$	$A \cup (B \cup C).$	
(c)	$A \cap B$	$=$	$B \cap A.$	*Commutativity*
	$A \cup B$	$=$	$B \cup A.$	
(d)	$A \cap (B \cup C)$	$=$	$(A \cap B) \cup (A \cap C).$	*Distributivity*
	$A \cup (B \cap C)$	$=$	$(A \cup B) \cap (A \cup C).$	
(e)	$(A \cap B)^c$	$=$	$A^c \cup B^c.$	*De Morgan's Laws*
	$(A \cup B)^c$	$=$	$A^c \cap B^c.$	
(f)	*If $A \subseteq B$, then* $A \cap B$	$=$	$A.$	*Absorption Rules*
	If $A \subseteq B$, then $A \cup B$	$=$	$B.$	

Theorem (Interactions with $\mathcal{U}$ and $\emptyset$)**.** *Let A be a set (in some universe $\mathcal{U}$). Then, the following identities hold:*

(a)	$\mathcal{U}^c$	$=$	$\emptyset.$
	$\emptyset^c$	$=$	$\mathcal{U}.$
(b)	$A \cap \mathcal{U}$	$=$	$A.$
	$A \cup \mathcal{U}$	$=$	$\mathcal{U}.$
(c)	$A \cap \emptyset$	$=$	$\emptyset.$
	$A \cup \emptyset$	$=$	$A.$
(d)	$A \cap A^c$	$=$	$\emptyset.$
	$A \cup A^c$	$=$	$\mathcal{U}.$

Section 1.5

Definition. (a) An **argument form** $p_1;\ p_2;\ \cdots;\ p_n;\ \therefore\ \ r$ is a sequence of (premise) statement forms $p_1, p_2, \ldots, p_n$ followed by a (conclusion) statement form r (preceded by the symbol $\therefore$ for "therefore").

(b) The argument form is considered to be **valid** if and only if the statement form $p_1 \wedge p_2 \wedge \cdots \wedge p_n \to r$ is a tautology. Otherwise, it is considered to be **invalid**.

Theorem (Basic Valid Argument Forms)**.**

(a)	$p \to q$	*Direct Implication*
	p	
	$\therefore\ \ q$	
(b)	$p \to q$	*Contrapositive Implication*
	$\neg q$	
	$\therefore\ \ \neg p$	
(c)	$p \to q$	*Transitivity of* $\to$
	$q \to r$	
	$\therefore\ \ p \to r$	
(d)	$p \to r$	*Two Separate Cases*
	$q \to r$	
	$p \vee q$	
	$\therefore\ \ r$	
(e)	$p \vee q$	*Eliminating a Possibility*
	$\neg p$	
	$\therefore\ \ q$	
(f)	$p \wedge q$	*In Particular*
	$\therefore\ \ p$	
(g)	p	*Obtaining Or*
	$\therefore\ \ p \vee q$	
(h)	p	*Obtaining And*
	q	
	$\therefore\ \ p \wedge q$	
(i)	$p \leftrightarrow q$	*Substitution of Equivalent*
	p	
	$\therefore\ \ q$	

Theorem (Principle of Specification)**.** *If the premises*

$$\begin{array}{ll} \forall\, x \in \mathcal{U},\ \ p(x) & \quad\textit{and} \\ a \in \mathcal{U} & \end{array}$$

hold, then the conclusion $p(a)$ also holds.

Theorem (Principle of Generalization)**.** *From the following steps:*

(i) Take an arbitrary element $a \in \mathcal{U}$.

(ii) Establish that $p(a)$ holds.

the conclusion $\forall\, x \in \mathcal{U},\ \ p(x)$ is obtained.

1.2 Chapter 2

Section 2.1

Existential Statements. To prove a statement of the form

$$\exists\, x \in \mathcal{U} \text{ such that } p(x)$$

it suffices to present an example of a particular element $x \in \mathcal{U}$ for which $p(x)$ holds.

Counterexamples. A statement of the form $\forall\, x \in \mathcal{U}, p(x)$ is disproved by presenting an example of a particular element $x \in \mathcal{U}$ for which $p(x)$ does not hold. Such an example is called a **counterexample**.

Universal Statements for Small Universes. If a universe $\mathcal{U}$ has a very small cardinality, then it may be reasonable to prove a statement of the form $\forall\, x \in \mathcal{U}, p(x)$ by verifying $p(x)$ for each individual element $x \in \mathcal{U}$.

Section 2.2

If-Then Statements. To prove

$$\forall\, x \in \mathcal{U}, \quad p(x) \to q(x)$$

we suppose that $p(x)$ is true and then show that $q(x)$ must be true under that assumption.

Subsets. To prove $S \subseteq T$, we suppose that we have an element $x \in S$ and show that we must have $x \in T$.

Example. For all sets A and B, $A \cap B \subseteq A$.

Example (Transitivity of the Subset Relation)**.** Let A, B, and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Set Equalities. One method of proving $S = T$ is to prove $\forall\, x \in \mathcal{U}, (x \in S \leftrightarrow x \in T)$ through a sequence of equivalences.

Proof-Writing Strategies.

- A natural start gets the ball rolling.
- Thinking backwards from our destination tells us how to proceed.
- Throughout, the unwinding of definitions provides the details with which and for which we work.

Section 2.3

If and Only If Statements. We can prove a statement of the form $p \leftrightarrow q$ by first proving $p \to q$ and then proving $q \to p$.

Set Equalities Revisited. We can prove $S = T$ by proving $S \subseteq T$ and $T \subseteq S$.

Section 2.4

Proofs by Contradiction. We suppose that the negation of the desired statement holds, and show that this leads to a contradiction.

Proving the Contrapositive. We can prove an if-then statement $p \to q$ by supposing $\neg q$ and establishing $\neg p$.

Theorem ($\emptyset$ is Well-Defined)**.** *There is a unique set with no elements, namely* $\emptyset$.

Section 2.5

If, in the course of a proof, we have an "or" statement $p \vee q$, then we can proceed by considering the cases
Case 1: p.
Case 2: q.

1.3 Chapter 3

Section 3.1

Definition. An integer n is said to be **even** if $n = 2k$ for some integer k, and **odd** if $n = 2k + 1$ for some integer k.

Definition. Given integers n and d, we say that d **divides** n, written $d \mid n$, if $n = dk$ for some integer k. In this case, we also say that n is **divisible** by d, that n is a **multiple** of d, that d is a **divisor** of n, or that d is a **factor** of n. When n is not divisible by d, we write $d \nmid n$.

Example (Transitivity of the Divides Relation)**.** Let a, b, and c be integers. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Theorem. *Let $a, b \in \mathbb{Z}$ with $b > 0$. If $a \mid b$, then $a \leq b$.*

Definition. An integer p is said to be **prime** if $p > 1$ and the only positive divisors of p are 1 and p. An integer $n > 1$ that is not prime is said to be **composite**.

An integer $n > 1$ is **composite** if and only if
$\exists\, r, s \in \mathbb{Z}$ such that $r > 1, s > 1$, and $rs = n$.

Definition. Given integers m and n not both zero, their **greatest common divisor**, denoted $\gcd(m, n)$, is the unique integer d such that

(i) $d > 0$,

(ii) $d \mid m$ and $d \mid n$, and

(iii) $\forall\, c \in \mathbb{Z}^+$, if $c \mid m$ and $c \mid n$, then $c \leq d$.

Definition. Two integers m and n are **relatively prime** if $\gcd(m, n) = 1$.

Example. Given any positive integer k, $\gcd(k, 0) = k$.

Lemma. *Let n be any integer. Then, n and $n + 1$ are relatively prime.*

Section 3.2

Theorem (Well-Ordering Principle for the Integers)**.** *Each nonempty subset of the nonnegative integers has a smallest element.*

Theorem (Existence of Prime Divisors)**.** *Every integer greater than 1 has a prime divisor.*

Theorem. *There are infinitely many primes.*

Theorem (Division Algorithm)**.** *Given any integer n and any positive integer d, there exist unique integers q and r such that $n = dq + r$ and $0 \leq r < d$.*

Definition. In the Division Algorithm, we say that q is the **quotient** and r is the **remainder** upon division of n by d. We also write $q = n$ **div** d and $r = n$ **mod** d.

Definition. Let x be any real number.

(a) The **floor** of x, denoted $\lfloor x \rfloor$, is the largest integer n such that $n \leq x$.

(b) The **ceiling** of x, denoted $\lceil x \rceil$, is the smallest integer n such that $x \leq n$.

Theorem. *Let x be any real number.*

(a) $\lfloor x \rfloor$ is the unique value $n \in \mathbb{Z}$ such that $n \leq x < n + 1$.

(b) $\lceil x \rceil$ is the unique value $n \in \mathbb{Z}$ such that $n - 1 < x \leq n$.

Theorem. *Given any integer n and positive integer d, $\lfloor \frac{n}{d} \rfloor = n$ div d.*

Example (Check Digit Formulas)**.**

(a) In a **Universal Product Code** (UPC) number

$$d_1 \quad d_2 \; d_3 \; d_4 \; d_5 \; d_6 \quad d_7 \; d_8 \; d_9 \; d_{10} \; d_{11} \quad d_{12}$$

d_{12} is determined by the requirement that

$$[3(d_1+d_3+d_5+d_7+d_9+d_{11})+d_2+d_4+d_6+d_8+d_{10}+d_{12}] \bmod 10 = 0.$$

(b) In an **International Standard Book Number** (ISBN)

$$d_1 \; - \; d_2 \; d_3 \; d_4 \; d_5 \; d_6 \; d_7 \; d_8 \; d_9 \; - \; d_{10}$$

d_{10} is determined by the requirement that

$$[10d_1+9d_2+8d_3+7d_4+6d_5+5d_6+4d_7+3d_8+2d_9+d_{10}] \bmod 11 = 0.$$

The **binary linear codes** we consider in this book are constructed as follows. Given a **message**, encoded as a binary string $b_1 b_2 \ldots b_k$, a binary linear code specifies a **code word** $b_1 b_2 \ldots b_k b_{k+1} \ldots b_n$ by appending **parity check digits**, which are determined by the sum of some of the binary digits in the message. The **weight** w of a binary linear code is the minimum number of ones that appear in a nonzero code word. In the method of **nearest neighbor decoding**, we use the first k digits of the code word $b_1 b_2 \ldots b_n$ differing from $c_1 c_2 \ldots c_n$ in the fewest number of digits.

A **shift cipher** has the form $y = (x + b) \bmod n$ for some choice of an integer b. To decrypt a message, the formula $x = (y - b) \bmod n$ is used.

Section 3.3

Theorem (Expressing the GCD as a Linear Combination)**.** *Given integers m and n not both zero, there exist integers x and y such that* $\gcd(m, n) = mx + ny$.

Corollary. *Two integers m and n are relatively prime if and only if there exist integers x and y such that $mx + ny = 1$.*

Theorem (GCD Reduction)**.** *Let n and m be integers such that $n \geq m > 0$. Write $n = mq + r$ where $q, r \in \mathbb{Z}$ with $0 \leq r < m$. Then,* $\gcd(n, m) = \gcd(m, r)$.

Algorithm 1 Euclid's Algorithm for finding $\gcd(n, m)$

Let $n, m \in \mathbb{Z}^+$ with $n \geq m$.

Algorithm.

While $m > 0$,
    `\begin`
    Let $r = n \bmod m$.
    Let $n = m$.
    Let $m = r$.
    `\end.`
Return n.

Theorem (Euclid's Lemma)**.** *Let m, n, and c be integers. If $c \mid mn$ and* $\gcd(c, m) = 1$*, then $c \mid n$.*

Section 3.4

Definition. A real number r is said to be **rational** if $r = \frac{a}{b}$ for some integers a and b with $b \neq 0$.

Theorem. $\mathbb{Z} \subseteq \mathbb{Q}$.

Theorem 1.1 (Field Properties of $\mathbb{Q}$)**.** *Let $r, s \in \mathbb{Q}$. Then,*

(a) $0, 1 \in \mathbb{Q}$.

(b) $r + s \in \mathbb{Q}$.

(c) $-s \in \mathbb{Q}$.

(d) $rs \in \mathbb{Q}$.

(e) if $s \neq 0$, then $\frac{1}{s} \in \mathbb{Q}$.

Theorem 1.2 (Expressing Rational Numbers in Lowest Terms)**.** *Given $r \in \mathbb{Q}$, there exist unique $a, b \in \mathbb{Z}$ such that $b > 0$,* $\gcd(a, b) = 1$*, and $r = \frac{a}{b}$. That is, $\frac{a}{b}$ expresses r in* ***lowest terms****.*

Theorem 1.3 (When Decimals are Rational)**.** *A real number written in decimal form represents a rational number if and only if the decimal part is either finite or repeating. Moreover, a rational number $r = \frac{a}{b}$ written in lowest terms has a finite decimal expansion if and only if 2 and/or 5 are the only prime divisors of b. Otherwise, the decimal part of r repeats.*

Definition. A real number that is not rational is said to be **irrational**.

Theorem. *$\sqrt{2}$ is irrational.*

Theorem (Rational Roots Theorem)**.** *Let $n \in \mathbb{Z}^+$ and let $f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ be a polynomial with integer coefficients $c_n, c_{n-1}, \ldots, c_1, c_0$ such that $c_n \neq 0$. If r is a rational root of f (i.e. $r \in \mathbb{Q}$ and $f(r) = 0$), then $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ such that $a \mid c_0$ and $b \mid c_n$.*

Definition. A real number r is said to be **algebraic** if r is a root of a polynomial with integer coefficients. Real numbers which are not algebraic are said to be **transcendental**.

Section 3.5

Definition. Given integers a, b, and n with $n > 1$, we say that a is **congruent** to b **modulo** n, written $a \equiv b \pmod{n}$, if $n \mid (a - b)$.

Theorem (Congruence is an Equivalence Relation)**.** *Let a, b, and n be integers with $n > 1$.*

(a) $a \equiv a \pmod{n}$.

(b) If $a \equiv b \pmod{n}$, *then* $b \equiv a \pmod{n}$.

(c) If $a \equiv b \pmod{n}$ *and* $b \equiv c \pmod{n}$, *then* $a \equiv c \pmod{n}$.

Theorem (Arithmetic Properties of Congruence)**.** *Let a_1, a_2, b_1, b_2, and n be integers with $n > 1$. If $a_1 \equiv a_2 \pmod{n}$ and $b_1 \equiv b_2 \pmod{n}$, then*

(a) $a_1 + b_1 \equiv a_2 + b_2 \pmod{n}$, *and*

(b) $a_1 b_1 \equiv a_2 b_2 \pmod{n}$.

Lemma. *Given integers n and d with $d > 1$, we have $n \bmod d \equiv n \pmod{d}$, and, moreover, $n \bmod d$ is the unique integer in $\{0, 1, \ldots, d-1\}$ congruent to n modulo d.*

Lemma (Modular Cancellation Rule)**.** *Let a, b_1, b_2, n be integers with $n > 1$. Suppose that $ab_1 \equiv ab_2 \pmod{n}$ and $\gcd(a, n) = 1$. Then, $b_1 \equiv b_2 \pmod{n}$.*

Definition. Given $a, n \in \mathbb{Z}$ with $n > 1$, a **multiplicative inverse of a modulo n** is an integer c such that $ac \equiv 1 \pmod{n}$.

Lemma. *Given $n \in \mathbb{Z}$ with $n > 1$, an integer a has a multiplicative inverse modulo n if and only if $\gcd(a, n) = 1$.*

A **linear cipher** has the form $y = (ax+b) \bmod n$, for some choice of integers a, b, and n. When a has a multiplicative inverse c modulo n, deciphering is accomplished with $x = c(y-b) \bmod n$.

For **RSA encryption**, an integer n is known to the sender and receiver of a secret message. Specifically, n is chosen by the receiver to be the product of two (large) primes p and q. Also, the receiver picks an integer a that has a multiplicative inverse c modulo $m = \operatorname{lcm}(p-1, q-1)$. The sender encrypts a message x using $y = x^a \bmod n$. The receiver decrypts the message using $x = y^c \bmod n$.

Theorem (Fermat's Little Theorem)**.** *If p is a prime, $a \in \mathbb{Z}$, and $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.*

Corollary. *Let $n \in \mathbb{Z}$ with $n > 1$.*
If there exists $a \in \mathbb{Z}$ such that $a^n \not\equiv a \pmod{n}$, then n is not prime.

Given integers a and $n > 1$, the **equivalence class of** a **modulo** n is $[a]_n = \{k \ : \ k \in \mathbb{Z} \text{ and } k \equiv a \pmod{n}\}$. For all $a, b \in \mathbb{Z}$, $[a]_n = [b]_n$ if and only if $a \equiv b \pmod{n}$. We define $\mathbb{Z}_n = \{[a]_n \ : \ a \in \mathbb{Z}\}$.

Theorem ($\mathbb{Z}_n$ Forms a Group Under $+$)**.** *Let n be an integer such that $n > 1$, and let $a, b, c \in \mathbb{Z}$. Then,*

(a)	$([a]_n + [b]_n) + [c]_n = [a]_n + ([b]_n + [c]_n)$,	*Associativity*
(b)	$[0]_n + [a]_n = [a]_n$,	*Identity*
(c)	$[-a]_n + [a]_n = [0]_n$.	*Inverse*

1.4 Chapter 4

Section 4.1

***n* factorial** is defined by $n! = n(n-1)(n-2)\cdots 2\cdot 1$. Given integers n and k with $0 \le k \le n$, the **binomial coefficient** is defined by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

We express a sequence in the form $\{s_n\}_{n\ge a}$ and mean that our sequence consists of the terms $s_a, s_{a+1}, s_{a+2}, s_{a+3}, s_{a+4}, s_{a+5}, \ldots$, where $a \in \mathbb{Z}$. That is, the sequence has been **indexed** by the integers $a, a+1, a+2, a+3, a+4, a+5, \ldots$.

An **arithmetic sequence** with given first term s_0 and common difference c is generated by the formula $\forall n \ge 0,\ \ s_n = s_0 + cn$. A **geometric sequence** with given first term s_0 and multiplying factor r is generated by the formula $\forall n \ge 0,\ \ s_n = s_0 r^n$.

Example (Recursion in Arithmetic and Geometric Sequences)**.**

(a) An arithmetic sequence is expressed recursively by specifying s_0 and a constant c for which $\forall n \ge 1,\ \ s_n = s_{n-1} + c$.

(b) A geometric sequence is expressed recursively by specifying s_0 and a constant r for which $\forall n \ge 1,\ \ s_n = rs_{n-1}$.

Section 4.2

Given a sequence $\{s_n\}$, the sum $S = s_a + s_{a+1} + s_{a+2} + \cdots + s_{b-1} + s_b$ is represented in **sigma notation** as $S = \displaystyle\sum_{i=a}^{b} s_i$.

Theorem. *Let $a, b \in \mathbb{Z}$, let $\{s_n\}$ and $\{t_n\}$ be sequences, and let $c \in \mathbb{R}$.*

(a) $\displaystyle\sum_{i=a}^{b}(s_i \pm t_i) = \sum_{i=a}^{b} s_i \pm \sum_{i=a}^{b} t_i$.

(b) $\displaystyle\sum_{i=a}^{b} cs_i = c\sum_{i=a}^{b} s_i$.

Theorem. *Let $n \in \mathbb{Z}$ with $n \ge 1$. Then,*

(a) $\displaystyle\sum_{i=1}^{n} 1 = n$.

(b) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

(c) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

(d) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$

Theorem. *Let* $r \in \mathbb{R}$ *with* $r \neq 1$ *and* $n \in \mathbb{Z}$ *with* $n \geq 0$*. Then,* $\displaystyle\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$.

Theorem. *Let* $m \geq 1$ *and* $n \geq 1$ *be integers. Then,*

$$\sum_{i=1}^{n} i^m = \frac{(n+1)((n+1)^m - 1) - \sum_{j=1}^{m-1}\left[\binom{m+1}{j}\sum_{i=1}^{n} i^j\right]}{m+1}.$$

In **product notation**, the product $P = s_a \cdot s_{a+1} \cdot s_{a+2} \cdot \quad \cdots \quad \cdot s_b$ is represented by $P = \displaystyle\prod_{i=a}^{b} s_i$.

Section 4.3

Outline (Proof by Mathematical Induction)**.** To show: $\forall\, n \geq a,\ P(n)$.
Proof by induction

1. Base cases:
   Show: $P(a), \ldots, P(b)$ are true.

2. Inductive step:
   Show: $\forall\, k \geq b$, if $P(k)$ is true, then $P(k+1)$ is true.
   That is,

   (a) Suppose $k \geq b$ and that $P(k)$ is true.

   (b) Show: $P(k+1)$ is true.

Theorem (Principle of Mathematical Induction)**.** *Let* $a \leq b$ *be integers, and let* $P(n)$ *be an expression that depends on the free integer variable* n*. If*

(i) $P(a), \ldots, P(b)$ *hold, and*

(ii) $\forall\, k \geq b$*, if* $P(k)$ *holds, then* $P(k+1)$ *holds,*

then, the statement $\forall\, n \geq a,\quad P(n)$ *holds.*

Section 4.4

Suppose we wish to prove by induction that $\forall\, n \geq 1,\ \displaystyle\sum_{i=1}^{n} s_i = p(n)$. In the inductive step, we split the sum $\displaystyle\sum_{i=1}^{k+1} s_i$ into $\left(\displaystyle\sum_{i=1}^{k} s_i\right) + s_{k+1}$ and use the inductive hypothesis to substitute $\displaystyle\sum_{i=1}^{k} s_i = p(k)$.

Section 4.5

Outline (Proof by Strong Induction)**.** To show: $\forall\, n \geq a,\ P(n)$.
Proof by strong induction

1. Base cases:
   Show: $P(a), \ldots, P(b)$ are true.

2. Inductive step:
   Show: $\forall\, k \geq b$, if $P(a), \ldots, P(k)$ are true, then $P(k+1)$ is true. That is,

   (a) Suppose $k \geq b$ and that $P(i)$ is true for all $a \leq i \leq k$.
   (b) Show: $P(k+1)$ is true.

Theorem (Principle of Strong Induction)**.** *Let $a \leq b$ be integers, and let $P(n)$ be an expression that depends on the free integer variable n. If*

(i) $P(a), \ldots, P(b)$ hold, and

(ii) $\forall\, k \geq b$, if $P(i)$ holds for each $a \leq i \leq k$, then $P(k+1)$ holds,

then, the statement $\forall\, n \geq a,\ \ P(n)$ holds.

Definition. The expression of an integer $n > 1$ as a product of the form $n = p_1^{e_1} \cdot p_2^{e_2} \cdot \quad \cdots \quad \cdot p_m^{e_m}$, where m is a positive integer, $p_1 < p_2 < \cdots < p_m$ are primes, and $e_1, e_2, \ldots, e_m$ are positive integers, is referred to as the **standard factorization** of n.

Theorem (Fundamental Theorem of Arithmetic)**.** *Every integer greater than 1 has a unique standard factorization.*

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is defined by $F_0 = 1$, $F_1 = 1$ and $\forall\, n \geq 2,\ \ F_n = F_{n-2} + F_{n-1}$.

Example. $\forall\, n \geq 2,\ \ F_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n+1}\right].$

Section 4.6

Pascal's triangle

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \vdots & & & &
\end{array}
$$

is characterized by the identities

$$\forall\, n \geq 0,\ \ \binom{n}{0} = \binom{n}{n} = 1,$$
$$\forall\, n \geq 2 \text{ and } 1 \leq k \leq n-1,\ \ \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

the second of which is known as **Pascal's identity**.

Theorem (The Binomial Theorem)**.** *Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Then,*

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

1.5 Chapter 5

Section 5.1

Definition. Given sets X and Y, a **relation from X to Y** is a subset R of $X \times Y$. When $(x, y) \in R$, we say that x **is related to** y (by R) and write $x\,R\,y$. Similarly, $(x, y) \notin R$ is denoted by $x \not R\, y$.

Definition. A **relation on a set** X is a relation from X to itself.

Definition. The **inverse** of a given relation R from a set X to a set Y is the relation $R^{-1} = \{(y, x)\ :\ y \in Y, x \in X, \text{ and } (x, y) \in R\}$ from Y to X. That is, $\forall\, y \in Y, x \in X, \quad y\,R^{-1}\,x \quad \text{if and only if} \quad x\,R\,y.$

An **arrow diagram** for R from X to Y is constructed by representing the sets X and Y in disjoint regions and drawing an arrow from an element $x \in X$ to an element $y \in Y$ if and only if $x\,R\,y$.

Given a relation R on a finite set X, a **directed graph**, or **digraph**, for R is obtained by displaying the elements of X and drawing an arrow from an element x to an element y if and only if $x\,R\,y$.

A relation R from a finite set X to a finite set Y may be represented by a $|X|$ by $|Y|$ **matrix**. For each $x \in X$ and $y \in Y$, the entry in row x and column y is assigned the value 1 if $x\,R\,y$ and 0 if $x \not R\, y$.

Since a relation on $\mathbb{R}$ is a subset of $\mathbb{R} \times \mathbb{R}$, its **graph** may be drawn in the Cartesian plane.

Definition. A relation R on a set X is said to be

(a) **reflexive** if $\forall\, x \in X,\ x\,R\,x$.

(b) **symmetric** if $\forall\, x, y \in X,\ x\,R\,y \ \rightarrow\ y\,R\,x$.

(c) **antisymmetric** if $\forall\, x, y \in X,\ x\,R\,y \ \text{ and }\ y\,R\,x \ \rightarrow\ x = y$.

(d) **transitive** if $\forall\, x, y, z \in X,\ x\,R\,y \ \text{ and }\ y\,R\,z \ \rightarrow\ x\,R\,z$.

Theorem. *Let R be any relation on a set X. Suppose R is symmetric and transitive and every $x \in R$ has some $y \in R$ to which it is related (i.e., $\forall\, x \in R, \exists\, y \in R$ such that $x\,R\,y$). Then, R is reflexive.*

Section 5.2

Definition. A **partial order relation** or **partial ordering** on a set X is a relation on X that is reflexive, antisymmetric, and transitive. In this case, we say that X is a **partially ordered set** or **poset** (under R). Given a partial order relation R on a set X and two elements $x, y \in X$, we say that x is **comparable** to y (under R) if and only if $x\,R\,y$ or $y\,R\,x$. Otherwise, x and y are said to be **incomparable**.

A **Hasse diagram** for a relation R on a set X is constructed as follows.

(1) Construct a digraph with the elements of X arranged so that all of the arrows point upward.

(2) Delete all loops.

(3) Delete any arrows that follow from transitivity.

(4) Delete the direction indicators from the arrows.

Definition. A partial order relation R on a set X is called a **total order relation** or **linear order relation** if every pair of elements is comparable. In this case, we say that X is a **totally ordered set** or **linearly ordered set** (under R).

Given a character set C, a **word** over C is a string $x_1x_2\cdots x_n$, where $n \in \mathbb{Z}^+$ and $x_1, x_2, \ldots, x_n \in C$. When C is partially ordered by some relation $\preceq$, we construct the **lexicographic ordering** $\trianglelefteq$ on the set of words over C. Given $\vec{x} = x_1x_2\cdots x_m$ and $\vec{y} = y_1y_2\cdots y_n$, let k be the largest index such that $x_1x_2\cdots x_k = y_1y_2\cdots y_k$, and define $\vec{x} \triangleleft \vec{y}$ if either $k = m < n$ or $k < m, n$ and $x_{k+1} \preceq y_{k+1}$. Take $\vec{x} \trianglelefteq \vec{y}$ precisely if $\vec{x} \triangleleft \vec{y}$ or $\vec{x} = \vec{y}$.

Definition. An **equivalence relation** on a set X is a relation on X that is reflexive, symmetric, and transitive. Given an equivalence relation R on a set X and two elements $x, y \in X$, we say that x is **equivalent** to y (under R) if and only if $x\ R\ y$.

Definition. Given an equivalence relation R on a set X, the **equivalence class** (under R) of an element $x \in X$ is the set $\{y\ :\ y \in X \text{ and } y\ R\ x\}$ of all elements of X that are equivalent to x. It is denoted by $[x]_R$, and the subscript is dropped from our notation if the relation R is understood in context. A **representative** of an equivalence class $[x]$ is an element $y \in [x]$.

Lemma. *Let X be any set and R be any equivalence relation on X. For all $x, y \in X$, $y\ R\ x$ if and only if $[y] = [x]$.*

Theorem. *Let X be any set and R be any equivalence relation on X. For all $x, y \in X$, $[y] \neq [x]$ if and only if $[y] \cap [x] = \emptyset$.*

Definition. Let $\mathcal{A}$ be a collection of sets from some universe $\mathcal{U}$.

(a) The **union** of $\mathcal{A}$, denoted $\displaystyle\bigcup_{A\in\mathcal{A}} A$, is the set defined by $\forall\ x \in \mathcal{U}$, $x \in \bigcup_{A\in\mathcal{A}} A$ iff $x \in A$ for some $A \in \mathcal{A}$.

(b) We say that $\mathcal{A}$ is a collection of **disjoint** sets if $\forall\ A, B \in \mathcal{A}$, if $A \neq B$, then $A \cap B = \emptyset$.

Definition. A **partition** of a set X is a collection $\mathcal{A}$ of disjoint nonempty subsets of X whose union is X.

Lemma. *Given an equivalence relation R on a nonempty set X, the collection of equivalence classes $\{[x] \ : \ x \in X\}$ is a partition of X, called the* ***partition of X corresponding to R****.*

Lemma. *Given a partition $\mathcal{A}$ of a set X, the relation R defined by $x \; R \; y$ if and only if $\exists \; A \in \mathcal{A}$ such that $x, y \in A$. is an equivalence relation on X, called the* ***equivalence relation on X corresponding to $\mathcal{A}$****.*

Theorem (Correspondence between Equivalence Relations and Partitions)**.** *Let X be a set, R be an equivalence relation on X, and $\mathcal{A}$ be a partition of X. Then, $\mathcal{A}$ is the partition of X corresponding to R if and only if R is the equivalence relation on X corresponding to $\mathcal{A}$.*

Section 5.3

Definition. A **function f from X to Y**, denoted $f : X \longrightarrow Y$, is a relation from X to Y such that each $x \in X$ is related to a *unique* $y \in Y$. In this context, we write $f(x) = y$ in place of $x \; f \; y$ or $(x, y) \in f$. When f is understood, we may also write $x \mapsto y$.

(a) When $f(x) = y$, so $x \mapsto y$, we say that f **maps** the element x to the element y or that y is the **image** of x (under f). We also say that f **maps** the set X to the set Y. In fact, functions are sometimes called maps.

(b) The **domain** of f is the set X. That is, domain$(f) = X$.

(c) The **codomain**, or **target**, of f is the set Y.

(d) The **range**, or **image**, of f is the set range$(f) = \{y \ : \ y \in Y \text{ and } f(x) = y \text{ for some } x \in X\}$.

Definition. Given functions $f : X \longrightarrow Y$ and $g : W \longrightarrow Z$ such that the range of f is a subset of the domain of g, their **composite**, denoted $g \circ f$, is the function $g \circ f : X \longrightarrow Z$ defined by the formula $\forall \; x \in X, \quad (g \circ f)(x) = g(f(x))$.

Theorem (Associativity of Function Composition)**.**
Given any functions $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$, and $h : Z \longrightarrow W$, $(h \circ g) \circ f = h \circ (g \circ f)$.

Definition. (a) A **real polynomial function** is a real function f for which there are $n \in \mathbb{N}$ and $c_n, c_{n-1}, \ldots, c_0 \in \mathbb{R}$ such that $\forall \; x \in \mathbb{R}, \quad f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$.

(b) An **exponential function** is a real function f for which there is some $b \in \mathbb{R}^+$ such that $\forall \; x \in \mathbb{R}, \quad f(x) = b^x$. The number b is called its **base**. The function $f(x) = e^x$ is called the **natural exponential** function.

The **graph of a function** $f : X \longrightarrow Y$ is the set $\{(x, y) \ : \ x \in X, y \in Y, \text{ and } f(x) = y\}$.

Test (Vertical Line Test)**.** Let $X, Y \subseteq \mathbb{R}$, and let R be a relation from X to Y. Consider all vertical lines in $\mathbb{R}^2$. If every such vertical line intersects R *at most once*, then R is a function. Moreover, each vertical line $x = a$ should intersect R *exactly once* if $a \in X$, and *not at all* if $a \notin X$.

Test (The Horizontal Line Range Test)**.** Let $X, Y \subseteq \mathbb{R}$, and let $f : X \longrightarrow Y$. Consider all horizontal lines in $\mathbb{R}^2$. The range of f consists of all values b such that the line $y = b$ intersects the graph of f *at least once.*

Section 5.4

Definition. Given a set X, the **identity function** on X, denoted id_X, is the function $\mathrm{id}_X : X \longrightarrow X$ given by the formula $\forall\, x \in X,\ \ \mathrm{id}_X(x) = x$.

Lemma. *Let $f : X \longrightarrow Y$ be any function. Then, $f \circ id_X = f$ and $id_Y \circ f = f$.*

Definition. Let a function $f : X \longrightarrow Y$ be given.

(a) We say that f is **one-to-one**, or **injective**, if $\forall\, x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

(b) We say that f is **onto**, or **surjective**, if $\text{range}(f) = Y$.

(c) We say that f is **bijective** if f is both one-to-one and onto. A bijective function is said to be a **bijection** or a **one-to-one correspondence**.

Test (Horizontal Line Tests)**.** Let $X \subseteq \mathbb{R}$, $Y \subseteq \mathbb{R}$, and $f : X \longrightarrow Y$. Consider all horizontal lines in $\mathbb{R}^2$ of the form $y = b$ for some $b \in Y$.

(a) If every such horizontal line intersects the graph of f *at most once*, then f is one-to-one.

(b) If every such horizontal line intersects the graph of f *at least once*, then f is onto.

Theorem (Composition Preserves One-to-one and Onto)**.** *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be functions.*

(a) If f and g are one-to-one, then $g \circ f$ is one-to-one.

(b) If f and g are onto, then $g \circ f$ is onto.

(c) If f and g are bijective, then $g \circ f$ is bijective.

Definition. Two functions $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ are said to be **inverses of one another** if $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Theorem. *Let $f : X \longrightarrow Y$ be any function.*

(a) If f is a bijection, then f^{-1} is a function and f and f^{-1} are inverses of one another.

(b) If there is a function $g : Y \longrightarrow X$ such that f and g are inverses of one another, then f is a bijection and $g = f^{-1}$.

Definition. If a function $f : X \longrightarrow Y$ satisfies the hypotheses of either part (a) or part (b) of the previous theorem, then the *function* f^{-1} defined by, $\forall\, x \in X$ and $y \in Y$, $f^{-1}(y) = x$ if and only if $f(x) = y$ is called the **inverse function** for f.

The inverse of the exponential function $f(x) = b^x$ is the **logarithm base** b function $g(x) = \log_b(x)$. That is, $\forall\, x \in \mathbb{R}$ and $y \in \mathbb{R}^+$, $\log_b(y) = x$ if and only if $b^x = y$.

Definition. The **natural logarithm** function is the function $\ln : \mathbb{R}^+ \longrightarrow \mathbb{R}$ given by $\forall\, x \in \mathbb{R}^+,\ \ln(x) = \log_e(x)$.

Section 5.5

Definition. Let a function $f : X \longrightarrow Y$ be given.

(a) Given a subset S of X, the **image of** S **under** f is the set given by $f(S) = \{t\ :\ t \in Y \text{ and } f(s) = t \text{ for some } s \in S\}$.

(b) Given a subset T of Y, the **inverse image of** T **under** f is the set given by $f^{-1}(T) = \{s\ :\ s \in X \text{ and } f(s) = t \text{ for some } t \in T\}$.

Definition. Let $\mathcal{U}$ be some fixed universal set. Given a set $\mathcal{I}$ and a function that assigns to each $i \in \mathcal{I}$ a set A_i in $\mathcal{U}$, we say that $\mathcal{I}$ is the **indexing set** for the **indexed collection** $\{A_i\}_{i\in\mathcal{I}}$ of sets.

Definition. Let $\mathcal{I}$ be the indexing set for an indexed collection of sets $\{A_i\}_{i\in\mathcal{I}}$ from some universe $\mathcal{U}$.

(a) The **union** of $\{A_i\}_{i\in\mathcal{I}}$, denoted $\bigcup\limits_{i\in\mathcal{I}} A_i$, is the set defined by $\forall\, x \in \mathcal{U},\quad x \in \bigcup\limits_{i\in\mathcal{I}} A_i \quad\leftrightarrow\quad x \in A_i$ for some $i \in \mathcal{I}$.

(b) The **intersection** of $\{A_i\}_{i\in\mathcal{I}}$, denoted $\bigcap\limits_{i\in\mathcal{I}} A_i$, is the set defined by $\forall\, x \in \mathcal{U},\quad x \in \bigcap\limits_{i\in\mathcal{I}} A_i \quad\leftrightarrow\quad x \in A_i$ for every $i \in \mathcal{I}$.

Section 5.6

Definition. (a) Two sets A and B are said to have the same **cardinality** if there is a bijection from A to B.

(b) Given $n \in \mathbb{N}$, a set A is said to have **cardinality** n if A has the same cardinality as the set $\{k\ :\ k \in \mathbb{Z} \text{ and } 1 \le k \le n\}$.

(c) A set A is said to be **finite**, or to have **finite cardinality**, if A has cardinality n for some $n \geq 0$. Otherwise, A is said to be **infinite**.

Theorem (Common Cardinality is an Equivalence Relation)**.** *Let A, B, and C be any sets in some fixed universal set $\mathcal{U}$.*

(a) A has the same cardinality as itself.

(b) If A has the same cardinality as B, then B has the same cardinality as A.

(c) If A has the same cardinality as B and B has the same cardinality as C, then A has the same cardinality as C.

Theorem (The Pigeon Hole Principle)**.** *If A is any set of cardinality n and B is any set of cardinality m with $n > m$, then there is no one-to-one function from A to B. That is, any function from A to B must send two distinct elements of A to the same element of B.*

Corollary. *Let A be a set with cardinality n, and let $m \in \mathbb{Z}$ with $m \neq n$. Then, A does not have cardinality m.*

Definition. Let A be any set.

(i) A is said to be **countably infinite** if A has the same cardinality as $\mathbb{Z}^+$.

(ii) A is said to be **countable** if A is finite or countably infinite.

(iii) A is said to be **uncountable** if A is not countable.

Theorem. $\mathbb{R}$ *is uncountable.*

Theorem. $\mathbb{Q}$ *is countably infinite.*

Theorem. *All intervals containing more than one element (including the interval $(-\infty, \infty) = \mathbb{R}$) have the same cardinality.*

1.6 Chapter 6

Section 6.1

Theorem (Multiplication Principle)**.** *Let A be a set of outcomes we wish to count. If there is a set of outcomes A_1, and, for each outcome in A_1, there is a set of outcomes A_2 such that*

(i) each outcome from A can be uniquely characterized by a pair of outcomes, the first from A_1 and the second from its corresponding set A_2, and

(ii) for each outcome from A_1, the number $|A_2|$ is the same,

then $|A| = |A_1| \cdot |A_2|$.

General Multiplication Principle. Suppose that $n \geq 2$ and each outcome in a set A is uniquely characterized by a sequence of outcomes, one from each of a sequence of sets $A_1, A_2, \ldots, A_n$. If, for each $2 \leq k \leq n$, the number $|A_k|$ does not depend on any of the sets A_i for $1 \leq i \leq k-1$, then $|A| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|$.

Section 6.2

A **permutation** of a set of objects is an ordering of those objects.

Theorem. *The number of ways to put n distinct items in order is $n!$.*

Definition. A **permutation** of k objects from a set of size n is an ordered list of k of the n objects. The number of permutations of k objects from n is denoted $P(n, k)$.

Theorem. *Let $n, k \in \mathbb{Z}$ with $0 \leq k \leq n$. Then,*
$P(n, k) = n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}$.

Definition. A **combination** of k elements from a set of size n is a subset of size k.

Theorem. *Let $n, k \in \mathbb{Z}$ with $0 \leq k \leq n$. Given a set of n distinct elements, the number of subsets of size k is given by the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.*

Section 6.3

Theorem (Addition Principle)**.** *Finite sets A and B are disjoint if and only if $|A \cup B| = |A| + |B|$.*

Theorem (Complement Principle)**.** *Given a subset A of a finite universal set $\mathcal{U}$, $|A| = |\mathcal{U}| - |A^c|$.*

Theorem (Basic Inclusion-Exclusion Principle)**.** *Given finite sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$.*

Corollary. *Given subsets A and B of a finite universal set $\mathcal{U}$, $|A^c \cap B^c| = |\mathcal{U}| - |A| - |B| + |A \cap B|$.*

Euler phi-function. For any positive integer n, the value $\phi(n)$ is defined to be the number of integers from 1 to n that are relatively prime to n.

Section 6.4

An **experiment** is a specific task of consideration. The possible results of the experiment are called **outcomes**, and the set S of all possible outcomes is called the **sample space** for the experiment. A subset E of S is said to be an **event**. Given a particular event E, the **probability** of E, denoted $P(E)$, is a value between 0 and 1 that gives the likelihood that an outcome in E will occur if the experiment is performed.

Definition. Let S be a finite sample space. The outcomes in S are said to be **equally likely** if $\forall\, x, y \in S, P(x) = P(y)$. That is, $\forall\, x \in S, P(x) = \frac{1}{|S|}$.

Definition (Probability when outcomes are equally likely)**.** If the outcomes in a finite sample space S are all equally likely, then the **probability** of an event E is given by $P(E) = \frac{|E|}{|S|}$.

Theorem (Probability Complement Principle)**.** *If E is an event in a sample space S, then $P(E) = 1 - P(E^c)$.*

Theorem (Basic Probability Inclusion-Exclusion)**.** *Given events E and F in a sample space S, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.*

Definition. Let E and F be events in a sample space S with $P(F) > 0$. The **conditional probability of** E **given** F, denoted $P(E \mid F)$, is given by $P(E \mid F) = \frac{P(E \cap F)}{P(F)}$.

Definition. Two events E and F in a sample space S are said to be **independent** if $P(E \cap F) = P(E) \cdot P(F)$.

Theorem. *Suppose $E, F_1, \ldots, F_n$ are events in a sample space S with $P(F_1)$, $\ldots$, $P(F_n)$ positive, and S is a disjoint union $S = F_1 \cup \cdots \cup F_n$. Then,* $P(E) = \displaystyle\sum_{i=1}^{n} P(E \mid F_i)P(F_i)$.

Corollary (Bayes' Formula)**.** *Suppose $E, F_1, \ldots, F_n$ are events in a sample space S with $P(E), P(F_1), \ldots, P(F_n)$ positive, and S is a disjoint union $S = F_1 \cup \cdots \cup F_n$. Then, for any $1 \leq k \leq n$,* $P(F_k \mid E) = \dfrac{P(E \mid F_k)P(F_k)}{\sum_{i=1}^{n} P(E \mid F_i)P(F_i)}$.

Section 6.5

A **standard deck** contains 52 cards. There are 13 of each **suit** (clubs $\clubsuit$, diamonds $\diamondsuit$, hearts $\heartsuit$, and spades $\spadesuit$), and each suit is numbered with the 13 **denominations** $2, 3, 4, 5, 6, 7, 8, 9, 10, \text{Jack}, \text{Queen}, \text{King}, \text{Ace}$ listed here in increasing order of value. Jacks, Queens, and Kings are called **face cards**. A **run** of 5 cards is any set of 5 cards whose denominations are either 5 consecutive denominations from the list above or the list $\text{Ace}, 2, 3, 4, 5$.

Hand	Description
Straight-Flush	A run of 5 cards of the same suit
Four of a Kind	4 cards of one denomination
Full House	3 cards of one denomination and 2 of one other
Flush	5 cards of the same suit, not forming a run
Straight	A run of 5 cards, not all of the same suit
Three of a Kind	3 cards of one denomination and 2 of others
Two Pairs	2 cards each of two denominations and 1 of one other
One Pair	2 cards of one denomination and 3 of others
Nothing	None of the hands listed above

Table 5: Order and Description of Poker Hands

Theorem. *The number of ways to distribute n identical items into c distinct categories is* $\binom{n+c-1}{n}$.

Section 6.6

A useful counting technique is to start with a count that is too large because it ignores a set of symmetries and then divide out by the size of that set of symmetries.

1.7 Chapter 7

Section 7.1

Theorem (Generalized Inclusion-Exclusion Principle). *Let n sets $A_1, A_2, \ldots, A_n$ be given. For each $1 \le i \le n$, define* $S_i = \displaystyle\sum_{1 \le j_1 < j_2 < \cdots < j_i \le n} |A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_i}|$.

Then, $|A_1 \cup A_2 \cup \cdots \cup A_n| = \displaystyle\sum_{i=1}^{n} (-1)^{i-1} S_i$.

E.g., the Inclusion-Exclusion Principle for $n = 4$.

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| \\ &\quad -(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\ &\quad +(|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) \\ &\quad -|A_1 \cap A_2 \cap A_3 \cap A_4| \end{aligned}$$

Corollary. *Given subsets $A_1, A_2, \ldots, A_n$ of $\mathcal{U}$ with $S_0 = |\mathcal{U}| \in \mathbb{N}$,*

$$|A_1{}^c \cap A_2{}^c \cap \cdots \cap A_n{}^c| = \sum_{i=0}^{n} (-1)^i S_i.$$

A **derangement** of a set is a permutation that leaves no element fixed.

Section 7.2

Given nonnegative integers $k_1, k_2, \ldots, k_m$ and $n = k_1 + k_2 + \cdots k_m$, the **multinomial coefficient** is defined by $\displaystyle\binom{n}{k_1, k_2, \cdots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}$.

Theorem 1.4. *Given nonnegative integers $k_1, k_2, \ldots, k_m$, and $n = k_1 + k_2 + \cdots + k_m$, the multinomial coefficient $\binom{n}{k_1, k_2, \cdots, k_m}$ counts the number of ways to split n distinct items into m distinct categories of sizes $k_1, k_2, \ldots, k_m$.*

Theorem 1.5 (The Multinomial Theorem). *Let $a_1, a_2, \ldots a_m \in \mathbb{R}$ and $n \in \mathbb{N}$. Then,*

$$(a_1 + a_2 + \cdots + a_m)^n = \sum_{\substack{0 \le k_1, k_2, \ldots, k_m \le n \\ k_1 + k_2 + \cdots + k_m = n}} \binom{n}{k_1, k_2, \cdots k_m} a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}.$$

The number of terms in the expansion of $(a_1 + a_2 + \cdots + a_m)^n$ is $\binom{n+m-1}{n}$.

Section 7.3

Definition. The **generating function** for a given sequence $c_0, c_1, c_2, \ldots$ of real numbers is the function $g(x) = \sum_{i=0}^{\infty} c_i x^i = c_0 + c_1 x + c_2 x^2 + \cdots$, where x is a real variable (and we identify $x^0 = 1$).

Theorem. *Let $n \in \mathbb{Z}^+$, and let $g(x) = c_0 + c_1x + c_2x^2 + \cdots$. Then,*

(a) $g(x) = \frac{1-x^{n+1}}{1-x}$ if and only if, for each $0 \le i \le n$, $c_i = 1$, and, for each $i > n$, $c_i = 0$.

(b) $g(x) = \frac{1}{(1-x)^n}$ if and only if, for each $i \ge 0$, $c_i = \binom{i+n-1}{i}$.

(c) $g(x) = (1+x)^n$ if and only if, for each $0 \le i \le n$, $c_i = \binom{n}{i}$, and, for each $i > n$, $c_i = 0$.

Section 7.4

Definition. A **group** is a set G together with a binary operation $\diamond$ (called composition) such that $\forall\, g, h \in G, g \diamond h \in G$ and the following conditions hold.

(i) $\forall\, g, h, k \in G, (g \diamond h) \diamond k = g \diamond (h \diamond k)$.

(ii) $\exists\, e \in G$ such that $\forall\, g \in G, e \diamond g = g \diamond e = g$.

(iii) $\forall\, g \in G, \exists\, g^{-1} \in G$ such that $g^{-1} \diamond g = g \diamond g^{-1} = e$.

Definition (Symmetry Groups for Regular n-gons)**.** Let $n \ge 3$, and let B be a regular n-gon. For each $i \in \mathbb{Z}$, let r_i denote the clockwise rotation of B about its center by $\frac{360i}{n}$ degrees. Let $f_1, f_2, \ldots, f_n$ denote the n reflections (flips) of B about lines through its center.
(a) $Z_n = \{r_0, r_1, \ldots, r_{n-1}\}$ forms the **cyclic group** of order n.
(b) $D_n = \{r_0, r_1, \ldots, r_{n-1}, f_1, f_2, \ldots, f_n\}$ forms the **dihedral group** of order $2n$.

Definition. Given a group G with composition $\diamond$ and a set X, we say that G **acts on** X via operation $*$ if $\forall\, g \in G, \forall\, x \in X, g * x \in X$ and the following conditions hold.

(i) $\forall\, x \in X, e * x = x$.

(ii) $\forall\, g, h \in G, \forall\, x \in X, h * (g * x) = (h \diamond g) * x$.

Definition. Let a group G act on a set X. For each $x \in X$, the **orbit of** x is the set $\mathrm{Orb}(x) = \{y \;:\; y \in X \text{ and } y = gx \text{ for some } g \in G\}$. An **orbit** is a set $\mathrm{Orb}(x)$ for some $x \in X$.

Theorem. *If a group G acts on a set X, then the orbits partition X.*

Theorem (Burnside's Formula)**.** *Let a group G act on a set X, and $\forall\, g \in G$, let $Fix(g) = \{x \;:\; x \in X \text{ and } gx = x\}$. Then, the number of orbits under this action is $N = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} |Fix(g)|$.*

Section 7.5

A **combinatorial proof** is a proof of a combinatorial identity by solving a counting problem in two different ways.

1.8 Chapter 8

Section 8.1

Definition. (a) A **graph** G consists of a pair of sets V_G and E_G together with a function $\epsilon_G : E_G \longrightarrow \mathcal{P}_2(V_G) \cup \mathcal{P}_1(V_G)$. We write $G = (V_G, E_G)$, and, rather than writing $\epsilon_G(e) = \{u, v\}$, we write $e \mapsto \{u, v\}$. An element of V_G is called a **vertex** of G, and an element of E_G is called an **edge** of G. When a particular graph G is clear in context, the subscripts are dropped from V, E, and ϵ.

(b) If $e \mapsto \{u, v\}$, then we say that the vertices u and v are the **endpoints** of the edge e, that u and v are **adjacent**, and that v is a **neighbor** of u. We say that an edge is **incident** with its endpoints.

(c) An edge e such that $e \mapsto \{v\}$, for some $v \in V$, is called a **loop**; it has a single endpoint. Two or more edges assigned the same set of endpoints are called **multiple edges** (or **parallel edges**).

(d) A **simple graph** is a graph $G = (V, E)$ that has no loops and no multiple edges.

Definition. A **drawing** of a graph $G = (V, E)$ in the plane is a one-to-one assignment of the vertices to points in the plane and, for each edge, the assignment of a curve joining the ends of the edge in such a way that (i) the only vertex points hit by a curve are the endpoints of the edge it represents, (ii) each curve is one-to-one (i.e., does not intersect itself) with the exception that the ends of a loop edge are assigned to a common point, and (iii) the images of curves associated with two distinct edges intersect in at most finitely many points. An intersection of two curves outside of their endpoints is called a **crossing**.

Definition. We say that a graph $H = (W, F)$ is a **subgraph** of a graph $G = (V, E)$ if $W \subseteq V$, $F \subseteq E$, and the endpoints of edges in F all lie in W and are the same as they are in G. Given a subset W of the vertex set V for a graph $G = (V, E)$, the **subgraph induced by** W is the subgraph whose edges set is $\{e \,:\, e \in E \text{ and the ends of } e \text{ are in } W\}$.

Definition. (a) A **walk** in a graph $G = (V, E)$ is an alternating list of vertices and edges $v_0, e_1, v_1, e_2, v_2, e_3, \ldots, v_{n-1}, e_n, v_n$ with $n \geq 0$ that **starts** at vertex v_0, **ends** at vertex v_n, and, in which, for each $1 \leq i \leq n$, $e_i \mapsto \{v_{i-1}, v_i\}$. The **length** of a walk is the number of edges it contains (counting multiple occurrences of the same edge), here n.

(b) A **circuit** is a walk of positive length that starts and ends at the same vertex.

(c) A **trail** is a walk with no repeated edges. In a graph with multiple edges, distinct multiple edges may be included.

(d) A **path** is a walk with no repeated vertices.

(e) A **cycle** is a circuit in which the only vertex repetition is $v_n = v_0$.

(f) The **distance** between two vertices u and v in G, denoted $\mathrm{dist}_G(u, v)$, is the length of the shortest walk in G between u and v. If there is no walk, then we assign $\mathrm{dist}_G(u, v) = \infty$. When G is clear in context, the subscripts may be dropped.

Definition. (a) A graph G is **connected** if, for any two vertices, there is a path between them. Otherwise, G is **disconnected**.

(b) A **component** of a graph G is a connected subgraph H that is not contained in any other connected subgraph of G.

Section 8.2

Definition. Let $n \in \mathbb{Z}^+$, and let $V = \{1, 2, \ldots, n\}$.

(a) The **path on n vertices** is the graph P_n with vertex set V and edge set $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}\}$.

(b) The **cycle on n vertices** is the graph C_n with vertex set V and edge set $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}$. When $n = 2$, we take C_2 to have two parallel edges.

Theorem. *Given a simple graph $G = (V, E)$,*

(a) if $|V| \leq 1$, then $|E| = 0$,

(b) if $|V| \geq 2$, then $0 \leq |E| \leq \binom{|V|}{2}$.

Definition. Given an integer $n \geq 1$, the **complete graph on n vertices** is the simple graph K_n with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E = \mathcal{P}_2(V)$. Any graph in which every pair of vertices is adjacent is said to be **complete**.

Definition. Given any $n \in \mathbb{N}$, the **empty graph on n vertices** is the graph Φ_n with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E = \emptyset$. Any graph in which no pair of vertices is adjacent is said to be **empty**.

Definition. A graph $G = (V, E)$ is **bipartite** if V can be expressed as a disjoint union $V_1 \cup V_2$ such that each edge of G has one endpoint in V_1 and one in V_2. In this case, the pair (V_1, V_2) is said to form a **bipartition** of G. Note that, for $i = 1$ or 2, the subgraph induced by V_i is empty.

Theorem. *Let G be any graph. Then, G is bipartite if and only if every cycle in G has even length.*

Definition. Given integers $m, n \geq 1$ and sets $V_1 = \{(1, 1), (1, 2), \ldots, (1, m)\}$ and $V_2 = \{(2, 1), (2, 2), \ldots, (2, n)\}$, the **complete bipartite graph** $K_{m,n}$ has vertex set $V = V_1 \cup V_2$ and edge set $E = \{\{v_1, v_2\} : v_1 \in V_1 \text{ and } v_2 \in V_2\}$.

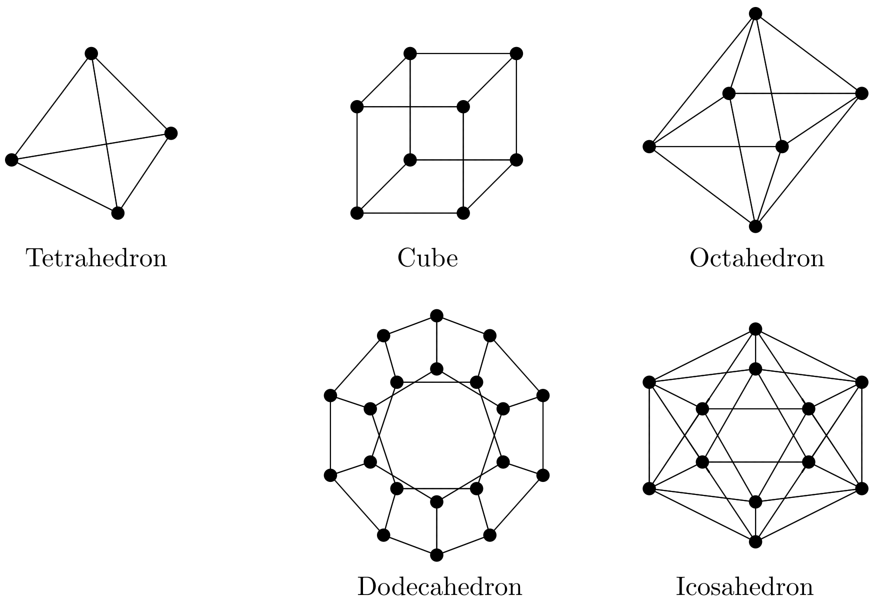


Figure 1: The Platonic Solids' Graphs

Definition. The n-**dimensional cube** Q_n is the simple graph whose vertex set is the set of binary sequences of length n and whose edges connect two vertices if and only if they differ in exactly one coordinate.

Section 8.3

Definition. An **adjacency matrix** for a graph G on n vertices is an n by n matrix $A = [a_{i,j}]$ obtained by fixing an ordering of the vertices, say $v_1, v_2, \ldots, v_n$, and, for each $1 \leq i \leq n$ and $1 \leq j \leq n$, taking $a_{i,j}$ to be the number of edges connecting v_i to v_j.

Definition (Matrix Multiplication)**.** Given an m by n matrix $A = [a_{i,j}]$ and an n by p matrix $B = [b_{i,j}]$, their product AB is defined to be the m by p matrix $C = [c_{i,j}]$ such that, for each $1 \leq i \leq m$ and $1 \leq j \leq p$, $c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$.

Definition. Let $n \in \mathbb{Z}^+$.

(a) The n by n **identity matrix** is the matrix $I_n = [a_{i,j}]$ such that $a_{i,j} = 1$ if $i = j$, and $a_{i,j} = 0$ otherwise.

(b) Given a permutation $p_1, p_2, \ldots, p_n$ of the integers $1, 2, \ldots, n$, the corresponding **permutation matrix** is the n by n matrix $P = [a_{i,j}]$ such that $a_{i,j} = 1$ if $p_i = j$, and $a_{i,j} = 0$ otherwise.

Theorem. *Let A be the adjacency matrix for a graph G obtained from the ordering $v_1, v_2, \ldots, v_n$ of its vertices, and let $m \in \mathbb{N}$. Then, the mth power of A, say $A^m = [b_{i,j}]$, has the property that, for each $1 \leq i \leq n$ and $1 \leq j \leq n$, the entry $b_{i,j}$ is the number of walks in G of length m from v_i to v_j.*

Definition. Let $v_1, v_2, \ldots, v_n$ be the vertices of a graph G without multiple edges. For each $1 \leq i \leq n$, an **adjacency list** for vertex v_i is a list of all of the neighbors of v_i. A listing, for each vertex, of its adjacency list, forms the **adjacency lists** for G.

Section 8.4

Definition. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs.

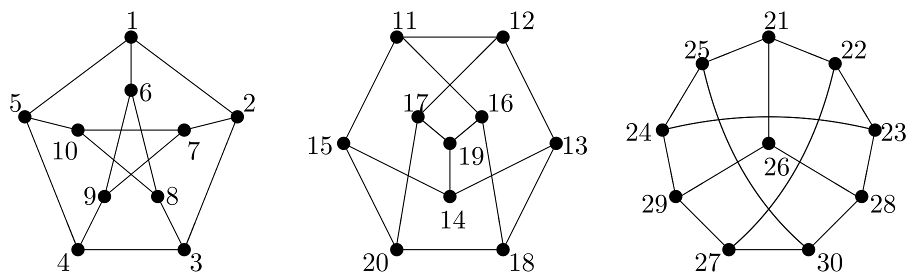
(a) A **graph isomorphism** from G to H, denoted $f : G \longrightarrow H$, is a pair of bijections $f_V : V_G \longrightarrow V_H$ and $f_E : E_G \longrightarrow E_H$ such that, for each $e \in E_G$, the bijection f_V maps the endpoints of e to the endpoints of $f_E(e)$. Both f_V and f_E may be denoted by just f.

(b) We say that G is **isomorphic** to H, written $G \cong H$, if there exists a graph isomorphism from G to H.

(c) A **graph automorphism** on G is a graph isomorphism from G to itself. A **nontrivial automorphism** is an automorphism $f : G \longrightarrow G$ that is not the identity map. The set of automorphisms on G is denoted $\mathrm{Aut}(G)$.

Definition. A graph $G = (V, E)$ is said to be **vertex transitive** if, for any $u, v \in V$, there is a graph automorphism f such that $f(u) = v$.

Example. Displayed are three drawings of the **Petersen Graph**.



Theorem (Graph Isomorphism is an Equivalence Relation)**.** *For all graphs G, H, and K,*

(a) $G \cong G$.

(b) if $G \cong H$, then $H \cong G$.

(c) if $G \cong H$ and $H \cong K$, then $G \cong K$.

Theorem. *For any graph G, $Aut(G)$ forms a group under composition.*

Definition. A **graph map** f from a graph $G = (V_G, E_G)$ to a graph $H = (V_H, E_H)$, denoted $f : G \longrightarrow H$, is a pair of functions $f_V : V_G \longrightarrow V_H$ and $f_E : E_G \longrightarrow E_H$ such that, for each edge $e \in E_G$, the function f_V maps the endpoints of e to the endpoints of $f_E(e)$.

Section 8.5

Theorem. *Let G, H be graphs. If $G \cong H$, then $|V_G| = |V_H|$ and $|E_G| = |E_H|$.*

Theorem. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs. Then $G \cong H$ if and only if there are orderings of V_G and V_H such that the corresponding adjacency matrices A_G and A_H are equal. In this case, we say that G and H have* a *common adjacency matrix.*

Definition. Let $G = (V, E)$ be a graph.

(a) The **degree of a vertex** v, denoted $\deg(v)$, is the number of non-loop edges incident with v plus twice the number of loops incident with v.

(b) The **maximum degree** (respectively, **minimum degree**) of G, denoted $\Delta(G)$ (respectively, $\delta(G)$), is the maximum (respectively, minimum) degree among all vertices in G.

(c) A **degree sequence** for G is a sequence $\deg(v_1), \deg(v_2), \ldots, \deg(v_n)$ obtained from some ordering $v_1, v_2, \ldots, v_n$ of V.

(d) If G has a constant degree sequence, then G is said to be **regular**. If each vertex has degree r, the G is called r-**regular**.

(e) A vertex of degree 0 is said to be an **isolated vertex**.

(f) A vertex of degree 1 is said to be a **pendant vertex**, or a **leaf**.

Lemma. *Let $f : G \longrightarrow H$ be a graph isomorphism and v a vertex of G. Then,* $\deg(f(v)) = \deg(v)$.

Theorem (Degree Invariants)**.** *Let G and H be graphs. If $G \cong H$, then G and H have a common degree sequence. In particular, $\Delta(G) = \Delta(H)$ and $\delta(G) = \delta(H)$.*

Theorem. *For any graph $G = (V, E)$, we have* $\sum_{v \in V} \deg(v) = 2|E|$.

Corollary. *Let $A = [a_{i,j}]$ be the adjacency matrix for a loopless graph $G = (V, E)$ obtained from the ordering $v_1, v_2, \ldots, v_n$ of its vertices. Then, for any $1 \leq k \leq n$, the sum of the entries in the kth row of A and the sum of the entries in the kth column of A both equal the degree of vertex v_k. That is,*

$\sum_{j=1}^{n} a_{k,j} = \sum_{i=1}^{n} a_{i,k} = \deg(v_k)$. *Moreover, the sum of all of the entries in A is twice the number of edges.*

Corollary. *In any graph, there must be an even number of odd degree vertices.*

Definition. The **complement** of a simple graph $G = (V_G, E_G)$ is the graph G^c with $V_{G^c} = V_G$ and $E_{G^c} = {E_G}^c$, where the complement of E_G is taken inside $\mathcal{P}_2(V_G)$.

Theorem. *Let G and H be simple graphs. If $G \cong H$, then $G^c \cong H^c$.*

Definition. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be any two graphs.

(a) To construct the **disjoint union** of G and H, the graph denoted $G + H$, we consider two cases. If V_G and V_H are disjoint, then we simply define $G + H = (V_G \cup V_H, E_G \cup E_H)$. If V_G and V_H are not disjoint, then we construct graphs $G' \cong G$ and $H' \cong H$ such that $V_{G'}$ and $V_{H'}$ are disjoint and define $G+H = (V_{G'} \cup V_{H'}, E_{G'} \cup E_{H'})$.

(b) If G and H are both subgraphs of the same graph, then

  (i) the **union** of G and H, denoted $G \cup H$, is the graph $(V_G \cup V_H, E_G \cup E_H)$.

  (ii) the **intersection** of G and H, denoted $G \cap H$, is the graph $(V_G \cap V_H, E_G \cap E_H)$.

(c) The **product** of G and H, denoted $G \times H$, is the graph with vertex set $V_G \times V_H$ and edge set $(E_G \times V_H) \cup (V_G \times E_H)$. An edge's endpoints are determined as follows. If $e \in E_G$, $v \in V_H$, and $e \mapsto \{x, y\}$, then $(e, v) \mapsto \{(x, v), (y, v)\}$. If $v \in V_G$, $e \in E_H$, and $e \mapsto \{x, y\}$, then $(v, e) \mapsto \{(v, x), (v, y)\}$.

Theorem. *We have $Q_0 \cong K_1$, $Q_1 \cong P_2$, and, for $n \geq 2$, $Q_n \cong Q_{n-1} \times P_2$.*

Section 8.6

Definition. (a) A **directed graph**, or **digraph**, G consists of a **vertex** set V_G, an **edge** set E_G, and a function $\epsilon : E_G \longrightarrow V_G \times V_G$. We write $G = (V_G, E_G)$, and, rather than writing $\epsilon_G(e) = (u, v)$, we write $e \mapsto (u, v)$. As in ordinary graphs, the subscripts may be dropped from our notation.

(b) If $e \mapsto (u, v)$, then the vertex u is called the **initial endpoint** or **tail** of e and the vertex v is called the **terminal endpoint** or **head** of e. We say that e goes from u to v.

(c) An edge e such that $e \mapsto (v, v)$ for some $v \in V$ is called a **loop**, and two or more edges assigned the same initial and terminal endpoints are called **multiple edges**.

(d) A **simple directed graph** is a directed graph $G = (V, E)$ that has no loops and no multiple edges. In this case, we can take E to be a subset of $V^2 \setminus \{(v, v) \ : \ v \in V\}$.

Definition. A **walk** in a directed graph $G = (V, E)$ is an alternating list of vertices and edges $v_0, e_1, v_1, e_2, v_2, e_3, \ldots, v_{n-1}, e_n, v_n$ with $n \geq 0$ that starts at vertex v_0, ends at vertex v_n, and, in which, for each $1 \leq i \leq n$, $e_i \mapsto (v_{i-1}, v_i)$. That is, edges must be traversed from tail to head.

Definition. Given a directed graph $G = (V, E)$, its **underlying graph**, denoted $\underline{G}$, is the ordinary graph with the same vertex set V and with the edge set $\underline{E}$ containing one undirected edge $\underline{e}$ for each directed edge $e \in E$. The ends of $\underline{e}$ are taken to be the head and tail of e.

Definition. Let G be a directed graph.

(a) G is said to be **strongly connected** if, for any two vertices u and v, there is a path from u to v and there is a path from v to u.

(b) G is said to be **weakly connected** if $\underline{G}$ is connected.

(c) A **strong component** of G is a strongly connected subgraph H that is not contained in any other strongly connected subgraph of G.

(d) A **weak component** of G is a subgraph H such that $\underline{H}$ is a component of $\underline{G}$.

Theorem. *For any directed graph $G = (V, E)$, we have*
$\displaystyle\sum_{v \in V} indeg(v) = \sum_{v \in V} outdeg(v) = |E|.$

Definition. The **adjacency matrix** for a directed graph G on the ordered list of vertices $v_1, v_2, \ldots, v_n$ is the $n \times n$ matrix $A = [a_{i,j}]$ such that $a_{i,j}$ is the number of edges from v_i to v_j.

Definition. A **Markov chain graph** is a directed graph $G = (V, E)$ without multiple edges, for which each edge (u, v) is assigned a value $p(u, v)$ in the interval $[0, 1]$. Moreover, for each vertex u, the sum of the values assigned to the edges with tail u must be 1.

Definition. Given a Markov chain graph and an ordering of its vertices, v_1, v_2, $\ldots$, v_n, its **transition matrix** is the $n \times n$ matrix $M = [p(v_i, v_j)]$. That is, the (i, j)th entry of M is the value $p(v_i, v_j)$, which may be denoted more compactly as $p(i, j)$.

Theorem. *Let M be the transition matrix for a Markov chain graph G obtained from the ordering v_1, v_2, $\ldots$, v_n of the vertices, and let $m \in \mathbb{N}$. Then, the mth power of M, say $M^m = [q_{i,j}]$, has the property that, for each $1 \leq i \leq n$ and $1 \leq j \leq n$, the entry $q_{i,j}$ is the probability of moving in G from v_i to v_j in a sequence of exactly m steps.*

A state v_i in a Markov chain graph is said to have **period** q if $q \in \mathbb{Z}^+$ and the length of every circuit starting at v_i is a multiple of q. We say that v_i is **periodic** if $q > 1$ and **aperiodic** if $q = 1$.

A **class** in a Markov chain is a set of states that corresponds to the vertex set for a strong component in the corresponding graph. If a Markov chain has just one class, then the Markov chain is said to be **irreducible**. A finite Markov chain is said to be **regular** if it is irreducible and every state is aperiodic.

Lemma. *A finite Markov chain with transition matrix M is regular if and only if there exists some $m \in \mathbb{Z}^+$ such that M^m has all positive entries.*

Theorem. *If M is a transition matrix for a finite regular Markov chain, then, as m increases, the powers M^m of M converge to a matrix, which we denote by M^∞, all of whose rows are the same. Moreover, for each j, the entry in the jth column of every row of M^∞ is the long-term probability of being in state v_j, independent of the initial state.*

In a Markov chain, a state v_i is said to be **absorbing** if $p(i,i) = 1$. A class is said to be **ergodic** if no edge in the Markov chain graph points from a vertex inside the class to a vertex outside the class. In this case, each of the states in the class is also said to be ergodic, or **recurrent**. A class for which there is an edge in the Markov chain graph pointing from a vertex inside the class to a vertex outside the class is said to be **transient**, as is each of its states. A Markov chain is said to be **absorbing** if each state is either absorbing or transient.

1.9 Chapter 9

Section 9.1

Definition. Let $G = (V, E)$ be any graph.

(a) Given subsets $W \subseteq V$ and $F \subseteq E$, the graph resulting from the **removal** of $W \cup F$, denoted $G \setminus (W \cup F)$, is the subgraph of G whose vertex set is $V \setminus W$ and whose edge set is
$E \setminus (F \cup \{e \in E \ : \ e \text{ is incident with some } v \in W\})$.

(b) A **disconnecting set** for G is a set D of vertices such that $G \setminus D$ is disconnected.

(c) The **connectivity** of G, denoted $\kappa(G)$, is the minimum number of vertices whose removal results in either a disconnected graph or a single vertex.

(d) A **κ-set** for G is a set of $\kappa(G)$ vertices whose removal results in either a disconnected graph or a single vertex.

Theorem. *(a) If $n \geq 2$, then $\kappa(P_n) = 1$.*

(b) If $n \geq 3$, then $\kappa(C_n) = 2$.

Theorem. *Let $G = (V, E)$ be any graph. If G is connected and 2-regular, then $G \cong C_n$, where $n = |V|$. So $\kappa(G) = 2$.*

Theorem. *Let $m, n \in \mathbb{Z}^+$. Then, $\kappa(K_{m,n}) = \min\{m, n\}$.*

Definition. Let $G = (V, E)$ be any graph.

(a) A **disconnecting set of edges** for G is a set F of edges such that $G \setminus F$ is disconnected.

(b) The **edge connectivity** of G, denoted $\lambda(G)$, is the minimum number of edges whose removal results in either a disconnected graph or a single vertex.

(c) A **λ-set** for G is a set of $\lambda(G)$ edges whose removal results in either a disconnected graph or a single vertex.

Theorem. *For any graph G, we have $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

The **connectivity of a directed graph** G, denoted $\kappa(G)$, is the minimum number of vertices whose removal results in a directed graph that is either not strongly connected or is a single vertex. The edge connectivity $\lambda(G)$ is defined analogously.

Section 9.2

Definition. Let G be a graph or a digraph. An **Euler circuit** (resp. **Euler trail**) in G is a circuit (resp. trail) which covers every edge exactly once and covers every vertex. Of course, unless G has isolated vertices, covering every edge will imply that every vertex is covered. If G contains an Euler circuit, then G is said to be **Eulerian**.

Theorem (Euler's Theorem)**.** *Given any graph G,*

(a) G has an Euler circuit if and only if G is connected and every vertex of G has even degree.

(b) G has an Euler trail that is not an Euler circuit if and only if G is connected and has exactly two vertices with odd degree.

Theorem. *Given any directed graph G on n vertices,*

(a) G has an Euler circuit if and only if G is strongly connected and every vertex v of G has $indeg(v) = outdeg(v)$.

(b) G has an Euler trail that is not an Euler circuit if and only if every vertex v of G has $indeg(v) = outdeg(v)$ except that one vertex v_1 has $outdeg(v_1) = 1 + indeg(v_1)$ and another vertex v_n has $indeg(v_n) = 1 + outdeg(v_n)$ and the directed graph obtained from G by adding the edge (v_n, v_1) is strongly connected.

Section 9.3

Definition. Let G be a graph or a digraph. A **Hamiltonian cycle** (respectively, **Hamiltonian path**) in G is a cycle (respectively, path) which covers every vertex. By definition, each vertex must be covered exactly once, with the exception that the starting and ending vertex of a Hamiltonian cycle is covered twice. If G contains a Hamiltonian cycle, then G is said to be **Hamiltonian**.

Theorem. *Let G be any graph. If G is Hamiltonian, then $\kappa(G) \geq 2$.*

Theorem. *If C is a Hamiltonian cycle in a graph $G = (V, E)$, then*

(i) C covers exactly two edges incident with each vertex, and

(ii) C has no subgraph which is a cycle on fewer than $|V|$ vertices.

In the case that G is directed, for each vertex v, C must cover exactly one edge whose head is v and one whose tail is v.

Theorem. *For any simple graph $G = (V, E)$ with $|V| \geq 3$, if $\delta(G) \geq \frac{|V|}{2}$, then G is Hamiltonian.*

Example. For each integer $n \geq 2$, the n-cube Q_n is Hamiltonian.

Definition. A **tournament** is a directed graph whose underlying graph is complete.

Theorem. *Every tournament has a Hamiltonian path.*

Section 9.4

Definition. (a) A **planar embedding** of a graph is a drawing of the graph such that the images of distinct edges do not intersect outside of their endpoints. That is, there are no crossings.

(b) A graph is said to be **planar** if it has a planar embedding.

Definition. Given a planar embedding of a graph $G = (V_G, E_G)$,

(a) a **region** is a maximal connected subset of the complement of the image of the embedding.

(b) we use R_G, or just R, to denote the set of regions.

(c) the **dual graph**, denoted $\mathrm{D}(G)$, is the graph with vertex set R_G and edge set E_G for which the endpoints of each edge e are taken to be the regions that, in the embedding, share the image of e as part of their boundary.

Theorem (Euler's Formula)**.** *Given any planar embedding of a connected graph* $G = (V, E)$*, we have* $|V| - |E| + |R| = 2$*.*

Corollary. *Given any planar simple graph* $G = (V, E)$ *with* $|V| \geq 3$*, we have* $|E| \leq 3|V| - 6$*.*

Proposition. K_5 *is not planar.*

Corollary. *Given any planar simple graph* $G = (V, E)$ *with* $|V| \geq 3$ *and no triangles (that is, no* 3*-cycles), we have* $|E| \leq 2|V| - 4$*.*

Proposition. $K_{3,3}$ *is not planar.*

Definition. Let $G = (V, E)$ be a graph.

(a) Given an edge $e \in E$, a new graph $G' = (V', E')$ is said to be obtained by **subdividing** $e = \{u, v\}$ if $V' = V \cup \{w\}$, where w is a new vertex not in V, and $E' = (E \setminus \{e\}) \cup \{e', e''\}$, where $e' \mapsto \{u, w\}$ and $e'' \mapsto \{w, v\}$ are new edges not in E. That is, e is subdivided by the new vertex w.

(b) We say that G' is a **subdivision** of G, or a G**-subdivision**, if G' is obtained from G by a (possibly empty) sequence of edge subdivisions. We also say that two graphs G' and G'' are **homeomorphic** if there is a graph G such that both G' and G'' are G-subdivisions.

Theorem (Kuratowski's Theorem)**.** *A graph is not planar if and only if it contains a subgraph that is a subdivision of either* K_5 *or* $K_{3,3}$*. Equivalently,* G *is not planar if and only if* G *contains a subgraph homeomorphic to* K_5 *or* $K_{3,3}$*.*

Definition. The **crossing number** of a graph G, denoted $\nu(G)$, is the minimum possible number of crossings in a drawing of G.

Section 9.5

Definition. Let G be a graph.

(a) A **coloring** of G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices have the same color.

(b) A **color class** for a coloring is a set of all the vertices of one color. The vertices are partitioned by the color classes.

(c) For any $k \in \mathbb{Z}^+$, a k**-coloring** of G is a coloring that uses k different colors.

(d) We say that G is k**-colorable** if there exists a coloring of G that uses at most k colors.

(e) The **chromatic number** of G, denoted $\chi(G)$, is the minimum possible number of colors in a coloring of G.

Theorem. *Let G be any graph. Then, G is bipartite if and only if $\chi(G) \leq 2$.*

Theorem. *Given any graphs G and H without loops, the chromatic number of their disjoint union is given by $\chi(G + H) = \max\{\chi(G), \chi(H)\}$.*

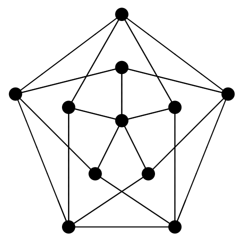
Definition. Let G be a graph.

(a) A **clique** in G is a subgraph that is complete.

(b) The **clique number** of G, denoted $\omega(G)$, is the maximum number of vertices in a clique of G.

Theorem. *Let G be any graph without loops. Then, $\chi(G) \geq \omega(G)$.*

Example. The pictured graph G is called the **Grötzsch graph**.



Definition. Let $G = (V, E)$ be a graph.

(a) An **independent set** in G is a subset W of V in which no two vertices are adjacent. That is, the subgraph induced by W is empty.

(b) The **independence number** of G, denoted $\alpha(G)$, is the maximum number of vertices in an independent set in G.

Theorem. *For any simple graph G, we have $\alpha(G^c) = \omega(G)$ and $\omega(G^c) = \alpha(G)$.*

Theorem. *For any graph $G = (V, E)$ without loops, if $|V| = n$, then $\chi(G) \geq \frac{n}{\alpha(G)}$.*

Algorithm (Greedy Coloring Algorithm)**.** Let a graph G on n vertices with no loops be given together with an ordering $v_1, v_2, \ldots, v_n$ of its vertices. In that order, color the vertices with positive integers so that, for each $1 \leq i \leq n$, vertex v_i is given the smallest possible color not assigned to a neighbor v_j of v_i with $j < i$. Note that v_1 has color 1.

Theorem. *For any graph G without loops, if $v_1, v_2, \ldots, v_n$ is a listing of its vertices, then $\chi(G) \leq 1 + \max\{\min\{\deg(v_i), i\} \;:\; 1 \leq i \leq n\}$.*

Theorem (Brooks' Theorem)**.** *Let G be any graph without loops. If G is not complete and not an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Corollary. *For any graph G without loops and any integer $r \geq 3$, if G is r-regular and no component of G is complete, then $\chi(G) \leq r$.*

Theorem. *Given any graphs G and H without loops, the chromatic number of their product is given by $\chi(G \times H) = \max\{\chi(G), \chi(H)\}$.*

Theorem (Four Color Theorem)**.** *If G is any planar graph, then $\chi(G) \leq 4$.*

1.10 Chapter 10

Section 10.1

When working with forests, a pendant vertex (i.e., a vertex of degree 1) is called a **leaf**. The vertices of degree at least two are said to be **internal vertices**.

Theorem. *Let G be a graph. If G is a tree, then there is a unique path between any pair of vertices in G.*

Theorem. *Every tree on two or more vertices has at least two leaves.*

Theorem. *Let $G = (V, E)$ be a graph. If G is a tree, then $|E| = |V| - 1$.*

A **spanning tree** for a graph $G = (V, E)$ is a subgraph $H = (V, F)$ that is a tree on the same vertex set V. It may be specified uniquely by its edge set F. More generally, a subgraph of a graph G is called a **spanning forest** if its intersection with each component of G is a spanning tree for that component.

Theorem. *If G is a connected graph, then G has a spanning tree.*

Definition. A **rooted tree** is a pair (T, v) where T is a tree and v is a vertex of T. The distinguished vertex v is said to be the **root** of T.

Definition. Let T be a rooted tree with root v.

(a) The **level** of a vertex u in T is its distance from v, namely $\mathrm{dist}(v, u)$.

(b) A **child** of a vertex u in T is a neighbor of u at a level greater than that of u; its level is always one greater.

(c) The **parent** of a vertex u in T is the unique neighbor of u with level less than that of u; its level is always one less. The root v is the only vertex in T without a parent.

(d) The **height** of T is the maximum level among its vertices.

(e) If each vertex of T has at most m children, then we say that T is an m**-ary tree**. If each internal vertex has exactly m children, then T is said to be a **full** m**-ary tree**. A 2-ary tree is also called a **binary tree**.

(f) If T has height h and all of its leaves are at levels h or $h-1$, then T is said to be **balanced**.

Theorem. *Let T be a full m-ary rooted tree with n vertices, l leaves, and i internal vertices. Then, $n = i + l$ and $n = mi + 1$.*

Theorem. *Let T be an m-ary rooted tree of height h with l leaves. Then, $l \leq m^h$ and $h \geq \lceil \log_m l \rceil$.*

Definition. (a) An **ordered rooted tree** is a rooted tree in which the set of children of each internal vertex is ordered.

(b) A **ordered binary tree** is a binary tree in which each child of an internal vertex is designated either as the **left child** or the **right child**, but not both. Moreover, at each internal vertex, the corresponding **left subtree** (respectively, **right subtree**) is the subtree rooted at its left child (respectively, right child).

The **quadtree** for a given image is an ordered full 4-ary tree constructed by placing a square bounding box around the image and recursively subdividing boxes within the image into four quadrants. Each box that is neither entirely black nor entirely white in color is itself subdivided into four quadrants, and this process is repeated until all boxes are monochromatic. Each box that gets subdivided is designated the parent of its resulting four quadrants, and an ordering for the quadrants is given by the standard convention for $\mathbb{R}^2$.

Section 10.2

Given a graph G, the tree T constructed by Algorithm 2 is called its **breadth-first search tree**.

Algorithm 2 Breadth-First Search

Let $G = (V, E)$ be a connected graph on n vertices and v a vertex of G.

Algorithm.

Let $M = [\,]$, $L = [v]$, and $F = \emptyset$.
While $|F| < n - 1$,
  `\begin`
  Let t be the first vertex in L and not in M.
  Add t to the end of M.
  While t has neighbors that are not in L,
    `\begin`
    Let e be an edge such that one end is t and the other
      is a vertex u outside of L.
    Let $F = F \cup \{e\}$.
    Add u to the end of L.
    `\end.`
  `\end.`
Return $T = (V, F)$.

Given a graph G, the tree T constructed by Algorithm 3 is called its **depth-first search tree**.

Algorithm 3 Depth-First Search

Let $G = (V, E)$ be a connected graph on n vertices and v a vertex of G.

Algorithm.

```
Let M = [v], L = [ ], and F = ∅.
Let t = v.
While t ≠ null,
    \begin
    While t has neighbors that are not in M,
        \begin
        Let e be an edge such that one end is t and the other
            is a vertex u outside of M.
        Let F = F ∪ {e}.
        Add u to the end of M.
        Let t = u.
        \end.
    Add t to the end of L.
    Replace t by the parent of t.
    \end.
Return T = (V, F).
```

Definition (Ordered Binary Tree Traversals)**.** Let T be an ordered binary tree. If T has no vertices, then list nothing. If T is just a root v, then list v. Otherwise, let T_L be the left subtree of v, and let T_R be the right subtree.

(a) A **postorder traversal** of T is accomplished by first performing a postorder traversal on T_L, second performing a postorder traversal on T_R, and third listing the root v for T.

(b) A **preorder traversal** of T is accomplished by first listing the root v for T, second performing a preorder traversal on T_L, and third performing a preorder traversal on T_R.

(c) An **inorder traversal** of T is accomplished by first performing an inorder traversal on T_L, second listing the root v for T, and third performing an inorder traversal on T_R.

An algebraic expression (involving binary operations) can be represented by an ordered binary tree. Its **postfix** notation is obtained by listing the vertices in the order L given by Depth-First Search. Similarly, its **prefix** notation is the list obtained from a preorder traversal, and its **infix** notation is the list obtained from an inorder traversal.

Section 10.3

A **weighted graph** is a graph $G = (V, E)$ for which each edge has been assigned a positive real number called the **weight** *of the edge*.

Definition 1.1. Let G be a weighted graph.

(a) The **weight** *of a subgraph* is the sum of the weights of the edges in that subgraph.

(b) A **minimum spanning tree** for G is a spanning tree with the minimum weight among all spanning trees.

Algorithm 1.1 Kruskal's Algorithm

Let $G = (V, E)$ be a weighted connected graph on $n > 1$ vertices.

Algorithm.

Let $F = \{e\}$, where e is a nonloop edge of minimum possible weight.
While $|F| < n - 1$,
`\begin`
Let e be an edge of minimum possible weight among all
edges in $E \setminus F$ for which $F \cup \{e\}$ contains no cycle.
Let $F = F \cup \{e\}$.
`\end.`
Return $T = (V, F)$.

Algorithm 1.2 Prim's Algorithm

Let $G = (V, E)$ be a weighted connected graph on $n > 1$ vertices.

Algorithm.

Let $F = \{e\}$, where e is a nonloop edge of minimum possible weight.
While $|F| < n - 1$,
`\begin`
Let e be an edge of minimum possible weight among all
edges in $E \setminus F$ that connect an endpoint of an edge from F
to a vertex that is not an endpoint of an edge from F.
Let $F = F \cup \{e\}$.
`\end.`
Return $T = (V, F)$.

Theorem 1.6. *Let G be any weighted connected graph. Each of Kruskal's Algorithm and Prim's Algorithm yield a minimum spanning tree for G.*

Definition 1.2. Let $G = (V, E)$ be a weighted graph.

(a) The **weighted distance** between two vertices u and v in G, denoted $\mathrm{Dist}_G(u, v)$, is the minimum weight of a path in G from u to v. If there is no path in G from u to v, then we assign $\mathrm{Dist}_G(u, v) = \infty$. When G is clear in context, the subscripts may be dropped.

(b) Suppose that a vertex v is specified in G. A **shortest path tree** for G from v is a spanning tree T such that, for each vertex w in G, the path in T from v to w has the minimum weight among all paths in G from v to w.

Algorithm 1.3 Dijkstra's Algorithm

Let G be a weighted connected graph on $n > 1$ vertices and v a vertex of G.

Algorithm.

```
Let F = {e}, where e is a nonloop edge incident with v
    of minimum possible weight.
While |F| < n − 1,
    \begin
    Let e be an edge such that
        one endpoint t is an endpoint of an edge from F, the other
        endpoint is not, and Dist_T(v, t) + ω(e) is as small as possible.
    Let F = F ∪ {e}.
    \end.
Return T = (V, F).
```

Theorem 1.7. *Let G be a weighted connected graph and v a vertex of G. Dijkstra's Algorithm yields a shortest path tree for G from v.*

Section 10.4

Algorithm 4 Sequential Search

Let x be a real number whose index is sought in an array A of length n. The index of x is returned in a variable called **location**. If x is not in A, then the algorithm returns location $= 0$.

Algorithm.

```
Let location = 0.
Let i = 1.
While location = 0 and i ≤ n,
    \begin
    If A[i] = x, then
        Let location = i.
    Otherwise,
        Let i = i + 1.
    \end.
Return location.
```

Algorithm 5 Binary Search

Let x be a real number whose index is sought in an *ordered* array A of length n. The index of x is returned in a variable called **location**. If x is not in A, then the algorithm returns location $= 0$. The variables **low** and **high** store the indices of the first and last entries, respectively, of the portion of A we are considering.

Algorithm.

Let location $= 0$.
Let low $= 1$.
Let high $= n$.
While low $<$ high,
  `\begin`
  Let mid $= \lfloor \frac{\text{low}+\text{high}}{2} \rfloor$.
  If $A[\text{mid}] < x$, then
    Let low $=$ mid $+ 1$.
  Otherwise,
    Let high $=$ mid.
  `\end.`
If $A[\text{low}] = x$, then
  Let location $=$ low.
Return location.

Definition. The **worst-case complexity** of an algorithm is a function $f(n)$ of the size n of the input to the algorithm. For each n, the value of $f(n)$ is the maximum number of operations performed in a run of the algorithm on an input of size n.

Example. The worst-case complexity of Sequential Search is n.

Example. The worst-case complexity of Binary Search is $1 + \lceil \log_2 n \rceil$.

Definition. Given a real function $g(x)$, **big-O of** $g(x)$, denoted $O(g(x))$, is the set of real functions $f(x)$ such that there exist positive constants C and d for which $\forall\, x > d,\ \ |f(x)| \leq C|g(x)|$. That the inequality $|f(x)| \leq C|g(x)|$ holds for all x greater than some fixed constant d can be described by saying that it holds **for large** x, or **eventually**.

Lemma. *Let f and g be real functions. Then, $f(x) \in O(g(x))$ if and only if $O(f(x)) \subseteq O(g(x))$.*

Lemma. *Let $r_1, r_2 \in \mathbb{Q}$ with $r_1 \leq r_2$. Then, $\forall\, x > 1,\ \ x^{r_1} \leq x^{r_2}$.*

Theorem. *Let $m \in \mathbb{N}$ and let $f(x) = c_m x^m + c_{m-1}x^{m-1} + \cdots + c_0$ be a polynomial of degree at most m. Then, $f(x) \in O(x^m)$.*

Definition. Given a real function $g(x)$, **big-Θ of** $g(x)$, denoted $\Theta(g(x))$, is the set of real functions $f(x)$ such that $f(x) \in O(g(x))$ and $g(x) \in O(f(x))$. Equivalently, $f(x) \in \Theta(g(x))$ if and only if there exist positive constants C_1, C_2, and d for which $\forall\, x > d,\ \ C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$. In this case, we say that $f(x)$ has the same **order** (or **order of growth**) as $g(x)$.

Lemma. *Let f and g be real functions. Then, $f(x) \in \Theta(g(x))$ if and only if $\Theta(f(x)) = \Theta(g(x))$ if and only if $O(f(x)) = O(g(x))$.*

Lemma. *Let f and g be real functions. Then, $f(x) \in O(g(x))$ and $g(x) \notin O(f(x))$ if and only if $O(f(x)) \subset O(g(x))$.*

Lemma. *Let $r_1, r_2 \in \mathbb{Q}$ with $r_1 < r_2$. Then, $x^{r_2} \notin O(x^{r_1})$.*

Definition. Let $g(n)$ be a function. An algorithm is $O(g(n))$ (respectively, $\Theta(g(n))$) if its worst-case complexity $f(n)$ is in $O(g(n))$ (respectively, $\Theta(g(n))$).

Theorem. *Let $b \in \mathbb{R}$ with $b > 1$ and $r \in \mathbb{Q}$ with $r > 2$.*
$O(1) \subset O(\log_b n) \subset O(n) \subset O(n \log_b n) \subset O(n^2) \subset O(n^r) \subset O(b^n) \subset O(n!)$.

An algorithm that is $\Theta(\log_b n)$ for some $b > 1$ is said to be a **logarithmic**. Those which are $\Theta(n^m)$ for some $m \in \mathbb{Z}^+$ are said to be **polynomial**, and those which are $\Theta(b^n)$ for some $b > 1$ are called **exponential**. Problems that can be solved by polynomial algorithms are said to be in class **P**. Problems for which a proposed solution can be checked for correctness with a polynomial algorithm are said to be in class **NP**, the class of nondeterministic polynomial algorithms. It is known that P $\subseteq$ NP.

Section 10.5

Example. The worst-case complexity of Insertion Sort is $\frac{n(n-1)}{2}$.

Example. Merge Sort is $O(n \log_2 n)$.

Theorem. *The worst-case complexity of every sorting algorithm is at least $\frac{n}{2} \log_2 \frac{n}{2}$.*

Algorithm 6 Insertion Sort

Let A be an array of length n that needs to be sorted (into nondecreasing order). Variable i is the index of the array entry to be inserted into its correct position relative to the previously sorted portion $A[1, \ldots, i-1]$, and j runs through that portion in search of this correct position.

Algorithm.

```
For i = 2 to n,
    \begin
    Let j = 1.
    While j < i and A[j] < A[i],
        Let j = j + 1.
    If j < i, then
        \begin
        Let temp = A[i].
        For k = i down to j + 1,
            Let A[k] = A[k − 1].
        Let A[j] = temp.
        \end.
    \end.
Return A.
```

Algorithm 7 Merge Sort

Let A be an array of length n that needs to be sorted (into nondecreasing order). Arrays A_1 and A_2 are used to store the first and second halves of A, respectively, after they have been sorted by this same algorithm.

Algorithm.

```
If n ≤ 1, then
    Return A.
Otherwise,
    \begin
    Let A_1 = Merge_Sort(A[1, ..., ⌊n/2⌋], ⌊n/2⌋).
    Let A_2 = Merge_Sort(A[⌊n/2⌋ + 1, ..., n], ⌈n/2⌉).
    Return Merge(A_1, ⌊n/2⌋, A_2, ⌈n/2⌉).
    \end
```

Algorithm 8 Merge

Given a sorted array A_1 of length n_1 and a sorted array A_2 of length n_2, this algorithm produces a sorted array A of length $n_1 + n_2$.

Algorithm.

```
Let i_1 = i_2 = i = 1.
While i_1 ≤ n_1 and i_2 ≤ n_2,
    \begin
    If A_1[i_1] < A_2[i_2], then
        \begin
        Let A[i] = A_1[i_1].
        Let i_1 = i_1 + 1.
        \end
    Otherwise,
        \begin
        Let A[i] = A_2[i_2].
        Let i_2 = i_2 + 1.
        \end
    Let i = i + 1.
    \end.
If i_1 > n_1, then
    While i_2 ≤ n_2,
        \begin
        Let A[i] = A_2[i_2].
        Let i = i + 1 and i_2 = i_2 + 1.
        \end
If i_2 > n_2, then
    While i_1 ≤ n_1,
        \begin
        Let A[i] = A_1[i_1].
        Let i = i + 1 and i_1 = i_1 + 1.
        \end
Return array A of length n_1 + n_2.
```

Algorithm 9 Bubble Sort

Variable i is the index of the entry to be determined based on the assumption that the portion $A[i+1,\ldots,n]$ is already set, while j runs through the lower indices to filter up the correct value for the i^{th} spot.

Algorithm.

```
For i = n down to 2,
    For j = 1 to i − 1,
        If A[j] > A[j + 1], then
            \begin
            Let temp = A[j + 1].
            Let A[j + 1] = A[j].
            Let A[j] = temp.
            \end.
Return A.
```

Algorithm 10 Selection Sort

Variable i is the index of the entry to be determined based on the assumption that the portion $A[1,\ldots,i-1]$ is already sorted, while j runs through the upper indices to find the correct value for the i^{th} spot. Variable min stores the index of the current best candidate for the i^{th} spot.

Algorithm.

```
For i = 1 to n − 1,
    \begin
    Let min = i.
    For j = i + 1 to n,
        If A[j] < A[min], then
            Let min = j.
    If min ≠ i, then
        \begin
        Let temp = A[i].
        Let A[i] = A[min].
        Let A[min] = temp.
        \end.
    \end.
Return A.
```

Algorithm 11 Quick Sort

Let A be an array of length n that needs to be sorted (into nondecreasing order). The variable **mid** will be a position chosen by the function Split that cuts the array into two pieces and partially sorts it so that
$A[1], \ldots, A[\text{mid}-1] < A[\text{mid}] \leq A[\text{mid}+1], \ldots, A[n]$.

Algorithm.

If $n \leq 1$, then
  Return A.
Otherwise,
  `\begin`
  Let $(A, \text{mid}) = \text{Split}(A, n)$.
  Let $A[1, \ldots, \text{mid}-1] = \text{Quick\_Sort}(A[1, \ldots, \text{mid}-1], \text{mid}-1)$.
  Let $A[1, \ldots, \text{mid}+1] = \text{Quick\_Sort}(A[\text{mid}+1, \ldots, n], n-\text{mid})$.
  Return A.
  `\end`

Algorithm 12 Split

Let A be an array of length n that needs to be reordered so that $A[1]$ is moved to the position, index **mid**, into which it will end up when the array is entirely sorted. When this function completes with A reordered, we want no larger entries than $A[\text{mid}]$ prior to position mid and no smaller entries than $A[\text{mid}]$ after position mid.

Algorithm.

Let mid $= 1$.
For $i = 2$ to n,
  If $A[i] < A[1]$, then
    `\begin`
    Let mid $=$ mid $+ 1$.
    Let temp $= A[\text{mid}]$. Let $A[\text{mid}] = A[i]$. Let $A[i] =$ temp.
    `\end`
Let temp $= A[\text{mid}]$. Let $A[\text{mid}] = A[1]$. Let $A[1] =$ temp.
Return (A, mid).

Sorting Algorithm	Insertion	Bubble	Selection	Merge	Quick
Worst-Case	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n \log_2 n)$	$\Theta(n^2)$
Average-Case	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n \log_2 n)$	$\Theta(n \log_2 n)$

Table 6: Time Complexities for Sorting Algorithms

Chapter 2

Answers to Selected Exercises

2.0 Chapter 0

1. 10.
$(1)2^3 + (0)2^2 + (1)2^1 + (0)2^0 = 10.$

3. 23.
$(1)2^4 + (0)2^3 + (1)2^2 + (1)2^1 + (1)2^0 = 23.$

5. 46.
$(1)2^5 + (0)2^4 + (1)2^3 + (1)2^2 + (1)2^1 + (0)2^0 = 46.$

7. 75.
$(1)2^6 + (0)2^5 + (0)2^4 + (1)2^3 + (0)2^2 + (1)2^1 + (1)2^0 = 75.$

9. 171.
$(1)2^7 + (0)2^6 + (1)2^5 + (0)2^4 + (1)2^3 + (0)2^2 + (1)2^1 + (1)2^0 = 171.$

11. 111011.

$$\begin{array}{lcrll}
59/2 & = & 29 & \text{remainder} & 1 \\
29/2 & = & 14 & \text{remainder} & 1 \\
14/2 & = & 7 & \text{remainder} & 0 \\
7/2 & = & 3 & \text{remainder} & 1 \\
3/2 & = & 1 & \text{remainder} & 1 \\
1/2 & = & 0 & \text{remainder} & 1
\end{array}$$

13. 1010100.

$$\begin{array}{lcrll}
84/2 & = & 42 & \text{remainder} & 0\\
42/2 & = & 21 & \text{remainder} & 0\\
21/2 & = & 10 & \text{remainder} & 1\\
10/2 & = & 5 & \text{remainder} & 0\\
5/2 & = & 2 & \text{remainder} & 1\\
2/2 & = & 1 & \text{remainder} & 0\\
1/2 & = & 0 & \text{remainder} & 1
\end{array}$$

15. 1110101.

$$\begin{array}{lcrll}
117/2 & = & 58 & \text{remainder} & 1\\
58/2 & = & 29 & \text{remainder} & 0\\
29/2 & = & 14 & \text{remainder} & 1\\
14/2 & = & 7 & \text{remainder} & 0\\
7/2 & = & 3 & \text{remainder} & 1\\
3/2 & = & 1 & \text{remainder} & 1\\
1/2 & = & 0 & \text{remainder} & 1
\end{array}$$

17. 100110000.

$$\begin{array}{lcrll}
304/2 & = & 152 & \text{remainder} & 0\\
152/2 & = & 76 & \text{remainder} & 0\\
76/2 & = & 38 & \text{remainder} & 0\\
38/2 & = & 19 & \text{remainder} & 0\\
19/2 & = & 9 & \text{remainder} & 1\\
9/2 & = & 4 & \text{remainder} & 1\\
4/2 & = & 2 & \text{remainder} & 0\\
2/2 & = & 1 & \text{remainder} & 0\\
1/2 & = & 0 & \text{remainder} & 1
\end{array}$$

19. 10000000000.
Note that $2^{10} = 1024$.

21. Using T for Tails and H for Heads, we see the 16 possibilities.

T	T	T	T
T	T	T	H
T	T	H	T
T	T	H	H
T	H	T	T
T	H	T	H
T	H	H	T
T	H	H	H
H	T	T	T
H	T	T	H
H	T	H	T
H	T	H	H
H	H	T	T
H	H	T	H
H	H	H	T
H	H	H	H

23. 115.
$(1)8^2 + (6)8^1 + (3)8^0 = 115.$

25. 1679.
$(3)8^3 + (2)8^2 + (1)8^1 + (7)8^0 = 1679.$

27. 16712.
$(4)8^4 + (0)8^3 + (5)8^2 + (1)8^1 + (0)8^0 = 16712.$

29. 3529.
$(13)16^2 + (12)16^1 + (9)16^0 = 3529.$

31. 23166.
$(5)16^3 + (10)16^2 + (7)16^1 + (14)16^0 = 23166.$

33. 50.
$(3)16^1 + (2)16^0 = 50.$

35. 73.

$$\begin{array}{lcrlr} 59/8 & = & 7 & \text{remainder} & 3 \\ 7/8 & = & 0 & \text{remainder} & 7 \end{array}$$

37. 165.

$$\begin{array}{lcrlr} 117/8 & = & 14 & \text{remainder} & 5 \\ 14/8 & = & 1 & \text{remainder} & 6 \\ 1/8 & = & 0 & \text{remainder} & 1 \end{array}$$

39. 214.

$$\begin{array}{lcrll} 140/8 & = & 17 & \text{remainder} & 4 \\ 17/8 & = & 2 & \text{remainder} & 1 \\ 2/8 & = & 0 & \text{remainder} & 2 \end{array}$$

41. $3b$.

$$\begin{array}{lcrlr} 59/16 & = & 3 & \text{remainder} & 11 \\ 3/16 & = & 0 & \text{remainder} & 3 \end{array}$$

43. 75.

$$\begin{array}{lcrll} 117/16 & = & 7 & \text{remainder} & 5 \\ 7/16 & = & 0 & \text{remainder} & 7 \end{array}$$

45. $acdc$.

$$\begin{array}{lcrlr} 44252/16 & = & 2765 & \text{remainder} & 12 \\ 2765/16 & = & 172 & \text{remainder} & 13 \\ 172/16 & = & 10 & \text{remainder} & 12 \\ 10/16 & = & 0 & \text{remainder} & 10 \end{array}$$

47. (a) 1463.

$$\underbrace{001}_{1}\underbrace{100}_{4}\underbrace{110}_{6}\underbrace{011}_{3}$$

(b) 333.

$$\underbrace{0011}_{3}\underbrace{0011}_{3}\underbrace{0011}_{3}$$

49. (a) 54613.

$$\underbrace{101}_{5}\underbrace{100}_{4}\underbrace{110}_{6}\underbrace{001}_{1}\underbrace{011}_{3}$$

(b) $598b$.

$$\underbrace{0101}_{5}\underbrace{1001}_{9}\underbrace{1000}_{8}\underbrace{1011}_{b}$$

51. 100111.

$$\underbrace{100}_{4}\underbrace{111}_{7}$$

53. 101011001100.

$$\underbrace{1010}_{a}\underbrace{1100}_{c}\underbrace{1100}_{c}$$

55. 2^m.
$(1)2^m + (0)2^{m-1} + \cdots + (0)2^0 = 2^m$.

57. A one followed by $2n$ zeros.
$(1)2^{2n} + (0)2^{2n-1} + \cdots + (0)2^0 = 2^{2n} = 4^n$.

59. It is divisible by 4 if and only if it ends in 00.
In the algorithm for converting numbers to binary, the first two divisions by 2 must result in remainder 0.

61. $8^m - 1$.
$(7)8^{m-1}+(7)8^{m-2}+\cdots+(7)8^0 = (7)[8^{m-1}+8^{m-2}+\cdots+8^0] = (7)\frac{8^m-1}{8-1} = 8^m-1$.
Alternatively, if 1 is added to this number, then the result in octal is a 1 followed by m zeros. That value is 8^m.

63. The number is divisible by 7 if and only if the sum of its digits is divisible by 7.
This is the analog of the divisibility by 9 test for base ten.

65. First rewrite the number in binary, and then group the digits into blocks of size 4 to convert to hexadecimal.

2.1 Chapter 1

Section 1.1

1. A true statement.
See the appendix in the textbook, that characterizes the integers.

3. Not a statement. It is a question.

5. Not a statement. It is not a declarative sentence.

7.

p	q	$\neg q$	$p \vee \neg q$
F	F	T	T
F	T	F	F
T	F	T	T
T	T	F	T

9.

p	q	r	$\neg p$	$q \wedge r$	$\neg p \to (q \wedge r)$
F	F	F	T	F	F
F	F	T	T	F	F
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	F	F	T
T	F	T	F	F	T
T	T	F	F	F	T
T	T	T	F	T	T

11.

p	q	r	$p \to q$	$(p \to q) \vee r$
F	F	F	T	T
F	F	T	T	T
F	T	F	T	T
F	T	T	T	T
T	F	F	F	F
T	F	T	F	T
T	T	F	T	T
T	T	T	T	T

13. They differ in the two rows in which q is true.
Use C2 = OR(A2,NOT(B2)) and D2 = OR(NOT(A2),NOT(B2)) to generate the following table.

	A	B	C	D
1	p	q	$p \vee \neg q$	$p \to \neg q$
2	FALSE	FALSE	TRUE	TRUE
3	FALSE	TRUE	FALSE	TRUE
4	TRUE	FALSE	TRUE	TRUE
5	TRUE	TRUE	TRUE	FALSE

15. They differ in the two rows in which p is true and q is false.
Use D2 = OR(NOT(A2),OR(B2,C2)) and
E2 = OR(NOT(A2),OR(B2,NOT(C2))) to generate the following table.

	A	B	C	D	E
1	p	q	r	$p \to (q \vee r)$	$p \to (q \vee \neg r)$
2	FALSE	FALSE	FALSE	TRUE	TRUE
3	FALSE	FALSE	TRUE	TRUE	TRUE
4	FALSE	TRUE	FALSE	TRUE	TRUE
5	FALSE	TRUE	TRUE	TRUE	TRUE
6	TRUE	FALSE	FALSE	FALSE	TRUE
7	TRUE	FALSE	TRUE	TRUE	FALSE
8	TRUE	TRUE	FALSE	TRUE	TRUE
9	TRUE	TRUE	TRUE	TRUE	TRUE

17.

p	q	$p \vee q$	$p \to p \vee q$
F	F	F	T
F	T	T	T
T	F	T	T
T	T	T	T

Observe that $p \to p \vee q$ is always true.

19. (a)

p	$\neg p$	$\neg\neg p$
F	T	F
T	F	T

The first and last columns are the same.
(b)

$\underline{t}$	$\neg\underline{t}$		$\underline{f}$	$\neg\underline{f}$
T	F		F	T

The first two columns show that $\neg\underline{t}$ is a contradiction. The last two columns show that $\neg\underline{f}$ is a tautology.

21.

p	$\underline{t}$	$\underline{t} \to p$	$p \to \underline{t}$
F	T	F	T
T	T	T	T

Columns 1 and 3 are the same, and columns 2 and 4 are the same.

23.

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \vee q$	$\neg(\neg p \wedge \neg q)$
F	F	T	T	T	F	F
F	T	T	F	F	T	T
T	F	F	T	F	T	T
T	T	F	F	F	T	T

The last two columns are the same.

25.

p	q	$p \vee q$	$p \to p \vee q$	$\underline{t}$
F	F	F	T	T
F	T	T	T	T
T	F	T	T	T
T	T	T	T	T

The last two columns are the same.

27.

p	q	r	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$	$(p \vee q) \vee r$	$p \vee (q \vee r)$
F	F	F	F	F	F	F
F	F	T	F	F	T	T
F	T	F	F	F	T	T
F	T	T	F	F	T	T
T	F	F	F	F	T	T
T	F	T	F	F	T	T
T	T	F	F	F	T	T
T	T	T	T	T	T	T

Columns 4 and 5 are the same, and columns 6 and 7 are the same.

29.

p	q	r	$p \oplus q$	$q \oplus r$	$(p \oplus q) \oplus r$	$p \oplus (q \oplus r)$
F	F	F	F	F	F	F
F	F	T	F	T	T	T
F	T	F	T	T	T	T
F	T	T	T	F	F	F
T	F	F	T	F	T	T
T	F	T	T	T	F	F
T	T	F	F	T	F	F
T	T	T	F	F	T	T

The last two columns are the same.

31.

p	q	r	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
F	F	F	F	F	F	F
F	F	T	F	F	F	F
F	T	F	F	F	F	F
F	T	T	F	F	T	T
T	F	F	F	F	T	T
T	F	T	T	T	T	T
T	T	F	T	T	T	T
T	T	T	T	T	T	T

Columns 4 and 5 are the same, and columns 6 and 7 are the same.

33.

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
F	F	T	T	T	T
F	T	T	T	F	F
T	F	T	T	F	F
T	T	F	F	F	F

Columns 3 and 4 are the same, and columns 5 and 6 are the same.

35.

p	q	$p \oplus q$	$\neg(p \oplus q)$	$p \leftrightarrow q$
F	F	F	T	T
F	T	T	F	F
T	F	T	F	F
T	T	F	T	T

The last two columns are the same.

37. They differ when p is false, q is true, and r is false.

p	q	r	$(p \to q) \to r$	$p \to (q \to r)$
F	T	F	F	T

This case, for example, shows that the truth tables are different. Hence, $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.

39. They differ when p is true, q is false, and r is true.

p	q	r	$p \oplus (q \wedge r)$	$(p \oplus q) \wedge (p \oplus r)$
T	F	T	T	F

This case, for example, shows that the truth tables are different. Hence, $p \oplus (q \wedge r)$ and $(p \oplus q) \wedge (p \oplus r)$ are not logically equivalent.

41.

p	q	$p \oplus q$	$\neg p \oplus \neg q$
F	F	F	F
F	T	T	T
T	F	T	T
T	T	F	F

The last two columns are the same. Hence, $p \oplus q$ and $\neg p \oplus \neg q$ are logically equivalent.

43. (a) $\neg q \to p$. (b) $\neg\neg q \to \neg p \equiv q \to \neg p$.
(c) $\neg p \to \neg\neg q \equiv \neg p \to q$. (d) $p \wedge \neg\neg q \equiv p \wedge q$.

45. (a) $r \to p \wedge \neg q$. (b) $\neg r \to \neg(p \wedge \neg q) \equiv \neg r \to \neg p \vee \neg\neg q \equiv \neg r \to \neg p \vee q$.
(c) $\neg(p \wedge \neg q) \to \neg r \equiv \neg p \vee q \to \neg r$. (d) $(p \wedge \neg q) \wedge \neg r \equiv p \wedge \neg q \wedge \neg r$.

47. (a) If Ted has a failing grade, then Ted's average is less than 60.
(b) If Ted has a passing grade, then Ted's average is at least 60.
(c) If Ted's average is at least 60, then Ted has a passing grade.
(d) Ted's average is less than 60, and Ted has a passing grade.

49. (a) If George is going to a movie or going dancing, then George feels well.
(b) If George is not going to a movie and not going dancing, then George does not feel well.
(c) If George does not feel well, then George is not going to a movie and not going dancing.
(d) George feels well, and George is not going to a movie and not going dancing.

51. $\neg(p \vee \neg q) \equiv \neg p \wedge \neg\neg q \equiv \neg p \wedge q$.

53. $\neg(\neg p \wedge (q \vee \neg r)) \equiv p \vee \neg(q \vee \neg r) \equiv p \vee (\neg q \wedge r)$.

55. Helen's average is less than 90, or Helen is not getting an A.

57. (a)

p	q	r	$p \wedge q$	$\neg p$	$q \to r$	$(p \wedge q) \to r$	$\neg p \vee (q \to r)$
F	F	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	T	F	F	T	F	T	T
F	T	T	F	T	T	T	T
T	F	F	F	F	T	T	T
T	F	T	F	F	T	T	T
T	T	F	T	F	F	F	F
T	T	T	T	F	T	T	T

The last two columns are the same.

(b)

$$\begin{array}{rcll} p \wedge q \to r & \equiv & \neg(p \wedge q) \vee r & \text{Example 1.10} \\ & \equiv & (\neg p \vee \neg q) \vee r & \text{De Morgan's Law} \\ & \equiv & \neg p \vee (\neg q \vee r) & \text{Associativity} \\ & \equiv & \neg p \vee (q \to r) & \text{Example 1.10} \end{array}$$

59.

$$\begin{array}{rcll} p \wedge (q \vee r \vee s) & \equiv & p \wedge (q \vee (r \vee s)) & \text{Associativity} \\ & \equiv & (p \wedge q) \vee (p \wedge (r \vee s)) & \text{Distributivity} \\ & \equiv & (p \wedge q) \vee ((p \wedge r) \vee (p \wedge s)) & \text{Distributivity} \\ & \equiv & (p \wedge q) \vee (p \wedge r) \vee (p \wedge s) & \text{Associativity} \end{array}$$

61.

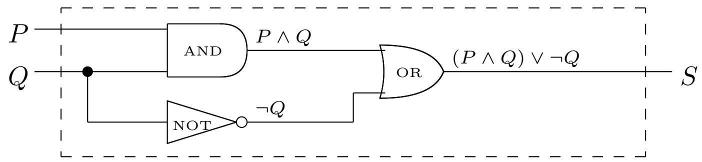
$$\begin{array}{ll} (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) & \\ \equiv (p \wedge (q \wedge \neg r)) \vee (p \wedge (\neg q \wedge r)) & \text{Associativity} \\ \equiv p \wedge ((q \wedge \neg r) \vee (\neg q \wedge r)) & \text{Distributivity} \\ \equiv p \wedge (q \oplus r) & \text{Definition of } \oplus \end{array}$$

63.
$$\begin{aligned} p \wedge (\neg(q \wedge r)) &\equiv p \wedge (\neg q \vee \neg r) && \text{De Morgan's Law} \\ &\equiv (p \wedge \neg q) \vee (p \wedge \neg r) && \text{Distributivity} \end{aligned}$$

65. Since $p \wedge q \wedge r \to p \wedge q$ is a tautology, the result follows from the Absorption Rule.
That is, think of the Absorption Rule as: If $u \to w$, then $u \vee w \equiv w$. Apply this with $u = p \wedge q \wedge r$ and $w = p \wedge q$.

67. (i) Given $\neg$ and $\wedge$, we have that $p \vee q \equiv \neg(\neg p \wedge \neg q)$ and $p \to q \equiv \neg(p \wedge \neg q)$.
(ii) Given $\neg$ and $\vee$, we have that $p \wedge q \equiv \neg(\neg p \vee \neg q)$ and $p \to q \equiv \neg p \vee q$.
(iii) Given $\neg$ and $\to$, we have that $p \wedge q \equiv \neg(p \to \neg q)$ and $p \vee q \equiv \neg p \to q$.

69. (a) $(P \wedge Q) \vee \neg Q = S$, as traced below.

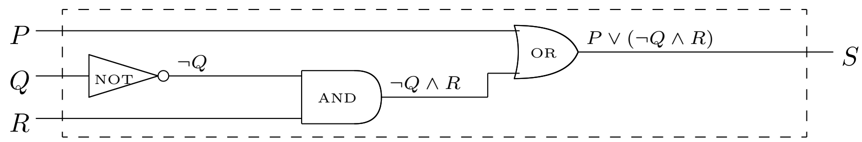


(b)

P	Q	S
0	0	1
0	1	0
1	0	1
1	1	1

(c) Yes, $S = P \vee \neg Q$ and can be done with two gates.

71. (a) $P \vee (\neg Q \wedge R) = S$, as traced below.

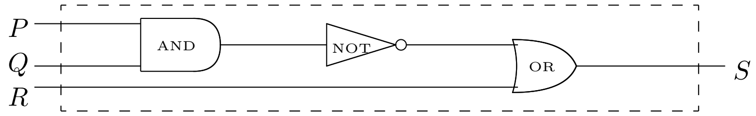


(b)

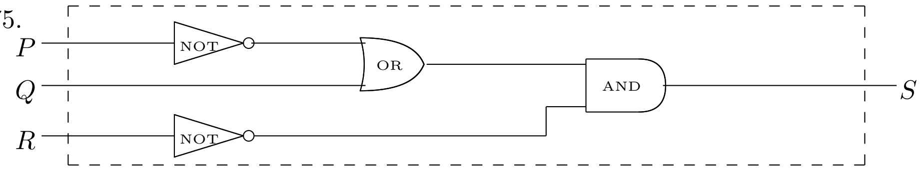
P	Q	R	S
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

(c) No. Note that there are not many ways to get a single output from three inputs and two gates, and none of them satisfy this table.

73.



75.



77. (a) $(P \vee Q) \wedge \neg(P \wedge Q) = S$, a 4 gate circuit.
(b) $(P \wedge \neg Q) \vee (\neg P \wedge Q) = S$, a 5 gate circuit.
So, the definition uses fewer gates than the alternative characterization.

Section 1.2

1. True.
Order does not matter.

3. True.
Repetition does not matter.

5. False.
$\{-1, 0, 1\} \neq \{\ldots, -2, -1, 0, 1\}$.

7. True.
Both are $\{-1, 1\}$.

9. $\{2, 4, 6\}$.

11. $\{\{1\}, \{4\}\}$.

13. $\{x : x \in \mathbb{R} \text{ and } x^3 - 4x^2 + 5x - 6 = 0\}$, which happens to equal $\{3\}$.
Note that $x^3 - 4x^2 + 5x - 6 = (x - 3)(x^2 - x + 2)$.

15. $\{n : n \in \mathbb{Z} \text{ and } n < -10\}$
or $\{n : n \in \mathbb{Z} \text{ and } n \leq -11\} = \{\ldots, -13, -12, -11\}$.

17. $(0, \infty)$
$= \{x : x \in \mathbb{R} \text{ and } x > 0\}$.

19. $[0, 0]$
$= \{x : x \in \mathbb{R} \text{ and } 0 \leq x \leq 0\}$.

21. $(1, \infty)$
$= \{x \ : \ x > 1\}$.

23. $(-1, 1)$
$= \{x \ : \ -1 < x < 1\}$.

25. True.
Note that $\sqrt{2} \approx 1.414$.

27. False.
$\{1\}$ is a set.

29. True.
$2 \in \{1, 2, 3\}$.

31. True.
$\emptyset$ is listed as an element on the right-hand side.

33. $\subset, \subseteq$.
Note that $\in$ does not work, since $\{1\}$ is not listed in $\{1, 2\}$, although 1 is. Note that $=$ does not work, since 2 is not in $\{1\}$.

35. $\subseteq, =$.
Note that $\in$ does not work, since the elements on the right-hand side are 6, 7, and 8. Note that $\subset$ does not work, since $=$ holds.

37. $\in, \subset, \subseteq$.
Note that $=$ does not work, since $\subset$ holds.

39. Finite. $|A| = 5$.

41. Infinite.
E.g., 5.9, 5.99, 5.999, ...

43. Finite. $|E| = 9$.
Namely, $E = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

45. Finite. $|G| = 2$.
Note that $\emptyset$ and $\{\emptyset\}$ are the two elements.

47. If yes, then he should not. If no, then he should. Hence, either way, there is a contradiction.
That is, if he shaves himself, then he is shaving someone who shaves himself, and he is not supposed to do that. If he does not shave himself, then he is someone who does not shave himself, and he is supposed to shave such a person.

49. `False`.
Use `{1,2,{2,1}} == {1,2,{1,2}}`.

51. `2`.
Use `Length[{{},{}}]`.

53. `true`.
Use `evalb({1,2,{2,1}} = {1,2,{1,2}})`.

55. `1`.
Use `nops({{},{}})`.

Section 1.3

1. $\forall\, x \in \mathbb{R}, x^2 + 1 > 0$.
See Appendix A, number 11.

3. $\exists\, n \in \mathbb{Z}$ such that $\frac{1}{n} \in \mathbb{Z}$.
E.g., $n = 1$ works.

5. $\exists\, n \in \mathbb{N}$ such that $\forall\, x \in \mathbb{R}, x^n \geq 0$.
E.g., $n = 2$ works.

7. $\exists\, x \in \mathbb{R}$ such that $\forall\, y \in \mathbb{R}$, if $2 \leq y \leq 3$, then $1 \leq xy < 2$.
Also, $\exists\, x \in \mathbb{R}$ such that $\forall\, y \in [2, 3], 1 \leq xy < 2$.
E.g., $x = \frac{1}{2}$ works.

9. $\exists\, x, y \in \mathbb{R}$ such that $x + y \in \mathbb{Z}$ and $xy \notin \mathbb{Z}$.
E.g., $x = \frac{1}{\sqrt{2}}$ and $y = \frac{-1}{\sqrt{2}}$ works.

11. $\forall\, x, y \in \mathbb{R}$, if $x < y$, then $e^x < e^y$.
See Definition 1.15, and use $f(x) = e^x$.

13. $\exists\, x \in [-2, 2]$ such that $x^3 \notin [0, 8]$. Negation is true.
E.g., $x = -2$ works.

15. $\exists\, x \in \mathbb{R}^+$ such that $x^2 > 4$ and $x \leq 2$. Original is true.
E.g., $x = -3$. Note that the contrapositive of the original is
$\forall\, x \in \mathbb{R}^+$, if $x \leq 2$, then $x^2 \leq 4$.

17. $\forall\, n \in \mathbb{Z}, \exists\, m \in \mathbb{Z}$ such that $nm \geq 1$. Original is true.
E.g., $n = 0$ works.

19. $\exists\, m, n \in \mathbb{Z}$ such that $m + n \notin \mathbb{Z}$. Original is true.
See Appendix A, number 1, with $S = \mathbb{Z}$.

21. $\forall\, n \in \mathbb{Z}, \frac{1}{n} \notin \mathbb{Z}$.
No integer's reciprocal is an integer.

23. $\exists\, x \in \mathbb{R}$ such that $x^2 + 1 \leq 0$.
There is a real number x such that $x^2 + 1 \leq 0$.

25. $\forall\, n \in \mathbb{N}, \exists\, x \in \mathbb{R}$ such that $x^n < 0$.
For every natural number n, there is a real number x such that $x^n < 0$.

27. $\forall\, x \in \mathbb{R}, \exists\, y \in \mathbb{R}$ such that $2 \leq y \leq 3$ and $xy \notin [1, 2)$.
For every real number x, there is a real number y such that
$2 \leq y \leq 3$ and $xy \notin [1, 2)$.

29. $\forall\, x, y \in \mathbb{R}, x + y \notin \mathbb{Z}$ or $xy \in \mathbb{Z}$.
For all real numbers x and y, either $x + y \notin \mathbb{Z}$ or $xy \in \mathbb{Z}$.

31. There is a student at Harvard University whose age is at most 17.
That is, there is a student at Harvard University whose age is not over 17.

33. There exists a truly great accomplishment that is immediately possible.

35. There is such a thing as bad publicity.

37. (a) A real function f is not constant iff $\forall\, c \in \mathbb{R}, \exists\, x \in \mathbb{R}$ such that $f(x) \neq c$.
The definition is $\exists\, c \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R}, f(x) = c$.
(b) $\exists\, x, y \in \mathbb{R}$ such that $f(x) \neq f(y)$.

39. $\exists\, x, y \in \mathbb{R}$ such that $x < y$ and $f(x) \geq f(y)$.
Recall that $\neg(p \to q) \equiv p \wedge \neg q$.

41. $\exists\, x, y \in \mathbb{R}$ such that $x \leq y$ and $f(x) > f(y)$.
Note that, in this case, we could replace $x \leq y$ with $x < y$.

43. $\forall\, M \in \mathbb{R}, \exists\, x \in \mathbb{R}$ such that $f(x) > M$.
That is, f is unbounded above.

45. True. An if-then statement is true whenever its hypothesis is false.
By Appendix A, number 11, $x^2 < 0$ never happens.

47. Say $\mathcal{U} = \{a, b\}$. The statement is equivalent to "$p(a) \wedge p(b)$."
That is, a and b are all of the x's.

49. Say $\mathcal{U} = \{a, b\}$. The logical equivalences are equivalent to

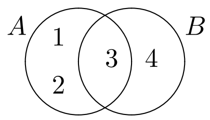
$$\begin{aligned} \neg[p(a) \wedge p(b)] &\equiv \neg p(a) \vee \neg p(b) \\ \neg[p(a) \vee p(b)] &\equiv \neg p(a) \wedge \neg p(b), \end{aligned}$$

since, in this case,

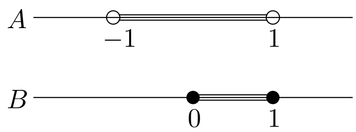
$$\begin{aligned} \forall\, x \in \mathcal{U}, p(x) &\equiv p(a) \wedge p(b) \\ \exists\, x \in \mathcal{U} \text{ such that } p(x) &\equiv p(a) \vee p(b). \end{aligned}$$

Section 1.4

1. $A^c = \{4\}$, $B^c = \{1, 2\}$, $A \cap B = \{3\}$, $A \cup B = \{1, 2, 3, 4\}$, $A \setminus B = \{1, 2\}$, $B \setminus A = \{4\}$, and $A \vartriangle B = \{1, 2, 4\}$.



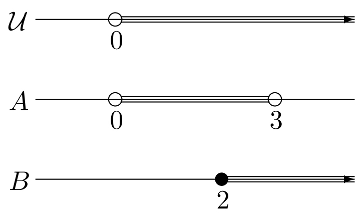
3. $A^c = (-\infty, -1] \cup [1, \infty)$, $B^c = (-\infty, 0) \cup (1, \infty)$, $A \cap B = [0, 1)$, $A \cup B = (-1, 1]$, $A \setminus B = (-1, 0)$, $B \setminus A = \{1\}$, and $A \vartriangle B = (-1, 0) \cup \{1\}$.



5. $A^c = \mathbb{Z}^-$, $B^c = \mathbb{Z}^- \cup \{0\}$, $A \cap B = \mathbb{Z}^+$, $A \cup B = \mathbb{N}$, $A \setminus B = \{0\}$, $B \setminus A = \emptyset$, and $A \vartriangle B = \{0\}$.

$$A = \mathbb{N} = \mathbb{Z}^+ \cup \{0\} = B \cup \{0\}.$$

7. $A^c = [3, \infty)$, $B^c = (0, 2)$, $A \cap B = [2, 3)$, $A \cup B = (0, \infty)$, $A \setminus B = (0, 2)$, $B \setminus A = [3, \infty)$, and $A \vartriangle B = (0, 2) \cup [3, \infty)$.



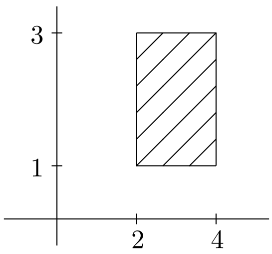
9. Yes. It is impossible to simultaneously have $n < 0$ and $n \geq 0$.

11. No, since 3.5 is in the intersection.

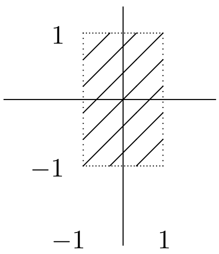
13. $\{(1,2),(1,4),(3,2),(3,4)\}$. Notice that it has $2 \cdot 2 = 4$ elements.

15. $\{(3,5),(5,5),(7,5),(9,5)\}$. Notice that it has $4 \cdot 1 = 4$ elements.

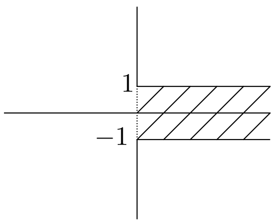
17.



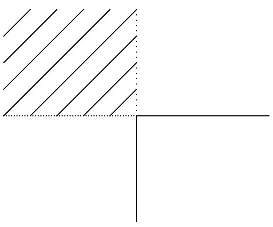
19.



21.



23.



25. $\{(1,2,1),(1,2,2),(1,4,1),(1,4,2),(3,2,1),(3,2,2),(3,4,1),(3,4,2)\}$.
Notice that it has $2 \cdot 2 \cdot 2 = 8$ elements.

27. $\{(a,a,a),(a,a,b),(a,b,a),(a,b,b),(b,a,a),(b,a,b),(b,b,a),(b,b,b)\}$.
Notice that it has $2^3 = 8$ elements.

29. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.
Notice that it has $2^3 = 8$ elements.

31. $\{\emptyset,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},$
$\{2,3,4\},\{1,2,3,4\}\}$.
Notice that it has $2^4 = 16$ elements.

33. 1024.
That is, for $A = \{n : n \in \mathbb{Z} \text{ and } 1 \leq n \leq 10\} = \{1,2,3,4,5,6,7,8,9,10\}$, we have $|A| = 10$, and hence $|\mathcal{P}(A)| = 2^{10} = 1024$.

35. $\emptyset$, $\{\pi\}$, $(-2,7]$, and $\mathbb{Z}$.
Many different answers are possible.

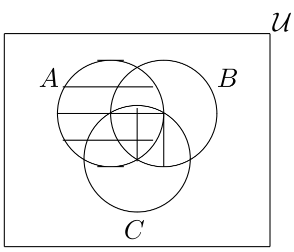
37. False.
1 is not an ordered pair.

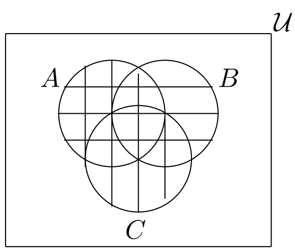
39. False.
$1 \notin \{3,4\}$.

41. $(A \vartriangle B) \vartriangle C = A \vartriangle (B \vartriangle C)$.
That is, $\oplus$ corresponds to $\vartriangle$.

43. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
That is, $\wedge$ corresponds to $\cap$, and $\vee$ corresponds to $\cup$.

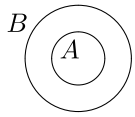
45.



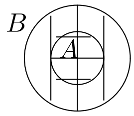


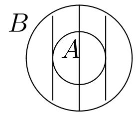
$A \cup (B \cap C)$ is the shaded portion of the left diagram. $(A \cup B) \cap (A \cup C)$ is the doubly shaded portion of the right diagram. Both portions are the same.

47. Since $A \subseteq B$, we use a Venn diagram of the following form.



Both desired equations can now be seen in the following shaded diagram.





$A \cup B$ is the shaded portion of the left diagram. B is shaded in the right diagram. Both portions are the same.

49.
$$\begin{array}{rcll} A \cap (A^c \cup B \cup C) & = & (A \cap A^c) \cup (A \cap B) \cup (A \cap C) & \text{Distributivity} \\ & = & \emptyset \cup (A \cap B) \cup (A \cap C) & \text{Theorem 1.6} \\ & = & (A \cap B) \cup (A \cap C) & \text{Theorem 1.6} \end{array}$$

51.
$$\begin{array}{ll} (A \cap B \cap C^c) \cup (A \cap B^c \cap C) & \\ = (A \cap (B \cap C^c)) \cup (A \cap (B^c \cap C)) & \text{Associativity} \\ = A \cap ((B \cap C^c) \cup (B^c \cap C)) & \text{Distributivity} \\ = A \cap ((B \setminus C) \cup (C \setminus B)) & \text{Definition of } \setminus \\ = A \cap (B \triangle C) & \text{Definition of } \triangle \end{array}$$

53.
$$\begin{array}{rcll} A \cap ((B \cap C)^c) & = & A \cap (B^c \cup C^c) & \text{De Morgan's Law} \\ & = & (A \cap B^c) \cup (A \cap C^c) & \text{Distributivity} \end{array}$$

55. Since $A \cap B \cap C \subseteq A \cap B$, the Absorption Rule yields the desired result. The Absorption Rule says: If $S \subseteq T$, then $S \cup T = T$. Apply this with $S = A \cap B \cap C$ and $T = A \cap B$, after invoking commutativity to conclude that $S \cup T = T \cup S$. That is,

$$(A \cap B) \cup (A \cap B \cap C) = (A \cap B \cap C) \cup (A \cap B) = A \cap B.$$

57. Yes, as sets, but not as lists.

```
In[1]:= setEq[x_,y_] := (Union[x] == Union[y])

In[2]:= setEq[{1,2},{2,1}]

Out[2]= True

In[3]:= setEq[{1,2,2},{2,1}]

Out[3]= True
```

59. `disjoint[x_, y_] := (Intersection[x, y] == {})`

```
In[1]:= disjoint[x_,y_] := (Intersection[x,y] == {})

In[1]:= disjoint[{1,2},{3,4}]

Out[2]= True

In[3]:= disjoint[{1,2},{1,4}]

Out[3]= False
```

61. `symmDiff[x_, y_] := Union[Complement[x, y], Complement[y, x]]`

```
In[1]:= symmDiff[x_,y_] := Union[Complement[x,y], Complement[y,x]]

In[2]:= symmDiff[{1,2},{3,4}]

Out[2]= {1,2,3,4}

In[3]:= symmDiff[{1,2},{1,4}]

Out[3]= {2,4}
```

63. In both cases, `intersect` is performed before `union`.

```
> {1,3} intersect {1,2} union {2,3};
                           {1, 2, 3}

> {1,3} union {1,2} intersect {2,3};
                           {1, 2, 3}
```

65. They test whether `A` is a subset of `B`. The first should be more efficient, since the power set can be much larger than the given set and thus expensive to compute.

```
> subset1 := (x,y) -> evalb(x = x intersect y) ;
         subset1 := (x, y) -> evalb(x = x intersect y)

> with(combinat, powerset);
                           [powerset]

> subset2 := (x,y) -> member(x,powerset(y)) ;
    subset2 := (x, y) -> member(x, (combinat:-powerset)(y))
```

```
> subset1({1},{1,2});
                                true

> subset2({1},{1,2});
                                true

> subset1({1},{2,3});
                               false

> subset2({1},{2,3});
                               false
```

67. `compU := x -> U minus x;`

```
> U := {0,1,2,3,4,5,6,7,8,9};
              U := {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

> compU := x -> U minus x;
                    compU := x -> U minus x

> compU({1,2,3});
                     {0, 4, 5, 6, 7, 8, 9}

> compU({2,3,9,12});
                     {0, 1, 4, 5, 6, 7, 8}
```

Section 1.5

1.

p	q	$p \to q$	$\neg q$	$\neg p$
F	F	T	T	T
F	T	T	F	
T	F	F	T	
T	T	T	F	

The first row demonstrates the validity of the argument form

$$\begin{array}{l} p \to q \\ \neg q \\ \therefore \neg p. \end{array}$$

3.

p	q	r	$p \to r$	$q \to r$	$p \vee q$	r
F	F	F	T	T	F	
F	F	T	T	T	F	
F	T	F	T	F	T	
F	T	T	T	T	T	T
T	F	F	F	T	T	
T	F	T	T	T	T	T
T	T	F	F	F	T	
T	T	T	T	T	T	T

Rows 4, 6, and 8 demonstrate the validity of the argument form

$$\begin{array}{l} p \to r \\ q \to r \\ p \vee q \\ \therefore r. \end{array}$$

5.

p	q	$p \wedge q$	p
F	F	F	
F	T	F	
T	F	F	
T	T	T	T

The last row demonstrates the validity of the argument form

$$\begin{array}{l} p \wedge q \\ \therefore p. \end{array}$$

7.

p	q	$p \wedge q$
F	F	
F	T	
T	F	
T	T	T

The last row demonstrates the validity of the argument form

$$\begin{array}{l} p \\ q \\ \therefore p \wedge q. \end{array}$$

9. Invalid. Consider when p is false and q is true.

p	q	$p \to q$	q	p
F	T	T	T	F

This row, for example, shows that the argument form is invalid.

11. Valid. The third, fourth, sixth, and eighth rows of the truth table

p	q	r	$p \vee q$	$p \to r$	$q \vee r$
F	F	F	F	T	F
F	F	T	F	T	T
F	T	F	T	T	T
F	T	T	T	T	T
T	F	F	T	F	F
T	F	T	T	T	T
T	T	F	T	F	T
T	T	T	T	T	T

show that the argument form is valid.

13. Invalid. Consider when p is true and q is true.

p	q	$p \vee q$	p	$\neg q$
T	T	T	T	F

This row, for example, shows that the argument form is invalid.

15. Valid.
The sixth, eighth, eleventh, twelfth, and fifteenth rows of the truth table

p	q	r	s	$p \vee q$	$p \to r$	$q \to s$	$r \vee s$
F	F	F	F	F	T	T	
F	F	F	T	F	T	T	
F	F	T	F	F	T	T	
F	F	T	T	F	T	T	
F	T	F	F	T	T	F	
F	T	F	T	T	T	T	T
F	T	T	F	T	T	F	
F	T	T	T	T	T	T	T
T	F	F	F	T	F	T	
T	F	F	T	T	F	T	
T	F	T	F	T	T	T	T
T	F	T	T	T	T	T	T
T	T	F	F	T	F	F	
T	T	F	T	T	F	T	
T	T	T	F	T	T	F	
T	T	T	T	T	T	T	T

show that the argument form is valid.

17. (a)

p	q	r	$p \to r$	$q \to r$	$p \vee q \to r$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	F	
F	T	T	T	T	T
T	F	F	F	T	
T	F	T	T	T	T
T	T	F	F	F	
T	T	T	T	T	T

Rows 1, 2, 4, 6, and 8 demonstrate the validity of the argument form.

(b)

	Statement Form	Justification
1.	$p \to r$	Given
2.	$q \to r$	Given
3.	$p \vee q$	Given
4.	$p \vee q \to r$	(1), (2), Part (a)
5.	$\therefore\ r$	(3), (4), Direct Implication

(c)

	Statement Form	Justification
1.	$p \to s$	Given
2.	$q \to s$	Given
3.	$r \to s$	Given
4.	$p \vee q \vee r$	Given
5.	$p \vee q \to s$	(1), (2), Part (a)
6.	$p \vee q \vee r \to s$	(3), (5), Part (a)
7.	$\therefore\ s$	(4), (6), Direct Implication

19. Invalid, since its argument *form*

$$\begin{array}{l} p \vee q \\ \therefore\ p \end{array}$$

is invalid.

p	q	$p \vee q$	p
F	T	T	F

This row, for example, shows that the argument form is invalid.

21. Invalid, since its argument *form*

$$\begin{array}{l} p \vee \neg p \\ \neg p \\ \therefore\ q \end{array}$$

is invalid.

p	q	$p \vee \neg p$	$\neg p$	q
F	F	T	T	F

This row, for example, shows that the argument form is invalid.

23.

	Statement Form	Justification
1.	$p \to q$	Given
2.	$q \to r$	Given
3.	p	Given
4.	$p \to r$	(1), (2), Transitivity of $\to$
5.	$\therefore\ r$	(3), (4), Direct Implication

25.

	Statement Form	Justification
1.	$p \to r$	Given
2.	$p \wedge q$	Given
3.	p	(2), In Particular
4.	$\therefore\ r$	(1), (3), Direct Implication

27.

	Statement Form	Justification
1.	$p \wedge (q \vee r)$	Given
2.	$(p \wedge q) \to s$	Given
3.	$(p \wedge r) \to s$	Given
4.	$(p \wedge q) \vee (p \wedge r)$	(1), Distributivity
5.	$\therefore\ s$	(2), (3), (4), Two Separate Cases

29.

	Statement Form	Justification
1.	$p \to q$	Given
2.	p	Given
3.	$\neg p \vee q$	(1), Substitution of Equivalent
4.	$\neg\neg p$	(2), Double Negative
5.	$\therefore\ q$	(3), (4), Eliminating a Possibility

31.

	Statement Form	Justification
1.	$\forall\, x \in \mathcal{U}, p(x) \to q(x)$	Given
2.	$a \in \mathcal{U}$	Given
3.	$\neg q(a)$	Given
4.	$p(a) \to q(a)$	(1),(2), Principle of Specification
5.	$\therefore\ \neg p(a)$	(3),(4), Contrapositive Implication

33.

	Statement Form	Justification
1.	$\forall\, x \in \mathcal{U}, p(x) \to q(x)$	Given
2.	$\forall\, x \in \mathcal{U}, \neg q(x)$	Given
3.	Let $a \in \mathcal{U}$ be arbitrary.	Assumption
4.	$p(a) \to q(a)$	(1),(3), Principle of Specification
5.	$\neg q(a)$	(2),(3), Principle of Specification
6.	$\neg p(a)$	(4),(5), Contrapositive Implication
7.	$\therefore\ \forall\, x \in \mathcal{U}, \neg p(x)$	(3),(6), Principle of Generalization

35.

	Statement Form	Justification
1.	$\forall\, x \in \mathcal{U}, p(x)$	Given
2.	$\forall\, x \in \mathcal{U}, q(x)$	Given
3.	$a \in \mathcal{U}$	Given
4.	$p(a)$	(1),(3), Principle of Specification
5.	$q(a)$	(2),(3), Principle of Specification
6.	$\therefore\ \ p(a) \wedge q(a)$	(4),(5), Obtaining And

37.

	Statement Form	Justification
1.	$\forall\, x \in \mathcal{U}, p(x)$	Given
2.	$\forall\, x \in \mathcal{U}, q(x)$	Given
3.	Let $a \in \mathcal{U}$ be arbitrary.	Assumption
4.	$p(a) \wedge q(a)$	(1),(2),(3), Exercise 35
5.	$\therefore\ \ \forall\, x \in \mathcal{U}, p(x) \wedge q(x)$	(3),(4), Principle of Generalization

39.

	Statement Form	Justification
1.	$\forall\, x \in \mathcal{U}, p(x) \vee q(x)$	Given
2.	$a \in \mathcal{U}$	Given
3.	$q(a) \rightarrow r(a)$	Given
4.	$p(a) \vee q(a)$	(1),(2), Principle of Specification
5.	$p(a) \rightarrow p(a) \vee r(a)$	Tautology
6.	$r(a) \rightarrow p(a) \vee r(a)$	Tautology
7.	$q(a) \rightarrow p(a) \vee r(a)$	(3),(6),Transitivity of $\rightarrow$
8.	$\therefore\ \ p(a) \vee r(a)$	(4),(5),(7), Separate Cases

41. Let $\mathcal{U} = \mathbb{Z}$, $p(n) =$ "$n^2 < 0$", and $q(n) =$ "$n^2 \geq 0$".
In the resulting argument

$$\begin{array}{l} \forall\, n \in \mathbb{Z}, n^2 < 0 \text{ or } n^2 \geq 0 \\ \forall\, n \in \mathbb{Z}, n^2 \not< 0 \\ \therefore \forall\, n \in \mathbb{Z}, n^2 \not\geq 0, \end{array}$$

all of the premises are true, but the conclusion

$$\forall\, n \in \mathbb{Z}, n^2 < 0$$

is false.

43. Let $\mathcal{U} = \mathbb{R}$, $p(x) =$ "$x \geq 0$", and $q(x) =$ "$x \leq 0$", and $a = 0$.
In the resulting argument

$$\begin{array}{l} \forall\, x \in \mathbb{R}, x \geq 0 \text{ or } x \leq 0 \\ 0 \in \mathbb{R} \\ 0 \geq 0 \text{ or } 0 \leq 0 \\ \therefore \forall\, x \in \mathbb{R}, x \geq 0 \text{ and } x \leq 0, \end{array}$$

all of the premises are true, but the conclusion is false (since it fails for $x = 1$).

45. Invalid. The form of the argument

$$\begin{array}{l} \forall\, x \in \mathcal{U}, p(x) \to q(x) \\ r \\ q(2) \\ \therefore\;\; p(2) \end{array}$$

is invalid.
Let $\mathcal{U} = \mathbb{R}$, $p(x) =$ "$x > 2$", $q(x) =$ "$x \geq 2$", and $r =$ "$2 \in \mathbb{R}$" to see this.

47. Valid. The form of the argument

$$\begin{array}{l} \forall\, x \in \mathcal{U}, p(x) \to q(x) \\ a \in \mathcal{U} \\ p(a) \vee r(a) \\ \therefore\;\; q(a) \vee r(a) \end{array}$$

is valid. Note that $\mathcal{U} = \mathbb{Z}$, $p(n) =$ "$n < 0$," $q(n) =$ "$-n > 0$," and $r(n) =$ "$n = 0$."

	Statement Form	Justification
1.	$\forall\, x \in \mathcal{U}, p(x) \to q(x)$	Given
2.	$a \in \mathcal{U}$	Given
3.	$p(a) \vee r(a)$	Given
4.	$p(a) \to q(a)$	(1),(2), Principle of Specification
8.	$\therefore\;\; q(a) \vee r(a)$	(3),(4), Exercise 11

49. If $\forall\, x, y \in \mathcal{U}, p(x, y)$ holds and $a, b \in \mathcal{U}$, then $p(a, b)$ holds.

Review

1. It is a true statement.
An if-then statement is true, when its hypothesis is false.

2.

p	q	$\neg q$	$p \to \neg q$
F	F	T	T
F	T	F	T
T	F	T	T
T	T	F	F

3.

p	q	$\neg p$	$\neg p \wedge q$	$p \vee q$	$(\neg p \wedge q) \vee p$
F	F	T	F	F	F
F	T	T	T	T	T
T	F	F	F	T	T
T	T	F	F	T	T

The last two columns are the same.

4.

p	q	$\neg p$	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$	$\neg p \vee q$
F	F	T	T	F	T	T
F	T	T	F	F	T	T
T	F	F	T	T	F	F
T	T	F	F	F	T	T

The last two columns are the same.

5.

p	q	$p \to q$	$q \to p$	$(p \to q) \vee (q \to p)$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	T
T	T	T	T	T

The last column is all T.

6. Yes.

p	q	$\neg p$	$\neg p \to q$	$p \vee q$
F	F	T	F	F
F	T	T	T	T
T	F	F	T	T
T	T	F	T	T

The last two columns are the same.

7. Yes.

p	q	r	$q \vee r$	$p \to q$	$p \to (q \vee r)$	$(p \to q) \vee r$
F	F	F	F	T	T	T
F	F	T	T	T	T	T
F	T	F	T	T	T	T
F	T	T	T	T	T	T
T	F	F	F	F	F	F
T	F	T	T	F	T	T
T	T	F	T	T	T	T
T	T	T	T	T	T	T

The last two columns are the same.

8. (a) $p \vee \neg q \to p$.
(b) $\neg(p \wedge \neg q) \to \neg p \equiv \neg p \wedge q \to \neg p$.
(c) $\neg p \to \neg(p \wedge \neg q) \equiv \neg p \to \neg p \wedge q$.
(d) $p \wedge \neg(p \wedge \neg q) \equiv p \wedge \neg p \wedge q = \underline{f}$.

9. If the program compiles, then the program does not contain a syntax error.

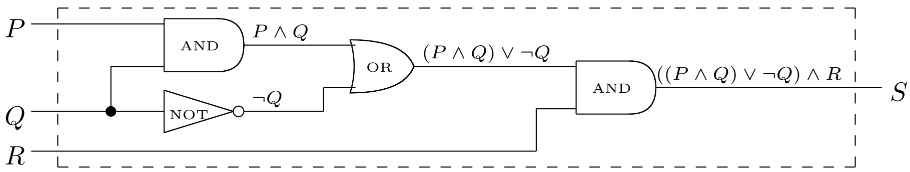
10. $\neg\neg p \wedge \neg(q \wedge \neg r) \equiv p \wedge (\neg q \vee r)$

11. Steve is not doing his homework and Steve is going to the basketball game.

12.
$$\begin{aligned} \neg p \wedge (q \vee \neg r) &\equiv (\neg p \wedge q) \vee (\neg p \wedge \neg r) && \text{Distributivity} \\ &\equiv (\neg p \wedge q) \vee \neg(p \vee r) && \text{De Morgan's Law} \end{aligned}$$

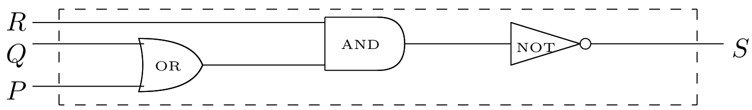
13.
$$\begin{aligned} & (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \\ & \equiv (p \wedge (q \wedge \neg r)) \vee (\neg p \wedge (q \wedge \neg r)) && \text{Associativity} \\ & \equiv (p \vee \neg p) \wedge (q \wedge \neg r) && \text{Distributivity} \\ & \equiv \underline{t} \wedge (q \wedge \neg r) && \text{Theorem 1.2} \\ & \equiv q \wedge \neg r && \text{Theorem 1.2} \end{aligned}$$

14. $((P \wedge Q) \vee \neg Q) \wedge R = S$,
as can be seen by tracing the circuit.

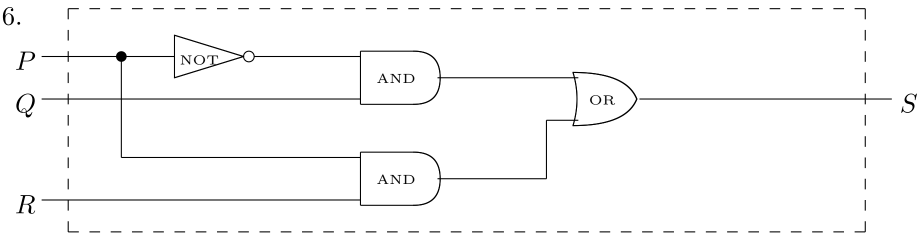


P	Q	R	$P \wedge Q$	$\neg Q$	$(P \wedge Q) \vee \neg Q$	S
0	0	0	0	1	1	0
0	0	1	0	1	1	1
0	1	0	0	0	0	0
0	1	1	0	0	0	0
1	0	0	0	1	1	0
1	0	1	0	1	1	1
1	1	0	1	0	1	0
1	1	1	1	0	1	1

15.



16.



17. True.
Both equal $(-\sqrt{2}, \sqrt{2})$.

18. True.
Order does not matter.

19. False.
$\{1\}$ is a subset.

20. False.
1 is an element.

21. True.
$\{1, 2\}$ is listed on the right-hand side.

22. False.
E.g., it contains $0.2, 0.22, 0.222, \ldots$.

23. True.
2 is listed on the right-hand side.

24. False.
1 is not listed on the right-hand side. Instead, $\{1\}$ is listed.

25. False.
0 is an element.

26. True.
Repetition does not matter, and $=$ is a special case of $\subseteq$.

27. False.
$|\{\emptyset\}| = 1$, since $\emptyset$ is the lone element of $\{\emptyset\}$.

28. $\{4, 6, 8, 10, 12\}$.

29. $\{x : x \in \mathbb{R} \text{ and } x^5 + x^4 + x^3 + x^2 + x + 1 = 0\}$. It happens to be $\{-1\}$.
Note that $x^5 + x^4 + x^3 + x^2 + x + 1 = (x + 1)(x^2 + x + 1)(x^2 - x + 1)$.

30. $\{x : x \in \mathbb{R} \text{ and } -3 < x \leq -1\}$.
Or, $\{x : -3 < x \leq -1\}$ if we understand that we are working in the context of real numbers.

31. 2.
$\frac{-1+\sqrt{5}}{2}$ and $\frac{-1-\sqrt{5}}{2}$ are the elements.

32. Read Example 1.22. $P = \{S \ : \ S \text{ is a set and } S \notin S\}$. If P is a set, then both $P \in P$ and $P \notin P$ lead to contradictions.

33. $\forall\, n \in \mathbb{Z},\ 2^n \in \mathbb{Z}$.

34. $\exists\, n \in \mathbb{Z}$ such that $2^n > 1000$.

35. $\forall\, x, y \in \mathbb{R}$, if $y \neq 0$, then $\frac{x}{y} \in \mathbb{R}$.
Or, $\forall\, x \in \mathbb{R}, \forall\, x \in \mathbb{R} \setminus \{0\}, \frac{x}{y} \in \mathbb{R}$.

36. $\forall\, x \in \mathbb{R}$, if $x \in (1, 4]$, then $\frac{1}{x} \in [\frac{1}{4}, 1)$.
Or, $\forall\, x \in (1, 4], \frac{1}{x} \in [\frac{1}{4}, 1)$.

37. $\forall\, m, n \in \mathbb{Z}, m + n \in \mathbb{Z}$.

38. $\forall\, x \in \mathbb{R}, \exists\, y \in \mathbb{R}$ such that $y^3 = x$.

39. $\exists\, x \in \mathbb{N}$ such that $x^2 \notin \mathbb{N}$ or $\frac{1}{2x} \in \mathbb{N}$.
$\exists\, x \in \mathbb{N}$ such that $\neg[x^2 \in \mathbb{N}$ and $\frac{1}{2x} \notin \mathbb{N}] \equiv \exists\, x \in \mathbb{N}$ such that $x^2 \notin \mathbb{N}$ or $\frac{1}{2x} \in \mathbb{N}$.
De Morgan's Law is used.

40. $\forall\, x \in \mathbb{R}, x^2 - x + 1 \neq 0$.

41. $\exists\, x \in \mathbb{R}$ such that $x^3 < 0$ and $x \geq 0$. Recall that $\neg(p \to q) \equiv p \wedge \neg q$.

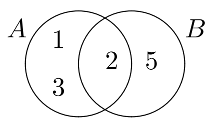
42. $\exists\, x, y \in \mathbb{R}$ such that $(x + y)^2 \neq x^2 + 2xy + y^2$.

43. $\forall\, n \in \mathbb{Z}, n < 0$ and $n^2 - 1 \leq 0$.
Recall that $\neg(p \to q) \equiv p \wedge \neg q$.

44. $\exists\, x \in \mathbb{R}$ such that $\forall\, n \in \mathbb{Z}, x^n \leq 0$.
$\exists\, x \in \mathbb{R}$ such that $\neg[\exists\, n \in \mathbb{Z}$ such that $x^n > 0] \equiv \exists\, x \in \mathbb{R}$ such that $\forall\, n \in \mathbb{Z}, x^n \leq 0$.

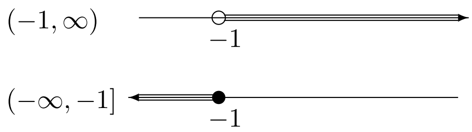
45. Truth is always popular, or it is sometimes wrong.
Note that 'but' is a form of 'and'.

46.

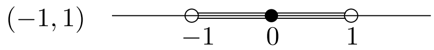


(a) $\{2\}$. (b) $\{1, 2, 3, 5\}$. (c) $\{5\}$. (d) $\{1, 3, 5\}$.
(e) $\{(1, 2), (1, 5), (2, 2), (2, 5), (3, 2), (3, 5)\}$. Note that it has $3 \cdot 2 = 6$ elements.
(f) $\{\emptyset, \{2\}, \{5\}, \{2, 5\}\}$. Note that it has $2^2 = 4$ elements.

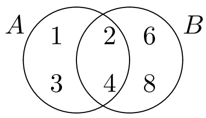
47. $(-\infty, -1]$.



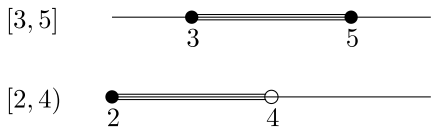
48. $\{0\}$.



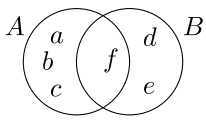
49. $\{1, 2, 3, 4, 6, 8\}$.



50. $[4, 5]$.



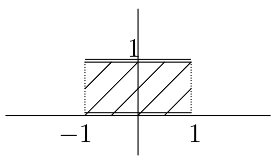
51. $\{a, b, c, d, e\}$.



52. No.
0 is in both.

53. $\{(x, p), (x, q), (y, p), (y, q), (z, p), (z, q)\}$.
Note that it has $3 \cdot 2 = 6$ elements.

54.



55. $\{(1,1)\}$.
Note that it has $1 \cdot 1 = 1$ element.

56. $\{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}\}$.
Note that it has $2^3 = 8$ elements.

57.
$$\begin{array}{rcll} (A^c \cap B^c)^c & = & (A^c)^c \cup (B^c)^c & \text{De Morgan's Law} \\ & = & A \cup B & \text{Double Complement Rule} \end{array}$$

58.
$$\begin{array}{rcll} A^c \cap (B \cup C^c) & = & (A^c \cap B) \cup (A^c \cap C^c) & \text{Distributivity} \\ & = & (A^c \cap B) \cup (A \cup C)^c & \text{De Morgan's Law} \end{array}$$

59.
$$\begin{array}{rcll} (A \cap B^c) \cup (A \cap B) & = & A \cap (B^c \cup B) & \text{Distributivity} \\ & = & A \cap \mathcal{U} & \text{Theorem 1.6} \\ & = & A & \text{Theorem 1.6} \end{array}$$

60.
$$\begin{array}{ll} \quad (A^c \cap B \cap C^c) \cup (A^c \cap (B^c \cup C)) & \\ = (A^c \cap (B \cap C^c)) \cup (A^c \cap (B^c \cup C)) & \text{Associativity} \\ = A^c \cap ((B \cap C^c) \cup (B^c \cup C)) & \text{Distributivity} \\ = A^c \cap ((B \cap C^c) \cup (B \cap C^c)^c) & \text{De Morgan's Law} \\ = A^c \cap \mathcal{U} & \text{Theorem 1.6} \\ = A^c & \text{Theorem 1.6} \end{array}$$

61.

p	q	r	$p \wedge q$	$p \to r$	$q \wedge r$
F	F	F	F	T	
F	F	T	F	T	
F	T	F	F	T	
F	T	T	F	T	
T	F	F	F	F	
T	F	T	F	T	
T	T	F	T	F	
T	T	T	T	T	T

The validity of the argument can be seen in the last row.

62.

p	q	r	$p \to q$	$q \to r$	$r \to p$	$p \leftrightarrow r$
F	F	F	T	T	T	T
F	F	T	T	T	F	
F	T	F	T	F	T	
F	T	T	T	T	F	
T	F	F	F	T	T	
T	F	T	F	T	T	
T	T	F	T	F	T	
T	T	T	T	T	T	T

The validity of the argument can be seen in the first and last rows.

63. Valid.

p	q	r	$p \vee q$	$q \to r$	$p \vee r$
F	F	F	F	T	
F	F	T	F	T	
F	T	F	T	F	
F	T	T	T	T	T
T	F	F	T	T	T
T	F	T	T	T	T
T	T	F	T	F	
T	T	T	T	T	T

The validity of the argument can be seen in rows 4, 5, 6, and 8.

64. Valid.

p	q	r	$p \wedge \neg q$	$q \vee r$	$\neg r \to p$
F	F	F	F	F	
F	F	T	F	T	
F	T	F	F	T	
F	T	T	F	T	
T	F	F	T	F	
T	F	T	T	T	T
T	T	F	F	T	
T	T	T	F	T	

The validity of the argument can be seen in row 6.

65. Invalid. Consider when p is false, q is true, and r is false.
The argument form is invalid as can be seen in this row of the truth table.

p	q	r	$p \to q$	$r \to q$	q	$p \vee r$
F	T	F	T	T	T	F

66. The argument is invalid, since its form

$$\begin{array}{l} p \vee q \\ q \\ \therefore \;\; \neg p \end{array}$$

is invalid,
as can be confirmed with a truth table.

p	q	$p \vee q$	q	$\neg p$
T	T	T	T	F

67.

	Statement Form	Justification
1.	$p \to (q \vee r)$	Given
2.	$\neg q \wedge \neg r$	Given
3.	$\neg(q \vee r)$	(2), De Morgan's Law
4.	$\therefore\ \neg p$	(1), (3), Contrapositive Implication

68.

	Statement Form	Justification
1.	$\neg r$	Given
2.	$p \to q$	Given
3.	$q \to r$	Given
4.	$p \to r$	(2), (3), Transitivity of $\to$
5.	$\therefore\ \neg p$	(1), (4), Contrapositive Implication

69.

	Statement Form	Justification
1.	$\forall\, x \in \mathcal{U}, p(x) \wedge q(x)$	Given
2.	Let $a \in \mathcal{U}$ be arbitrary.	Assumption
3.	$p(a) \wedge q(a)$	(1),(2), Principle of Specification
4.	$p(a)$	(3), In Particular
5.	$\therefore\ \forall\, x \in \mathcal{U}, p(x)$	(2),(4), Principle of Generalization

70.

	Statement Form	Justification
1.	$\forall\, x \in \mathcal{U}, p(x) \vee q(x)$	Given
2.	$a \in \mathcal{U}$	Given
3.	$\neg q(a)$	Given
4.	$p(a) \vee q(a)$	(1), (2), Principle of Specification
5.	$\therefore\ p(a)$	(3),(4), Eliminating a Possibility

71.

	Statement Form	Justification
1.	$\forall\, x \in \mathcal{U}, p(x) \vee \neg q(x)$	Given
2.	$\forall\, x \in \mathcal{U}, q(x)$	Given
3.	Let $a \in \mathcal{U}$ be arbitrary.	Assumption
4.	$p(a) \vee \neg q(a)$	(1),(3), Principle of Specification
5.	$q(a)$	(2),(3), Principle of Specification
6.	$\neg\neg q(a)$	(5), Double Negative
7.	$p(a)$	(4),(6), Eliminating a Possibility
8.	$\therefore\ \forall\, x \in \mathcal{U}, p(x)$	(3),(7), Principle of Generalization

72. It has the form

$$\begin{aligned}
&\forall\, x \in \mathcal{U}, p(x) \vee q(x) \\
&\forall\, x \in \mathcal{U}, q(x) \vee r(x) \\
&\therefore\ \ \forall\, x \in \mathcal{U}, p(x) \vee r(x)
\end{aligned}$$

which can be seen to be invalid when $\mathcal{U} = \mathbb{R}$, $p(x) =$ "$x \leq 0$", $q(x) =$ "$x \geq 0$", and $r(x) =$ "$x \leq 1$". That is, $\forall\, x \in \mathbb{R}, x \leq 0 \vee x \geq 0$ is true,
and $\forall\, x \in \mathbb{R}, x \geq 0 \vee x \leq 1$ is true, but $\forall\, x \in \mathbb{R}, x \leq 0 \vee x \leq 1$ is false.

2.2 Chapter 2

Section 2.1

1. *Proof.* Let L be the line given by the equation $y = 3x - 5$. Observe that

$$\begin{aligned} -8 &= 3(-1) - 5, \\ -2 &= 3(1) - 5, \text{ and} \\ 1 &= 3(1) - 5. \end{aligned}$$

Therefore, each of the points $(-1, -8)$, $(1, -2)$, and $(2, 1)$ lie on the common line L. $\square$

3. *Proof.* Let $A = \{2, 4\}$. Observe that $\{1, 2, 3, 4\} \setminus A = \{1, 2, 3, 4\} \setminus \{2, 4\} = \{1, 3\}$. $\square$

5. *Proof.* Let $A = B = \{1\}$. Observe that $A \cup B = \{1\} \cup \{1\} = \{1\} = A \cap B$. $\square$

7. *Proof.* Let $n = -3$. Observe that $10^n = 10^{-3} = .001$. $\square$

9. *Proof.* Let $m = -3, n = 2$. Observe that $3m + 5n = 3(-3) + 5(2) = 1$. $\square$

11. *Proof.* Let $A = B = \mathbb{Z}$. Observe that $A \setminus B = \mathbb{Z} \setminus \mathbb{Z} = B \setminus A$. $\square$

13. *Proof.* Let $x = -5$. Observe that $x \in \mathbb{Z}$ and $3(-5)^2 + 8(-5) = 35$. $\square$

15. *Proof.* The polynomial $x^2 - 1$ factors as $(x + 1)(x - 1)$. From the zero multiplication property, the solutions to the equation $(x + 1)(x - 1) = 0$ occur when $x + 1 = 0$ or $x - 1 = 0$. That is, $x = -1$ and $x = 1$ are the two distinct real roots of $x^2 - 1$. $\square$

17. *Proof.* Observe that $x^2 - 2x + 1 = (x - 1)^2 \geq 0$.
So $x^2 - 2x + 5 = x^2 - 2x + 1 + 4 \geq 0 + 4 = 4 > 0$.
Hence, the equation $x^2 - 2x + 5 = 0$ has no solution. $\square$

19. (a) $6000(1.075)^{10} = \$12{,}366.19$.
(b) Let $P = 4900$, and note that $4900(1.075)^{10} = 10099.05 > 10000$.
We used trial and error to find P.
Note that $P = 4800$ gives $A = 9892.95 < 10000$.

21. *Proof.* Let $A = \emptyset$. Observe that $A^2 = \emptyset^2 = \emptyset = A$. $\square$

23. $(-2, -1) \cup (1, 2)$ is not an interval.
Note that each type of interval in Definition 1.9 is a set I with the property that, if $x, y \in I$ and $x \leq z \leq y$, then $z \in I$. Here, for $I = (-2, -1) \cup (1, 2)$, we have $-1.5, 1.5 \in I$ and $-1.5 \leq 0 \leq 1.5$, but $0 \notin I$.

25. *Counterexample.* Let $x = \frac{1}{2}$. Observe that $x^2 = (\frac{1}{2})^2 = \frac{1}{4} \not> \frac{1}{2} = x$. $\Box$

27. *Counterexample.* Let $n = 3$. Observe that $n^2 = 9 \not\leq 8 = 2^n$. $\Box$

29. *Proof.* Let $x = -11$. Observe that $x = -11 < 10$ and $x^2 = 121 > 100$. $\Box$

31. *Counterexample.* Let $x = 0$ and $y = 2\pi$.
Observe that $x < y$, but $\sin(x) \not< \sin(y)$.
So the sine function is not increasing. (See Definition 1.15.) $\Box$

33. False, since 0 is nonnegative but not positive.
Observe that $0 \geq 0$ and $0 \not> 0$.

35. False.
Counterexample. Let $A = \{1\}$ and $B = \{2, 3\}$.
Observe that $|A| = 1 \leq 2 = |B|$, but $A \not\subseteq B$. $\Box$
Recall that $\neg[p \to q] \equiv p \wedge \neg q$.

37. *Counterexample.* Let $A = \{1, 2\}, B = \{1, 3\}$, and $C = \{1\}$.
Observe that $A \neq B$, and $A \cap C = \{1\} = B \cap C$. $\Box$
Recall that $\neg[p \to q] \equiv p \wedge \neg q$.

39. *Proof.* Observe that $\emptyset \cup \{3\} = \{3\} = \emptyset \bigtriangleup \{3\}$, $\{1\} \cup \{3\} = \{1, 3\} = \{1\} \bigtriangleup \{3\}$, and $\{1, 2\} \cup \{3\} = \{1, 2, 3\} = \{1, 2\} \bigtriangleup \{3\}$. $\Box$
That is, we checked $A \cup \{3\} = A \bigtriangleup \{3\}$, for each of $A = \emptyset, \{1\}$, and $\{1, 2\}$.

41. Observe that

$$\begin{array}{lllll}
9(2) = 18 & \text{and} & 1+8 & = 9, & \\
9(3) = 27 & \text{and} & 2+7 & = 9, & \\
9(4) = 36 & \text{and} & 3+6 & = 9, & \\
9(5) = 45 & \text{and} & 4+5 & = 9, & \\
9(6) = 54 & \text{and} & 5+4 & = 9, & \\
9(7) = 63 & \text{and} & 6+3 & = 9, & \\
9(8) = 72 & \text{and} & 7+2 & = 9, & \text{and} \\
9(9) = 81 & \text{and} & 8+1 & = 9. &
\end{array}$$

43.

n	$11n$	hundreds - tens + ones
19	209	$2-0+9=11=11(1)$
20	220	$2-2+0=0=11(0)$
21	231	$2-3+1=0=11(0)$
22	242	$2-4+2=0=11(0)$
23	253	$2-5+3=0=11(0)$
24	264	$2-6+4=0=11(0)$
25	275	$2-7+5=0=11(0)$
26	286	$2-8+6=0=11(0)$
27	297	$2-9+7=0=11(0)$
28	308	$3-0+8=11=11(1)$
29	319	$3-1+9=11=11(1)$

In each case, the relevant alternating sum is seen to be a multiple of 11.

45. Observe that

$$\begin{aligned} 2^2-1 &= 3, \\ 2^3-1 &= 7, \\ 2^5-1 &= 31, \text{ and} \\ 2^7-1 &= 127 \end{aligned}$$

are all prime.

47. Here are the sequences, each ending in 1.
1,
$2 \mapsto 1$,
$3 \mapsto 10 \mapsto 5 \mapsto 16 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1$,
$4 \mapsto 2 \mapsto 1$,
$5 \mapsto 16 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1$, and
$6 \mapsto 3 \mapsto 10 \mapsto 5 \mapsto 16 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1$.

Section 2.2

1. *Proof.* Let $x \in \mathbb{R}^+$. So $x > 0$. Multiplying by -1 gives $-x < 0$. So $-x \in \mathbb{R}^-$. $\square$

3. *Proof.* Suppose $x \in \mathbb{R}$ and $x \in (2,4)$. That is, $2 < x < 4$. Multiplication by 2 gives $4 < 2x < 8$. Thus, $2x \in (4,8)$. $\square$

5. Counterexample: When $x = \frac{1}{4}$, we have $\sqrt{\frac{1}{4}} \not< \frac{1}{4}$.
That is, $\sqrt{\frac{1}{4}} = \frac{1}{2} \geq \frac{1}{4}$.

7. Counterexample: When $x = -3$, we have $x < 2$ and $x^2 \geq 4$.
Recall that $\neg[p \to q] \equiv p \wedge \neg q$.

9. *Proof.* Suppose $x \in \mathbb{R}$ and $x < -2$. Since $x < 0$, multiplication by x gives $x^2 > -2x$. Since $-2 < 0$, multiplying $x < -2$ by -2 gives $-2x > (-2)^2$. Transitivity of $>$ gives $x^2 > (-2)^2$. That is, $x^2 > 4$. $\square$

11. *Proof.* Suppose $0 < x < y$. Since $x > 0$, we have $x^2 = x \cdot x < x \cdot y$. Since $y > 0$, we have $x \cdot y < y \cdot y = y^2$. Hence, $x^2 < x \cdot y < y^2$. $\square$

13. *Proof.* Suppose $R > 2$. So $2I < RI = 10$. Division by 2 gives $I < 5$. $\square$

15. *Proof.* Suppose f is a constant real function. So we have $c \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R},\ f(x) = c$. Observe that $\forall\, x \in \mathbb{R},\ (2f)(x) = 2f(x) = 2c$. So $2f$ is constant. $\square$
See the definition of *constant* in Definition 1.15. Also, note that $2f$ is to be interpreted as an example of a *constant multiple* as defined in Definition 1.16.

17. *Proof.* Suppose f is a periodic real function. So we have $p \in \mathbb{R}^+$ such that $\forall\, x \in \mathbb{R},\ f(x+p) = f(x)$. Observe that $\forall\, x \in \mathbb{R},\ f^2(x+p) = [f(x+p)]^2 = [f(x)]^2 = f^2(x)$. So f^2 is periodic. $\square$
See the definition of *periodic* in Definition 1.15. Also, the proper way to interpret f^2 is explained in the paragraph after Definition 1.16.

19. *Proof.* Suppose f and g are nondecreasing real functions. Suppose $x \leq y$ are real numbers. Observe that $(f+g)(x) = f(x) + g(x) \leq f(y) + g(y) = (f+g)(y)$. So $f+g$ is nondecreasing. $\square$
See the definition of *nondecreasing* in Definition 1.15. Also, note that $f+g$ is to be interpreted as a *sum* as defined in Definition 1.16.

21. *Proof.* Suppose f and g are constant. So we have $c, d \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R},\ f(x) = c$ and $g(x) = d$. Observe that
$\forall\, x \in \mathbb{R},\ (fg)(x) = f(x)g(x) = cd$. So fg is constant. $\square$
Note that fg is to be interpreted as a *product* as defined in Definition 1.16.

23. *Proof.* Suppose f is periodic. So we have $p \in \mathbb{R}^+$ such that
$\forall\, x \in \mathbb{R},\ f(x+p) = f(x)$. Observe that $\forall\, x \in \mathbb{R},\ (f+c)(x+p) = f(x+p)+c = f(x) + c = (f+c)(x)$. So $f+c$ is periodic. $\square$
The proper way to interpret $f+c$ is explained in the paragraph after Definition 1.16. In the string of equalities above, the first results from the definition of $f+c$, the second from the fact that f is periodic, and the third from the definition of $f+c$ (again).

25. *Proof.* Suppose f is increasing. Suppose $x < y$ are real numbers. So, $f(x) < f(y)$. Since $c > 0$, multiplication by c gives $cf(x) < cf(y)$. That is, $(cf)(x) < (cf)(y)$. So cf is increasing. $\square$
To show that cf is increasing, we must show that, if $x < y$ are real numbers, then $(cf)(x) < (cf)(y)$. This is accomplished in sentences two through five of

the proof.

27. *Proof.* Let A be a square. So A is a rectangle. Hence, A is a parallelogram. $\Box$
By definition, a square is a (special) rectangle. Also, a rectangle is a (special) parallelogram.

29. *Proof.* Let A be a rectangle and a rhombus. Hence, A is a rectangle with all sides congruent. That is, A is a square. $\Box$

31. *Proof.* Suppose $a \in \mathbb{Z} \cap \mathbb{R}^+$. So $a \in \mathbb{Z}$ and $a > 0$. Since $a \geq 0$, we have $a \in \mathbb{N}$. $\Box$

33. *Proof.* Suppose $x \in \mathbb{R}^+$. So $x > 0$. That is, $x \not\leq 0$. In particular, $x \not< 0$. Hence, $x \notin \mathbb{R}^-$. That is, $x \in (\mathbb{R}^-)^c$. $\Box$
This is a subset proof. Recall that A^c denotes $\{x : x \in \mathcal{U} \text{ and } x \notin A\}$. In this case, the understood universe is $\mathcal{U} = \mathbb{R}$.

35. *Proof.* Suppose $x \in A$. Hence, $x \in A$ or $x \in B$. So $x \in A \cup B$. $\Box$
The *Obtaining Or* argument form from Theorem 1.7 is invoked here with $p =$ "$x \in A$" and $q =$ "$x \in B$".

37. *Proof.* Suppose $A \subseteq A \cap B$ and suppose $x \in A$. It follows that $x \in A \cap B$. That is, $x \in A$ and $x \in B$. In particular, $x \in B$. Therefore $A \subseteq B$. $\Box$
The primary structure of the statement is that of an if-then statement. Hence, our proof starts by supposing its hypothesis. However, since its conclusion is the subset fact $A \subseteq B$, we immediately initiate a proof of that by further supposing $x \in A$. Note that the *In Particular* argument form from Theorem 1.7 is invoked here with $q =$ "$x \in A$" and $p =$ "$x \in B$".

39. *Proof.* Suppose $A \subseteq B$. Suppose $x \in A \cap C$. So, $x \in A$ and $x \in C$. Since $x \in A$ and $A \subseteq B$, we get $x \in B$. So, $x \in B$ and $x \in C$. Thus, $x \in B \cap C$. $\Box$

41. *Proof.* Let $x \in \mathcal{U}$. From the string of logical equivalences

$$\begin{aligned}
x \in (A \cap B) \cap C &\leftrightarrow (x \in A \cap B) \wedge x \in C \\
&\leftrightarrow (x \in A \wedge x \in B) \wedge x \in C \\
&\leftrightarrow x \in A \wedge (x \in B \wedge x \in C) \\
&\leftrightarrow x \in A \wedge x \in B \cap C \\
&\leftrightarrow x \in A \cap (B \cap C).
\end{aligned}$$

it follows that $x \in (A \cap B) \cap C \leftrightarrow x \in A \cap (B \cap C)$. $\Box$.

43. *Proof.* Let $x \in \mathcal{U}$. From the string of logical equivalences

$$\begin{aligned} x \in A \cap B &\leftrightarrow x \in A \wedge x \in B \\ &\leftrightarrow x \in B \wedge x \in A \\ &\leftrightarrow x \in B \cap A. \end{aligned}$$

it follows that $x \in A \cap B \leftrightarrow x \in B \cap A$. $\square$.

45. Counterexample: Let $A = \{1\}$ and $B = \{2\}$. Since $A \times B = \{(1,2)\}$ and $B \times A = \{(2,1)\}$, we see that $A \times B \neq B \times A$.
Recall that the elements of a product of two sets are *ordered pairs*. At the heart of this proof is the fact that *order* is indeed important. That is $(1,2) \neq (2,1)$, as ordered pairs.

47. *Proof.* Let $x \in \mathcal{U}$. From the string of logical equivalences

$$\begin{aligned} x \in A \cup (B \cap C) &\leftrightarrow x \in A \vee x \in B \cap C \\ &\leftrightarrow x \in A \vee (x \in B \wedge x \in C) \\ &\leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \\ &\leftrightarrow (x \in A \cup B) \wedge (x \in A \cup C) \\ &\leftrightarrow x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

it follows that $x \in A \cup (B \cap C) \leftrightarrow x \in (A \cup B) \cap (A \cup C)$. $\square$.

49. *Proof.* Let $x \in \mathcal{U}$. From the string of logical equivalences

$$\begin{aligned} x \in (A \cap B)^c &\leftrightarrow x \notin A \cap B \\ &\leftrightarrow \neg(x \in A \cap B) \\ &\leftrightarrow \neg(x \in A \wedge x \in B) \\ &\leftrightarrow \neg(x \in A) \vee \neg(x \in B) \\ &\leftrightarrow x \notin A \vee x \notin B \\ &\leftrightarrow x \in A^c \vee x \in B^c \\ &\leftrightarrow x \in A^c \cup B^c. \end{aligned}$$

it follows that $x \in (A \cap B)^c \leftrightarrow x \in A^c \cup B^c$. $\square$.

Section 2.3

1. *Proof.* $(\rightarrow)$ Suppose $x \in \mathbb{R}^-$. So $x < 0$. Multiplication by -1 gives $-x > 0$. That is, $-x \in \mathbb{R}^+$. $(\leftarrow)$ Suppose $-x \in \mathbb{R}^+$. So $-x > 0$. Multiplication by -1 gives $x = (-1)(-x) < 0$. That is, $x \in \mathbb{R}^-$. $\square$

3. *Proof.* Let $x \in \mathbb{R}$. $(\rightarrow)$ Suppose $x = 2x$. So $0 = 2x - x$. That is, $x = 0$. $(\leftarrow)$ Suppose $x = 0$. Observe that $0 = 2(0)$. $\square$

5. *Proof.* ($\rightarrow$) Suppose $x^3 > 0$. Note $x \neq 0$ (since $0^3 = 0$). So $x^2 > 0$ and $\frac{1}{x^2} > 0$. Multiplying both sides of $x^3 > 0$ by $\frac{1}{x^2}$ gives $x > 0$. ($\leftarrow$) Suppose $x > 0$. Since $x^2 > 0$, multiplication by x^2 gives $x^3 > 0$. □

7. *Proof.* Let $x \in \mathbb{R}$. ($\rightarrow$) Suppose $4 - x < 2$. So $4 < 2 + x$. Hence, $2 < x$. ($\leftarrow$) Suppose $x > 2$. So $x + 2 > 4$. Thus, $2 > 4 - x$. □

9. *Proof.* Let $x \in \mathbb{R}$. ($\rightarrow$) Suppose $x^4 - 16 = 0$. So $(x^2 + 4)(x^2 - 4) = 0$. Since $x^2 + 4 > 0$, it must be that $x^2 - 4 = 0$. ($\leftarrow$) Suppose $x^2 - 4 = 0$. So $(x^2 + 4)(x^2 - 4) = 0$. That is, $x^4 - 16 = 0$. □

11. *Proof.* ($\rightarrow$) Done in Exercise 15 from Section 2.2. ($\leftarrow$) Suppose $2f$ is constant. So we have $c \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R},\ 2f(x) = c$. Therfore, $\forall\, x \in \mathbb{R},\ f(x) = \frac{c}{2}$. So, f is constant. □

13. *Proof.* ($\rightarrow$) Suppose f is bounded above. So we have $M \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R},\ f(x) \leq M$. Observe that $\forall\, x \in \mathbb{R},\ (f + 1)(x) = f(x) + 1 \leq M + 1$. So $f + 1$ is bounded above. ($\leftarrow$) Suppose $f + 1$ is bounded above. So we have $M \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R},\ (f + 1)(x) \leq M$. That is, $\forall\, x \in \mathbb{R},\ f(x) + 1 \leq M$. So $\forall\, x \in \mathbb{R},\ f(x) \leq M - 1$. Thus, f is bounded above. □

15. *Proof.* ($\rightarrow$) Suppose f is bounded above and below. So we have $M, L \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R},\ L \leq f(x) \leq M$. Let $U = \max\{L^2, M^2\}$. It can be shown that $\forall\, x \in \mathbb{R},\ f^2(x) \leq U$. (Hint: First argue that $\forall\, x \in \mathbb{R},\ |f(x)| \leq \max\{|L|, |M|\}$.) So f^2 is bounded above. ($\leftarrow$) Suppose f^2 is bounded above. So we have $M \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R},\ f^2(x) \leq M$. It follows that $\forall\, x \in \mathbb{R},\ -\sqrt{M} \leq f(x) \leq \sqrt{M}$. So f is bounded above and below. □

17. *Proof.* ($\rightarrow$) Suppose f is periodic. So we have $p \in \mathbb{R}^+$ such that, $\forall\, x \in \mathbb{R}$, $f(x+p) = f(x)$. Suppose $x \in \mathbb{R}$. So $(2f)(x+p) = 2 \cdot f(x+p) = 2 \cdot f(x) = (2f)(x)$. Hence, $2f$ is periodic. ($\leftarrow$) Suppose $2f$ is periodic. So we have $p \in \mathbb{R}^+$ such that, $\forall\, x \in \mathbb{R}, f(x + p) = f(x)$. Suppose $x \in \mathbb{R}$. So $f(x + p) = \frac{1}{2} \cdot (2f)(x + p) = \frac{1}{2} \cdot (2f)(x) = f(x)$. Hence, f is periodic. □

19. *Proof.* ($\rightarrow$) Suppose 20 points are scored. Let t be the number of touchdowns and f be the number of field goals scored. If $t \geq 3$, then $7t \geq 21$ points are scored. So $t \leq 2$. If $t = 0$, then $3f = 20$ is impossible. If $t = 1$, then $7 + 3f = 20$ is impossible, since $3f = 13$ is impossible. So $t = 2$, and it must be that $f = 2$ to give $7t + 3f = 7(2) + 3(2) = 20$. ($\leftarrow$) Suppose $t = 2$ touchdowns and $f = 2$ field goals are scored. Hence, $7t + 3f = 7(2) + 3(2) = 20$ points are scored. □

21. *Proof.* ($\subseteq$) Suppose $x \in \mathbb{R}^+ \cap [-2, 2]$, So $x > 0$ and $-2 \leq x \leq 2$. Hence, $0 < x \leq 2$. Thus, $x \in (0, 2]$. ($\supseteq$) Suppose $x \in (0, 2]$. So $-2 < 0 < x \leq 2$. Hence, $x > 0$ and $-2 \leq x \leq 2$. Thus, $x \in \mathbb{R}^+ \cap [-2, 2]$. □

23. *Proof.* ($\subseteq$) Suppose $x \in \mathbb{N} \setminus (-1,1)$. So $x \in \mathbb{Z}$, $x \geq 0$, and $x \notin (-1,1)$. Since $x \not\leq -1$, it must be that $x \geq 1$. Hence, $x \in \mathbb{Z}^+$. ($\supseteq$) Suppose $x \in \mathbb{Z}^+$. So $x \in \mathbb{Z}$ and $x > 0$. In particular, $x \geq 1$. Thus, $x \in \mathbb{N}$ and $x \notin (-1,1)$. Hence, $x \in \mathbb{N} \setminus (-1,1)$. $\square$

25. *Proof.* ($\subseteq$) Suppose $x \in [1,3) \cap [2,4)$. So $1 \leq x < 3$ and $2 \leq x < 4$. Hence, $2 \leq x < 3$. That is, $x \in [2,3)$. ($\supseteq$) Suppose $x \in [2,3)$. So $2 \leq x < 3$. Hence, $1 \leq x < 3$ and $2 \leq x < 4$. Thus, $x \in [1,3) \cap [2,4)$. $\square$

27. *Proof.* Suppose $A \subseteq B$. ($\subseteq$) Suppose $x \in A \cap B$. So $x \in A$ and $x \in B$. In particular, $x \in A$. Hence $A \cap B \subseteq A$. ($\supseteq$) Suppose $x \in A$. Since $A \subseteq B$, we get $x \in B$. Thus $x \in A$ and $x \in B$. So $x \in A \cap B$. Hence $A \subseteq A \cap B$. It follows that $A \cap B = A$. $\square$

29. *Proof.* ($\subseteq$) Suppose $x \in (A \setminus C) \cap B$. So $x \in A \setminus C$ and $x \in B$. Hence $x \in A$, $x \notin C$, and $x \in B$. Since $x \in B$ and $x \notin C$, we have $x \in B \setminus C$. Thus $x \in A$ and $x \in B \setminus C$. That is $x \in A \cap B \setminus C$. ($\supseteq$) Essentially, reverse the previous argument. $\square$

31. *Proof.* Suppose $A \cap B = A \cap C$. ($\subseteq$) Suppose $x \in A \cap B \cap C$. So $x \in A$, $x \in B$, and $x \in C$. Since $x \in A$ and $x \in B$, we have $x \in A \cap B$. ($\supseteq$) Suppose $x \in A \cap B$. Since $A \cap B = A \cap C$, we have $x \in A \cap C$. So $x \in A$ and $x \in B$. Also, $x \in A$ and $x \in C$. Thus $x \in A$, $x \in B$, and $x \in C$. So $x \in A \cap B \cap C$. $\square$

33. *Proof.* ($\subseteq$) Suppose $x \in (A \setminus B) \setminus C$. So $x \in A \setminus B$ and $x \notin C$. So $x \in A$ and $x \in B^c \cap C^c$. Since $x \notin (B^c \cap C^c)^c$, De Morgan's Law tells us that $x \notin B \cup C$. Since $x \in A$ and $x \notin B \cup C$, we have $x \in A \setminus (B \cup C)$. ($\supseteq$) Essentially, reverse the previous argument. $\square$

35. *Proof.* Suppose $A \subseteq B$ and $(x,y) \in A^2$. Since $x, y \in A$ and $A \subseteq B$, it follows that $x, y \in B$. Hence, $(x,y) \in B^2$. $\square$
Recall that $A^2 = A \times A$. So elements of A^2 are ordered pairs.

37. *Proof.* ($\subseteq$) Suppose $(x,y) \in (\mathcal{U} \times B) \setminus (A \times B)$. So $(x,y) \in \mathcal{U} \times B$ and $(x,y) \notin A \times B$. We have ($x \in \mathcal{U}$ and) $y \in B$. Since $(x,y) \notin A \times B$, it must be that $x \notin A$. That is $x \in A^c$. Hence $(x,y) \in A^c \times B$. ($\supseteq$) Suppose $(x,y) \in A^c \times B$. So $x \in A^c$ and $y \in B$. Since $x \notin A$, we get $(x,y) \notin A \times B$. Since $(x,y) \in \mathcal{U} \times B$, we have $(x,y) \in (\mathcal{U} \times B) \setminus (A \times B)$. Thus, $(\mathcal{U} \times B) \setminus (A \times B) = A^c \times B$. $\square$

39. *Proof.* Let $x, y \in \mathcal{U}$. From the string of equivalences

$$\begin{aligned}
(x,y) \in A \times (B \cap C) &\leftrightarrow x \in A \;\wedge\; y \in B \cap C \\
&\leftrightarrow x \in A \;\wedge\; y \in B \;\wedge\; y \in C \\
&\leftrightarrow x \in A, y \in B \;\wedge\; x \in A, y \in C \\
&\leftrightarrow (x,y) \in A \times B \;\wedge\; (x,y) \in A \times C \\
&\leftrightarrow (x,y) \in (A \times B) \cap (A \times C),
\end{aligned}$$

it follows that

$$(x,y) \in A \times (B \cap C) \leftrightarrow (x,y) \in (A \times B) \cap (A \times C).$$

Hence, $A \times (B \cap C) = (A \times B) \cap (A \times C)$. □

41. *Proof.* ($\rightarrow$) Suppose $A \times C = B \times C$. ($\subseteq$) Suppose $x \in A$. Since $(x, c) \in A \times C$, it follows that $(x, c) \in B \times C$. So $x \in B$. ($\supseteq$) Similar. ($\leftarrow$) Suppose $A = B$. ($\subseteq$) Suppose $(x, y) \in A \times C$. So $x \in A$ and $y \in C$. Since $A = B$, we have $x \in B$. So $(x, y) \in B \times C$. ($\supseteq$) Similar. □

43. *Proof.* Suppose $S \in \mathcal{P}(A^c) \setminus \{\emptyset\}$. So $S \subseteq A^c$ and $S \neq \emptyset$. Since $S \neq \emptyset$, we have some $x \in S$. Since $S \subseteq A^c$, we have $x \in A^c$. That is, $x \in S$ and $x \notin A$. So $S \nsubseteq A$. Therefore $S \notin \mathcal{P}(A)$. That is, $S \in \mathcal{P}(A)^c$. □

45. *Proof.* Suppose $S \in \mathcal{P}(A \cap B)$. That is, $S \subseteq A \cap B$. Since $A \cap B \subseteq A$, we get $S \subseteq A$. Since $A \cap B \subseteq B$, we get $S \subseteq B$. So $S \in \mathcal{P}(A)$ and $S \in \mathcal{P}(B)$. Thus $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Hence $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$. □

47. Let r be the average speed over the entire trip, let r_1 be the average speed over the first lap, and let r_2 be the average speed over the second lap. Observe that $\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. (a) $r = 48$ mph. (b) $\frac{2r_1r_2}{r_1+r_2} = r = 60$ if and only if $r_1 = \frac{30r_2}{r_2-30} > 30$.
That is, let t be time over the entire trip, let t_1 be the time over the first lap, and let t_2 be the time over the second lap. Note that 2 is the distance over the entire trip, and 1 is the distance over each lap. Now use the fact that rate times time equals distance to substitute for time in the equation $t = t_1 + t_2$. The rest is algebra.

Section 2.4

1. *Proof.* Suppose not. Let s be the smallest element of $(1, 2)$. Observe that $\frac{s+1}{2}$ is a smaller element of $(1, 2)$. (Think about it.) This is a contradiction. □
That is, since $1 < s$, it follows that $2 = 1 + 1 < s + 1 < s + s = 2s$. So $1 = \frac{2}{2} < \frac{s+1}{2} < \frac{2s}{2} = s$.

3. *Proof.* Suppose not. Let L be the largest element of $\mathbb{N}$. However, $L + 1$ is a larger element of $\mathbb{N}$. This is a contradiction. □

5. *Proof.* Let A be a set, and suppose $A \cap \emptyset \neq \emptyset$. So we have an element $x \in A \cap \emptyset$. Hence, $x \in A$ and $x \in \emptyset$. However, $x \in \emptyset$ is impossible. This is a contradiction. So it must be that $A \cap \emptyset = \emptyset$. □

7. *Sketch.* Suppose $(0, 1]$ has finite cardinality n. The list $\frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^{n+1}}$ of numbers in $(0, 1]$ is then too long. □
There cannot be $n + 1$ elements in a set of cardinality n.

9. *Sketch.* Suppose $\{(x, y) \ : \ x, y \in \mathbb{R} \text{ and } y = \sqrt{x}\}$ has finite cardinality n. The list of elements $(1, 1), (4, 2), \dots, ((n + 1)^2, n + 1)$ is then too long. □
There cannot be $n + 1$ elements in a set of cardinality n.

11. *Proof.* Suppose $(1, 0) \neq \emptyset$. So there is a real number x such that $1 < x < 0$. In particular, $1 < 0$. This is a contradiction. □

13. *Proof.* Suppose $\mathbb{R}^+ \cap \mathbb{R}^- \neq \emptyset$. So there is a real number x such that $x \in \mathbb{R}^+$ and $x \in \mathbb{R}^-$. However, it is impossible to have both $x > 0$ and $x < 0$. □

15. (a) *Sketch.* Suppose to the contrary that Tracy wins the election. So every other candidate must have also received fewer than $\frac{1}{n}$ of the votes. However, the total of the fractions of the votes for the n candidates would then be less than $n \cdot \frac{1}{n} = 1$, which is impossible. □
(b) *Sketch.* Suppose to the contrary that Tracy comes in last. So every other candidate must have also received more than $\frac{1}{n}$ of the votes. However, the total of the fractions of the votes for the n candidates would then be more than $n \cdot \frac{1}{n} = 1$, which is impossible. □
That is, say there are m candidates, and for each $1 \leq i \leq m$, the fraction of the votes received by candidate i is f_i. Hence, $1 = f_1 + f_2 + \cdots + f_m$.

17. *Proof.* Suppose $b > a$ are real numbers. Since $\frac{a+b}{2} \in (a, b)$, we see that $(a, b) \neq \emptyset$. □

19. *Proof.* Suppose $A \neq \emptyset$. So we have an element $x \in A$. Thus $(x, x) \in A^2$. Hence $A^2 \neq \emptyset$. □
Recall that elements of A^2 have the form (x, y). Here, $x = y$ is chosen.

21. *Proof.* Suppose $A \subseteq B$. Suppose $(x, y) \in A^2$. Since $x \in A$ and $y \in A$ and $A \subseteq B$, we get $x \in B$ and $y \in B$. That is, $(x, y) \in B^2$. □

23. *Proof.* Suppose $A \times B \neq \emptyset$. So we have some element $(x, y) \in A \times B$. In particular, $x \in A$. So $A \neq \emptyset$. □

25. *Proof.* Suppose $A = B$. So $A \subseteq B$ and $B \subseteq A$. In particular $A \subseteq B$. □

27. The contrapositive "if $A = B$ then $\mathcal{P}(A) = \mathcal{P}(B)$" is easy to see.

29. *Proof.* Suppose that A is finite and B is finite. Say A has m elements $A = \{a_1, a_2, \ldots, a_m\}$, and B has n elements $B = \{b_1, b_2, \ldots, b_n\}$. Observe that $A \times B =$

$$\begin{array}{cccc}
\{(a_1, b_1), & (a_1, b_2), & \cdots, & (a_1, b_n), \\
(a_2, b_1), & (a_2, b_2), & \cdots, & (a_2, b_n), \\
\vdots & & & \vdots \\
(a_m, b_1), & (a_m, b_2), & \cdots, & (a_m, b_n)\}
\end{array}$$

has mn elements. Thus, $A \times B$ is finite. □

31. *Proof.* Suppose to the contrary that $x \in A$, $x \notin A \cap B$, and $x \in B$. Thus, $x \in A \cap B$ and $x \notin A \cap B$, a contradiction. □

33. *Proof.* Suppose to the contrary that there are two distinct lines l and m that intersect in two or more points. That is, we have distinct points P and Q in their intersection. Since both l and m contain P and Q, we must have $l = m$, by the uniqueness assertion in Euclid's First Postulate. □

35. (a) *Proof.* Suppose f is not decreasing. So we have $x, y \in \mathbb{R}$ with $x < y$ and $0 < f(x) \leq f(y)$. Hence, $f^2(x) \leq f^2(y)$. So f^2 is not decreasing. □
(b) Yes, by Exercise 18.

37. *Proof.* Suppose f is bounded above. So we have $M \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R},\ f(x) \leq M$. Observe that $\forall\, x \in \mathbb{R},\ (f + 100)(x) = f(x) + 100 \leq M + 100$. So $f + 100$ is bounded above. □

39. *Proof.* Suppose f is increasing. (Goal: f is not periodic.) Suppose $p \in \mathbb{R}^+$. Since $f(0 + p) > f(0)$, it cannot be that f is periodic. □
Note that a direct proof is also straightforward.

41. The converse "if f is constant, then f^2 is constant" is easy to prove.
Proof. Suppose f is constant. So we have some $c \in \mathbb{R}$ such that, $\forall\, x \in \mathbb{R}, f(x) = c$. Suppose $x \in \mathbb{R}$. So $f^2(x) = f(x)f(x) = c^2$. Since $c^2 \in \mathbb{R}$, we see that f^2 is constant. □

43. We prove the contrapositive. *Proof.* Let $x \in \mathbb{R}$. Suppose $x \neq 0$. So $x^2 > 0$. In particular, $x^2 \neq 0$. □

45. *Proof.* Suppose not. So, we have some $x > 0$ with $\frac{1}{x} < 0$. Hence $1 = x \cdot \frac{1}{x} < 0$. This is a contradiction. □

47. *Proof.* Suppose not. So we have some $0 < x < y$ with $0 < \frac{1}{x} \leq \frac{1}{y}$. Hence, $1 = \frac{1}{x} \cdot x < \frac{1}{x} \cdot y$ and $\frac{1}{x} \cdot y \leq \frac{1}{y} \cdot y = 1$. Thus, $1 < \frac{1}{x} \cdot y \leq 1$, a contradiction. □

49. *Proof.* Suppose not. So, we have some $x \in (-\infty, -1) \cap (1, \infty)$. That is,

$x < -1$ and $x > 1$. Hence, $1 < x < -1$, which is a contradiction. □

Section 2.5

1. *Proof.* Suppose $A \subseteq B$ and $C \subseteq D$. (Goal: $A \cup C \subseteq B \cup D$.) Suppose $x \in A \cup C$. That is, $x \in A$ or $x \in C$. *Case 1*: $x \in A$. Since $A \subseteq B$, we get $x \in B$. So $x \in B$ or $x \in D$. Hence, $x \in B \cup D$. *Case 2*: $x \in C$. Since $C \subseteq D$, we get $x \in D$. So $x \in B$ or $x \in D$. Hence, $x \in B \cup D$. In either case, $x \in B \cup D$. Thus, $A \cup C \subseteq B \cup D$. □

3. *Proof.* Suppose $A \subseteq B$. ($\subseteq$) Suppose $x \in A \cup B$. So $x \in A$ or $x \in B$. *Case 1*: $x \in A$. Since $A \subseteq B$, we get $x \in B$. *Case 2*: $x \in B$. We have $x \in B$. In either case, $x \in B$. Thus, $A \cup B \subseteq B$. ($\supseteq$) Suppose $x \in B$. So $x \in A$ or $x \in B$. Hence, $x \in A \cup B$. □

5. *Proof.* ($\subseteq$) $A \cup A^c \subseteq \mathcal{U}$ since everything is in $\mathcal{U}$. ($\supseteq$) Suppose $x \in \mathcal{U}$. *Case 1*: $x \in A$. We have $x \in A$ or $x \in A^c$. So $x \in A \cup A^c$. *Case 2*: $x \notin A$. So $x \in A^c$. We have $x \in A$ or $x \in A^c$. So $x \in A \cup A^c$. In either case, $x \in A \cup A^c$. Hence, $\mathcal{U} \subseteq A \cup A^c$. □

7. *Proof.* ($\subseteq$) Suppose $x \in A \cup \mathcal{U}$. So $x \in A$ or $x \in \mathcal{U}$. In both cases, $x \in \mathcal{U}$. ($\supseteq$) Suppose $x \in \mathcal{U}$. So $x \in A$ or $x \in \mathcal{U}$. Hence, $x \in A \cup \mathcal{U}$. □

9. *Proof.* ($\rightarrow$) Suppose $A \cup B \subseteq C$. Since $A \subseteq A \cup B$, we have $A \subseteq C$. Since $B \subseteq A \cup B$, we have $B \subseteq C$. Thus $A \subseteq C$ and $B \subseteq C$. ($\leftarrow$) Suppose $A \subseteq C$ and $B \subseteq C$. Suppose $x \in A \cup B$. So $x \in A$ or $x \in B$. In the case that $x \in A$, since $A \subseteq C$, we get $x \in C$. In the case that $x \in B$, since $B \subseteq C$, we get $x \in C$. In both cases, $x \in C$. So $A \cup B \subseteq C$. □

11. *Proof.* Suppose $x \in A \triangle B$. We use the characterization of $\triangle$ displayed after Definition 1.18. That is, $x \in (A \setminus B) \cup (B \setminus A)$. So, $x \in A \setminus B$ or $x \in B \setminus A$. If $x \in A \setminus B$, then $x \in A$. If $x \in B \setminus A$, then $x \in B$. In either case, $x \in A \cup B$. □

13. *Sketch.* Suppose $x \in (A \cup B) \setminus C$. If $x \in A$, then $x \in A$. If $x \in B$, then $x \in B \setminus C$. □
In either case, $x \in A \cup (B \setminus C)$.

15. *Proof.* ($\subseteq$) Suppose $x \in A \cup (B \setminus C)$. So $x \in A$ or $x \in B \setminus C$. *Case 1*: $x \in A$. Since $A \subseteq A \cup B$, $x \in A \cup B$. Since $x \in A$, we get $x \notin C \setminus A$. So $x \in (A \cup B) \setminus (C \setminus A)$. *Case 2*: $x \in B \setminus C$. So $x \in B$ and $x \notin C$. Since $x \in B$ and $B \subseteq A \cup B$, we get $x \in A \cup B$. Since $x \notin C$, we have $x \notin C \setminus A$. So $x \in (A \cup B) \setminus (C \setminus A)$. In both cases, $x \in (A \cup B) \setminus (C \setminus A)$. ($\supseteq$) Suppose $x \in (A \cup B) \setminus (C \setminus A)$. So $x \in A \cup B$ and $x \notin C \setminus A$. Thus $x \in A$ or $x \in B$, and $x \notin C$ or $x \in A$. *Case 1*: $x \in A$. Since $A \subseteq A \cup (B \setminus C)$, we get $x \in A \cup (B \setminus C)$. *Case 2*: $x \notin A$. So $x \in B$ and $x \notin C$. That is, $x \in B \setminus C$. Since $B \setminus C \subseteq A \cup (B \setminus C)$, we get $x \in A \cup (B \setminus C)$. In both cases, $x \in A \cup (B \setminus C)$. □

17. *Proof.* Let $x, y \in \mathcal{U}$. From the string of equivalences

$$\begin{aligned}
(x,y) \in (A \times B)^c &\leftrightarrow \quad x \notin A \quad \vee \quad y \notin B \\
&\leftrightarrow \quad x \in A^c \quad \vee \quad y \in B^c \\
&\leftrightarrow \quad (x,y) \in A^c \times \mathcal{U} \quad \vee \quad (x,y) \in \mathcal{U} \times B^c \\
&\leftrightarrow \quad (x,y) \in (A^c \times \mathcal{U}) \cup (\mathcal{U} \times B^c),
\end{aligned}$$

it follows that

$$(x,y) \in (A \times B)^c \leftrightarrow (x,y) \in (A^c \times \mathcal{U}) \cup (\mathcal{U} \times B^c).$$

Hence $(A \times B)^c = (A^c \times \mathcal{U}) \cup (\mathcal{U} \times B^c)$. □
Note that our use of the string of equivalences avoids a need for cases as well as a need to handle two subset arguments.

19. *Proof.* Let $x \in \mathcal{U}$. From the string of equivalences

$$\begin{aligned}
x \in A \cap (B \vartriangle C) &\leftrightarrow \quad x \in A \quad \wedge \quad (x \in B \quad \oplus \quad x \in C) \\
&\leftrightarrow \quad (x \in A \quad \wedge \quad x \in B) \quad \oplus \quad (x \in A \quad \wedge \quad x \in C) \\
&\leftrightarrow \quad x \in A \cap B \quad \oplus \quad x \in A \cap C \\
&\leftrightarrow \quad x \in (A \cap B) \vartriangle (A \cap C).
\end{aligned}$$

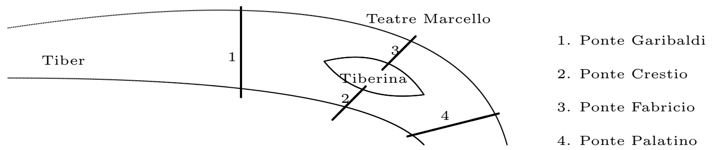
it follows that

$$x \in A \cap (B \vartriangle C) \leftrightarrow x \in (A \cap B) \vartriangle (A \cap C).$$

Hence $A \cap (B \vartriangle C) = (A \cap B) \vartriangle (A \cap C)$. □
Note that our use of the string of equivalences avoids a need for cases as well as a need to handle two subset arguments.

21. *Proof.* Suppose $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$. So $S \in \mathcal{P}(A)$ or $S \in \mathcal{P}(B)$. That is, $S \subseteq A$ or $S \subseteq B$. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, in either case, $S \subseteq A \cup B$. So $S \in \mathcal{P}(A \cup B)$. □

23. *Sketch.* If bridge 2 is taken next, then bridge 3 must follow with bridge 4 after that, leaving the tourist on the wrong side of the Tiber with no way to return. It bridge 4 is taken next, then bridge 3 must follow, with bridge 2 after that, leaving the tourist on the wrong side of the Tiber with no way to return. □



25. *Sketch.*

$$\begin{aligned} -x \geq 0 \quad &\leftrightarrow \quad x \leq 0. \\ -x < 0 \quad &\leftrightarrow \quad x > 0. \end{aligned}$$

Note that $-(-x) = x$. □

27. *Sketch.*

$$\begin{aligned} 1 - 2x \geq 0 \quad &\leftrightarrow \quad x \leq \frac{1}{2}. \\ 1 - 2x < 0 \quad &\leftrightarrow \quad x > \frac{1}{2}. \end{aligned}$$

In the second case, $-(1 - 2x) = 2x - 1$. □

29. *Sketch.*

$$\begin{aligned} x^2 + 2ax + a^2 = 0 \quad &\leftrightarrow \quad x = \frac{-2a \pm \sqrt{4a^2 - 4a^2}}{2} = -a. \\ x + 3 = 0 \quad &\leftrightarrow \quad x = -3. \end{aligned}$$

Either $-a = -3$ or not. □
That is, the only way for there to be exactly one root is to have $a = 3$.

31. *Proof.* If $x \geq 0$, then $|x|^2 = x^2$. If $x < 0$, then $|x|^2 = (-x)^2 = (-1)^2 x^2 = 1 \cdot x^2 = x^2$. □

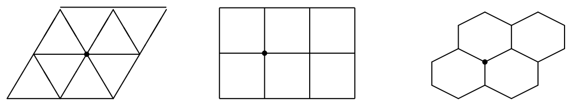
33. *Sketch.* If $x, y \geq 0$, then $|xy| = xy = |x||y|$. If $x \leq 0, y \geq 0$, then $|xy| = -xy = (-x)y = |x||y|$. If $x \geq 0, y \leq 0$, then $|xy| = -xy = x(-y) = |x||y|$. If $x, y \leq 0$, then $|xy| = xy = (-x)(-y) = |x||y|$. □
In the left-most equalities above, we are using the facts that the product of two positives is positive, the product of two negatives is positive, and the product of a positive with a negative is negative.

35. *Proof.* ($\rightarrow$) We prove the contrapositive. Suppose $-1 \leq x \leq 1$. In both of the cases, $x \geq 0$ and $x < 0$, we get that $x^2 \leq 1$. Squaring both sides again gives $x^4 \leq 1$. ($\leftarrow$) Suppose $x < -1$ or $x > 1$. In both cases, we get $x^2 > 1$. Hence $x^4 > 1$. □

37. *Proof.* Suppose $x^2 = y^2$. So $x^2 - y^2 = 0$. Hence, $(x + y)(x - y) = 0$. So $x + y = 0$ or $x - y = 0$. Therefore, $x = -y$ or $x = y$. That is, $x = \pm y$. □

39. *Sketch.* If $xy \geq 0$, then $|x + y| = |x| + |y|$. If $xy < 0$, then $|x + y| \leq \max\{|x|, |y|\} \leq |x| + |y|$. □
The first case happens when $x, y \geq 0$ or $x, y \leq 0$. The second case happens when $x \geq 0, y \leq 0$ or $x \leq 0, y \geq 0$.

41. *Sketch.* Let $A_n = \frac{180(n-2)}{n}$. If $n \geq 7$, then $128 < A_n < 180$ and no multiple of A_n can equal 360. If $n = 5$, then no multiple of $A_5 = 108$ can equal 360. Equilateral triangles ($n = 3$), squares ($n = 4$), and regular hexagons ($n = 6$) certainly do tile the floor as shown. $\square$



Review

1. *Sketch.* $x^2 + y^2 = 25$ fits each point. $\square$
Plugging the points into the general form $(x-h)^2 + (y-k)^2 = r^2$, we get
$(3-h)^2 + (4-k)^2 = r^2$, so $25 - 6h + h^2 - 8k + k^2 = r^2$,
$(4-h)^2 + (-3-k)^2 = r^2$, so $25 - 8h + h^2 + 6k + k^2 = r^2$,
$(-5-h)^2 + (-k)^2 = r^2$, so $25 + 10h + h^2 + k^2 = r^2$.
The first equation minus the third equation gives $-16h - 9k = 0$. The second equation minus the third equation gives $-18h + 6k = 0$. Hence $h = k = 0$. Substituting this into any of the equations gives $25 = r^2$. So $r = 5$.

2. *Sketch.* $x^4 - 2x^2 - 8 = (x^2+2)(x^2-4)$ and $x^2 + 2 \neq 0$. $\square$
The point is that the roots of $x^4 - 2x^2 - 8$ are the roots of $x^2 + 2$ together with the roots of $x^2 - 4$. However, $x^2 + 2$ has no roots. The roots of $x^2 - 4$ are certainly 2 and -2.

3. Notice that $n = 25 - m$. So $m(25 - m) = 100$. This becomes the quadratic equation $0 = m^2 - 25m + 100$, which has solutions $m = 5$ or 20.
Proof. Let $m = 20, n = 5$.
Observe that $mn = 20 \cdot 5 = 100$ and $m + n = 20 + 5 = 25$. $\square$

4. *Proof.* Let $x = 2$. Observe that $2^x = 2^2 = x^2$. $\square$

5. *Sketch.* Let $A = \emptyset, B = C = \mathbb{Z}$. $\square$
There are infinitely many different answers that will work. In fact, any choice with $A \subset B \subseteq C$ will work.

6. *Sketch.* Let $A = \{1, 2\}$. $\square$
Note that $|\mathcal{P}(A)| = 2^{|A|}$ and $|A^2| = |A|^2$.
So, by Exercise 4, we can pick $|A| = 2$. Hence, any set A with 2 elements will work.

7. An 85 on the third test yields an average of 75.
We want $\frac{80+60+t_3}{3} = 75$. This gives $t_3 = 85$.

8. False. Consider $f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$

Observe that $f^2(x) = \begin{cases} 1^2 = 1 & \text{if } x \geq 0, \\ (-1)^2 = 1 & \text{if } x < 0. \end{cases}$.

Since $\forall\ x \in \mathbb{R}, f^2(x) = 1$, we see that f^2 is constant. Certainly, f is not constant.

9. *Sketch.* Let $A = B = \mathbb{Z}, C = \emptyset$. $\square$

The point is that we can make $A \subseteq B \cup C$ by just forcing $A \subseteq B$. As long as we pick $A \neq \emptyset$, we can then pick C so that $A \nsubseteq C$.

10. *Sketch.* Let $A = B = \{1\}, C = \emptyset$. Observe that $A \vartriangle (B \cap C) = \{1\}$ and $(A \vartriangle B) \cap (A \vartriangle C) = \emptyset$. $\square$

11. *Proof.* Observe that $(-1)^4 = 1 = (-1)^2$, $0^4 = 0 = 0^2$, and $1^4 = 1 = 1^2$. $\square$

12. *Proof.* If $A = \{1\}$, then $A^3 = \{(1,1,1)\}$ and $|A^3| = 1$. If $A = \{2\}$, then $A^3 = \{(2,2,2)\}$ and $|A^3| = 1$. $\square$

13. *Proof.* Since $(-1,0) \in \mathbb{Z} \times \mathbb{N}$, and $(-1,0) \notin \mathbb{N} \times \mathbb{Z}$ (because $-1 \notin \mathbb{N}$), it follows that $\mathbb{Z} \times \mathbb{N} \neq \mathbb{N} \times \mathbb{Z}$.

14. *Proof.* Let $n \in \mathbb{Z}^+$. So $n \geq 1$. Hence, $n \cdot n \geq n \cdot 1$. That is, $n^2 \geq n$. $\square$

15. *Proof.* Suppose $x \in [2,4]$. So $2 \leq x \leq 4$. Hence, $4 = 2^2 \leq x^2 \leq 4^2 = 16$. That is, $x^2 \in [4,16]$. $\square$

16. *Proof.* Suppose f is constant. So we have $c \in \mathbb{R}$ such that $\forall\ x \in \mathbb{R},\ f(x) = c$. Observe that $\forall\ x \in \mathbb{R},\ (f+1)(x) = f(x) + 1 = c + 1$. So $f + 1$ is constant. $\square$

17. *Proof.* Suppose f is periodic and g is constant. So we have $p \in \mathbb{R}^+$ and $c \in \mathbb{R}$ such that $\forall\ x \in \mathbb{R},\ f(x+p) = f(x)$ and $g(x) = c$. Observe that $\forall\ x \in \mathbb{R},\ (f+g)(x+p) = f(x+p)+g(x+p) = f(x)+c = f(x)+g(x) = (f+g)(x)$. So $f + g$ is periodic. $\square$

18. *Proof.* Suppose f is bounded above and g is bounded below.
So we have $M, L \in \mathbb{R}$ such that $\forall\ x \in \mathbb{R},\ f(x) \leq M$ and $g(x) \geq L$.
Note that $\forall\ x \in \mathbb{R},\ -g(x) \leq -L$. Observe that
$\forall\ x \in \mathbb{R},\ (f-g)(x) = f(x) - g(x) = f(x) + (-g(x)) \leq M + (-L) = M - L$.
So $f - g$ is bounded above. $\square$

19. *Proof.* Let t_1, t_2, t_3 represent the test scores, in order. Suppose $t_1 \leq 40$. Since $t_2 \leq 100$ and $t_3 \leq 100$, we have an average of at most $\frac{40+100+100}{3} = 80$. $\square$

20. *Proof.* Suppose $A \subseteq C$. Suppose $x \in A \cap B$. So $x \in A$ and $x \in B$. Since $x \in A$ and $A \subseteq C$, we get $x \in C$. Thus $A \cap B \subseteq C$. $\square$

21. *Proof.* Suppose $x \in A \setminus B$. So $x \in A$ and $x \notin B$. In particular, $x \in A$. $\square$

22. *Proof.* Let $x \in \mathcal{U}$. From the string of equivalences

$$\begin{aligned} x \in (A \setminus B)^c &\leftrightarrow \neg[x \in A \setminus B] \\ &\leftrightarrow \neg[x \in A \wedge x \notin B] \\ &\leftrightarrow x \notin A \vee x \in B \\ &\leftrightarrow x \in A^c \vee x \in B \\ &\leftrightarrow x \in A^c \cup B. \end{aligned}$$

it follows that

$$x \in (A \setminus B)^c \leftrightarrow x \in A^c \cup B.$$

Hence $(A \setminus B)^c = A^c \cup B$. $\square$

23. *Proof.* Suppose $A \subset B$. Hence, we have $x \in B$ with $x \notin A$. That is, $x \in B \setminus A$. So $B \setminus A \neq \emptyset$. $\square$

24. *Proof.* Let $x \in \mathbb{R}$.
($\rightarrow$) Suppose $x \in [1, 2]$. So $1 \leq x \leq 2$.
Multiplication by 2 gives $2 \leq 2x \leq 4$. Hence $2x \in [2, 4]$.
($\leftarrow$) Suppose $2x \in [2, 4]$. So $2 \leq 2x \leq 4$.
Division by 2 gives $1 \leq x \leq 2$. Hence $x \in [1, 2]$. $\square$

25. *Sketch.* $3x - 2 \in (1, 4)$ iff $1 < 3x - 2 < 4$ iff $1 < x < 2$ iff $1 < 5 - 2x < 3$ iff $5 - 2x \in (1, 3)$. $\square$

26. *Sketch.* $x^2 = y^2$ iff $x^2 - y^2 = 0$ iff $(x+y)(x-y) = 0$ iff $x+y = 0$ or $x-y = 0$ iff $x = -y$ or $x = y$. Since $x, y \in \mathbb{R}^+$, it is not possible that $x = -y$. $\square$
Note that when $y \in \mathbb{R}^+$, we have $-y \in \mathbb{R}^-$. Hence $x = -y$ cannot happen when $x \in \mathbb{R}^+$.

27. *Sketch.* The Trichotomy Law in Appendix A tells us that
$\forall\, x, y \in \mathbb{R},\ \ x = y \ \oplus\ x > y \ \oplus\ y > x$. From this it follows that
$\forall\, x, y \in \mathbb{R},\ \ x \neq y \ \leftrightarrow\ x > y$ or $y > x$. Negating both sides of this equivalence gives the desired result. $\square$
It is important to realize that

$$\neg[x > y] \ \equiv\ x \leq y \ \equiv\ x < y \ \vee\ x = y \ \equiv\ x < y \ \oplus\ x = y.$$

28. Here it is more convenient to use the characterization of constant functions given in Exercise 37(b) from Section 1.3 (and proven in Exercise 12 from Section 2.3).

Proof. ($\rightarrow$) Suppose f is constant. Since, $\forall\, x, y \in \mathbb{R}, f(x) = f(y)$, it follows that f is both nondecreasing and nonincreasing. ($\leftarrow$) Suppose f is nondecreasing and nonincreasing. So, $\forall\, x, y \in \mathbb{R}$, if $x \leq y$, then $f(x) \leq f(y)$ and $f(x) \geq f(y)$. By Exercise 27 it follows that $\forall\, x, y \in \mathbb{R}, f(x) = f(y)$. So, f is constant. $\square$

29. *Proof.* ($\rightarrow$) Done in Exercise 17 from Section 2.2. ($\leftarrow$) Suppose f^2 is periodic. So we have $p \in \mathbb{R}^+$ such that $\forall\, x \in \mathbb{R}$, $f^2(x+p) = f^2(x)$. Since f is nonnegative, $\forall\, x \in \mathbb{R}$, $f(x+p) \geq 0$ and $f(x) \geq 0$. From Exercise 26 it follows that $\forall\, x \in \mathbb{R}$, $f(x+p) = f(x)$. So f is periodic. $\square$

30. *Proof.* ($\rightarrow$) Suppose $A^2 = B^2$. ($\subseteq$) Suppose $x \in A$. So $(x, x) \in A^2 = B^2$. Hence, $x \in B$. ($\supseteq$) Similar. So $A = B$. ($\leftarrow$) Suppose $A = B$. ($\subseteq$) Suppose $(x, y) \in A^2$. So $x \in A = B$ and $y \in A = B$. Hence $(x, y) \in B^2$. ($\supseteq$) Similar. So $A^2 = B^2$. $\square$
Recall that $A^2 = A \times A$ and $(u, w) \in C \times D$ iff $u \in C$ and $w \in D$.

31. *Proof.* ($\rightarrow$) Suppose $A \setminus B \subseteq C$. Suppose $x \in A$. *Case 1*: $x \in B$. We have $x \in B \cup C$. *Case 2*: $x \notin B$. So $x \in A \setminus B$. Hence $x \in C$. We have $x \in B \cup C$. In both cases $x \in B \cup C$. ($\leftarrow$) Suppose $A \subseteq B \cup C$. Suppose $x \in A \setminus B$. So $x \in A$ and $x \notin B$. Since $x \in A$, we have $x \in B \cup C$. Since $x \notin B$, it must be that $x \in C$. Hence $A \setminus B \subseteq C$. $\square$

32. *Proof.* Let $t_1, t_2, \ldots, t_n$ be the test scores. ($\rightarrow$) Suppose some test score t_k is less than 100. Then the average $\frac{t_1+t_2+\cdots+t_n}{n}$ is at most $\frac{(n-1)100+t_k}{n} < \frac{(n-1)100+100}{n} = 100$. ($\leftarrow$) Suppose $t_1 = t_2 = \cdots = t_n = 100$. The average is then $\frac{n(100)}{n} = 100$. $\square$

33. *Proof.* ($\subseteq$) Suppose $x \in (A \cap B) \setminus C$. So $x \in A$ and $x \in B$ and $x \notin C$. Since $x \in A$ and $x \notin C$, we have $x \in A \setminus C$. Since $x \in B$ and $x \notin C$, we have $x \in B \setminus C$. So $x \in (A \setminus C) \cap (B \setminus C)$. ($\supseteq$) Suppose $x \in (A \setminus C) \cap (B \setminus C)$. So $x \in A$ and $x \notin C$ and $x \in B$ (and $x \notin C$). Since $x \in A$ and $x \in B$, we have $x \in A \cap B$. Since $x \in A \cap B$ and $x \notin C$, we have $x \in (A \cap B) \setminus C$. $\square$

34. *Proof.* ($\subseteq$) Suppose $(x, y) \in A \times (B \cup C)$. So $x \in A$ and $y \in B \cup C$. That is, $y \in B$ or $y \in C$. *Case 1*: $y \in B$. We get $(x, y) \in A \times B$. *Case 2*: $y \in C$. We get $(x, y) \in A \times C$. In both cases, $(x, y) \in (A \times B) \cup (A \times C)$. ($\supseteq$) Suppose $(x, y) \in (A \times B) \cup (A \times C)$. So $(x, y) \in A \times B$ or $(x, y) \in A \times C$. *Case 1*: $(x, y) \in A \times B$. We get $y \in B$. So $y \in B \cup C$. Thus, $(x, y) \in A \times (B \cup C)$. *Case 2*: $(x, y) \in A \times C$. We get $y \in C$. So $y \in B \cup C$. Thus, $(x, y) \in A \times (B \cup C)$. In both cases, $(x, y) \in A \times (B \cup C)$. $\square$

35. *Proof.* Suppose $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$. So $S \in \mathcal{P}(A)$ or $S \in \mathcal{P}(B)$. *Case 1*: $S \in \mathcal{P}(A)$. We have $S \subseteq A$. Since $A \subseteq A \cup B$, we get $S \subseteq A \cup B$. *Case 2*: $S \in \mathcal{P}(B)$. We have $S \subseteq B$. Since $B \subseteq A \cup B$, we get $S \subseteq A \cup B$. In both cases, $S \subseteq A \cup B$. That is $S \in \mathcal{P}(A \cup B)$. $\square$

36. *Sketch.* Suppose not. Let L be the largest element. Observe that $\frac{L}{2}$ is a larger element of $\mathbb{R}^-$. (Think about it.) This is a contradiction. $\square$
Note that $L < \frac{L}{2} < 0$, since $2L < L < 0$.

37. *Proof.* Suppose not. Let s be the smallest element of $(-1, 1)$. However, $\frac{-1+s}{2}$ is a smaller element of $(-1, 1)$. This is a contradiction. $\square$
If $-1 < s < 1$, then $-1 = \frac{-1+(-1)}{2} < \frac{-1+s}{2} < \frac{-1+1}{2} = 0 < 1$.

38. *Proof.* Suppose $p \in \mathbb{R}^+$. Observe that $f(0+p) = p \neq 0 = f(0)$. So f cannot be periodic. $\square$
That is, p cannot be its period (and p is arbitrary).

39. *Proof.* Suppose $M \in \mathbb{R}$. Let $L = \max\{M, 2\}$. Note that $L > 1$. Observe that $f(L) = L^2 > L \geq M$. So f cannot be bounded above and therefore cannot be bounded. $\square$
That is, M cannot be an upper bound (and M is arbitrary).

40. *Proof.* Suppose $x < 0$. Repeated multiplication by x gives $x^2 > 0$, $x^3 < 0$, $x^4 > 0$, and finally $x^5 < 0$. $\square$

41. *Proof.* Suppose $A \times \emptyset \neq \emptyset$. So we have $(x, y) \in A \times \emptyset$. Thus, in particular, $y \in \emptyset$. This is a contradiction. $\square$

42. *Proof.* Suppose $A \cap B \neq \emptyset$. So we have some $x \in A \cap B$. Hence, in particular, $x \in A$. So $A \neq \emptyset$. $\square$

43. *Proof.* Suppose $A \neq \emptyset$ and $B \neq \emptyset$. So we have some $x \in A$ and $y \in B$. Hence, $(x, y) \in A \times B$. Therefore, $A \times B \neq \emptyset$. $\square$

44. *Proof.* Suppose $B = C$. ($\subseteq$) Suppose $(x, y) \in A \times B$. So $x \in A$ and $y \in B = C$. Hence, $(x, y) \in A \times C$. ($\supseteq$) Similar. Therefore, $A \times B = A \times C$. $\square$

45. We prove the contrapositive.
Proof. Let A be a set. Suppose A is finite. So $|A| = n$ for some $n \in \mathbb{N}$. Since $|\mathcal{P}(A)| = 2^n \in \mathbb{N}$, we see that $\mathcal{P}(A)$ is finite.

46. *Proof.* Let f be a real function, and suppose that f is constant. Hence, we have $c \in \mathbb{R}$ such that

$$\forall\, x \in \mathbb{R},\ f(x) = c.$$

Observe that

$$\forall\, x \in \mathbb{R},\ f^2(x) = [f(x)]^2 = c^2.$$

Since $c^2 \in \mathbb{R}$ and $\forall\, x \in \mathbb{R},\ f^2(x) = c^2$, it follows that f^2 is constant. $\square$

47. *Proof.* Let t_1, t_2, t_3, t_4 be the test grades. Suppose Erik has no test grade of

at least 60. Since $t_1, t_2, t_3, t_4 < 60$, Erik's average is $\frac{t_1+t_2+t_3+t_4}{4} < \frac{4(60)}{4} = 60$. So Erik does not pass. $\square$

48. *Proof.* Let $x \in \mathcal{U}$. From the string of logical equivalences

$$\begin{aligned} x \in (A \cap B \cap C)^c &\leftrightarrow x \notin A \cap B \cap C \\ &\leftrightarrow \neg(x \in A \cap B \cap C) \\ &\leftrightarrow \neg(x \in A \wedge x \in B \wedge x \in C) \\ &\leftrightarrow \neg(x \in A) \vee \neg(x \in B) \vee \neg(x \in C) \\ &\leftrightarrow x \notin A \vee x \notin B \vee x \notin C \\ &\leftrightarrow x \in A^c \vee x \in B^c \vee x \in C^c \\ &\leftrightarrow x \in A^c \cup B^c \cup C^c \end{aligned}$$

it follows that $x \in (A \cap B \cap C)^c \leftrightarrow x \in A^c \cup B^c \cup C^c$. $\square$.

49. *Proof.* ($\subseteq$) Suppose $x \in (A \setminus C) \cup (B \setminus C)$. So $x \in A \setminus C$ or $x \in B \setminus C$. *Case 1*: $x \in A \setminus C$. So $x \in A$ and $x \notin C$. Since $A \subseteq A \cup B$, we have $x \in A \cup B$. Thus, $x \in (A \cup B) \setminus C$. *Case 2*: $x \in B \setminus C$. So $x \in B$ and $x \notin C$. Since $B \subseteq A \cup B$, we have $x \in A \cup B$. Thus, $x \in (A \cup B) \setminus C$.
($\supseteq$) Suppose $x \in (A \cup B) \setminus C$. So $x \in A \cup B$ and $x \notin C$. That is $x \in A$ or $x \in B$. *Case 1*: $x \in A$. We have $x \in A \setminus C$. Hence, $x \in (A \setminus C) \cup (B \setminus C)$. *Case 2*: $x \in B$. We have $x \in B \setminus C$. Hence, $x \in (A \setminus C) \cup (B \setminus C)$. $\square$

50. *Proof.* Suppose $x \in (A \vartriangle B) \cap (A \vartriangle C)$. So $x \in A \vartriangle B$ and $x \in A \vartriangle C$.
Case 1: $x \in A$. It must be that $x \notin B$ (and $x \notin C$). In particular, $x \notin B \cap C$.
Case 2: $x \notin A$. It must be that $x \in B$ and $x \in C$. Hence, $x \in B \cap C$.
In both cases, $x \in A \vartriangle (B \cap C)$. $\square$

51. *Sketch.* Suppose $A = \emptyset$ or $B = \emptyset$. *Case 1*: $A = \emptyset$. So $A \times B = \emptyset \times B = \emptyset$. *Case 2*: $B = \emptyset$. So $A \times B = A \times \emptyset = \emptyset$. $\square$
OR (contrapositive): Suppose $A \times B \neq \emptyset$. So we have $(x, y) \in A \times B$. Since $x \in A$ and $y \in B$, we have $A \neq \emptyset$ and $B \neq \emptyset$.

52. *Proof.* ($\rightarrow$) Suppose $xy > 0$. So $x \neq 0$ and $y \neq 0$. *Case 1*: $x > 0$. We see that $y = \frac{xy}{x} > 0$. *Case 2*: $x < 0$. We see that $y = \frac{xy}{x} < 0$. Thus, either $x, y > 0$ or $x, y < 0$. ($\leftarrow$) Suppose $x, y > 0$ or $x, y < 0$. *Case 1*: $x, y > 0$. We get $xy > 0$. *Case 2*: $x, y < 0$. We get $xy = (-x)(-y) > 0$. In both cases, $xy > 0$. $\square$

53. Since $x^2 \geq 0$, the definition of absolute value gives that $|x^2| = x^2$.

54. *Sketch.* $x^2 - 1 < 0$ iff $x^2 < 1$ iff $-1 < x < 1$. Also $-(x^2 - 1) = 1 - x^2$. $\square$
Recall that $|y| = -y$ if $y < 0$, and $|y| = y$ if $y \geq 0$. Use $y = x^2 - 1$, which is negative when $-1 < x < 1$ and nonnegative otherwise.

55. *Sketch.* Since $|x|^2 = x^2$ and $|x| \geq 0$, it follows that $|x| = \sqrt{x^2}$. $\square$
The point is that, for $y \geq 0$, $\sqrt{y}$ is the nonnegative number z such that $z^2 = y$. Here $y = x^2$ and $x = |x|$.

56. *Sketch.* $x^2 - 6x + 8 = (x-4)(x-2)$ and, by Exercise 52,
$(x-4)(x-2) > 0 \leftrightarrow [(x-4) > 0$ and $(x-2) > 0]$ or $[(x-4) < 0$ and $(x-2) < 0]$
$\leftrightarrow [x > 4$ and $x > 2]$ or $[x < 4$ and $x < 2] \leftrightarrow x > 4$ or $x < 2$. $\square$
In fact, $x^2 - 6x + 8 = 0$ if and only if $x = 2$ or $x = 4$. We are simply determining the sign of $x^2 - 6x + 8$ on each of the intervals $(-\infty, 2)$, $(2, 4)$, and $(4, \infty)$.

57. *Proof.* Let $x \in \mathbb{R}$. $(\rightarrow)$ Suppose $x = \frac{1}{x}$. So $x^2 = 1$. Hence $x = \pm 1$. $(\leftarrow)$ Suppose $x = \pm 1$. In both cases, observe that $x^2 = 1$. Hence $x = \frac{1}{x}$. $\square$

2.3 Chapter 3

Section 3.1

1. *Proof.* Let m be even and n be odd. So $m = 2j$ and $n = 2k+1$ for some $j, k \in \mathbb{Z}$. Observe that $mn = (2j)(2k+1) = 2(j(2k+1))$. Since $j(2k+1) \in \mathbb{Z}$, we see that mn is even. □

3. *Proof.* Suppose n is odd. So $n = 2k+1$ for some $k \in \mathbb{Z}$. Observe that $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2k \in \mathbb{Z}$, we see that n^2 is odd. □

5. *Proof.* Suppose n is odd. So $n = 2k+1$ for some $k \in \mathbb{Z}$. Observe that $\frac{n+1}{2} = \frac{2k+2}{2} = k+1 \in \mathbb{Z}$. □

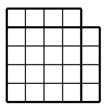
7. *Proof.* Suppose n is an even integer. So $n = 2k$ for some $k \in \mathbb{Z}$. Observe that $(-1)^n = (-1)^{2k} = ((-1)^2)^k = 1^k = 1$. □

9. On.
Off $= -1$ and On $= 1$.

11. *Proof.* Observe that $a \cdot k = 0$ when $k = 0 \in \mathbb{Z}$. Hence $a \mid 0$. □

13. *Proof.* Suppose $a \mid 1$. So $1 = ak$ for some $k \in \mathbb{Z}$. Since $a, k \in \mathbb{Z}$, this is only possible if $a = k = \pm 1$. (Under any other conditions, $|ak| > 1$.) □
Since $1 = ak$, it follows that $a, k \neq 0$. In particular, $|k| \geq 1$. If it were the case that $|a| \geq 2$, then $|ak| = |a| \cdot |k| \geq 2 \cdot 1 = 2$, which is impossible. Hence, $|a| \leq 1$, and it follows that $|a| = 1$.

15. (a) *Proof.* Note that $a-1, a^2-1 \in \mathbb{Z}$. Observe that $a^2 - 1 = (a+1)(a-1)$ and $(a+1) \in \mathbb{Z}$. Hence, $(a-1) \mid (a^2-1)$. □
(b) R breaks into two $(a-1) \times 1$ rectangles and an $(a-1) \times (a-1)$ square, as we see in the case below, when $a = 5$.



17. *Proof.* Suppose $a \mid b$ and $b \mid a$. So $b = aj$ and $a = bk$ for some $j, k \in \mathbb{Z}$. So $a = bk = a(jk)$. So $1 = jk$. Thus $k \mid 1$. By Exercise 13, it follows that $k = \pm 1$. Therefore, $a = bk = \pm b$. □

19. *Proof.* Suppose n is even. So $n = 2k$ for some $k \in \mathbb{Z}$. Observe that $n^2 = (2k)^2 = 4k^2$ and $k^2 \in \mathbb{Z}$. Hence $4 \mid n^2$. □

21. No, since $8 \nmid 420$. Yes, since $7 \mid 420$.

23. *Proof.* Suppose $a \mid b$ and $a \mid c$. So $b = aj$ and $c = ak$ for some $j, k \in \mathbb{Z}$. Note that $b - c = aj - ak = a(j - k)$. Since $j - k \in \mathbb{Z}$, we see that $a \mid (b - c)$. $\square$

25. Yes, when $a = 2, b = c = 1$.

27. $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71.$

29. *Proof.* Let p be a prime with $3 \mid p$. So $p = 3k$ for some $k \in \mathbb{Z}$. In fact, $k > 0$. Since p is prime and $3 \neq 1$, it must be that $k = 1$. Thus, $p = 3$. $\square$

31. *Proof.* Let $p \in \mathbb{Z}$ with $p > 1$. $(\rightarrow)$ Suppose p is prime. Suppose $r > 1$ and $s > 1$. Since the only positive divisors of p are 1 and p, we cannot have $rs = p$, so $rs \neq p$. $(\leftarrow)$ Suppose $\forall\, r, s \in \mathbb{Z}$, if $r > 1$ and $s > 1$, then $rs \neq p$. Suppose t is a positive divisor of p. So $p = tu$, for some $u \in \mathbb{Z}$. Moreover, $u > 0$. Since $tu = p$, we must have $t \leq 1$ or $u \leq 1$. This forces $t = 1$ or $u = 1$. If $u = 1$, then $t = p$. So $t = 1$ or $t = p$. $\square$

33. Negate the characterization given in Exercise 31.
That is, the proof for Exercise 31 also proves this result, since an integer greater than 1 is composite iff it is not prime.

35. *Proof.* Suppose n is composite. So $n = rs$ for some $r, s \in \mathbb{Z}$ with $1 < r, s < n$. Suppose to the contrary that both $r, s > \sqrt{n}$. Then $n = rs > \sqrt{n} \cdot \sqrt{n} = n$, a contradiction. Hence, it must be that one of r or s is less than or equal to $\sqrt{n}$. $\square$

37.

	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22
23	24	25	26	27	28	29	30	31	32	33
34	35	36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63	64	65	66
67	68	69	70	71	72	73	74	75	76	77
78	79	80	81	82	83	84	85	86	87	88
89	90	91	92	93	94	95	96	97	98	99
100	101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120	121

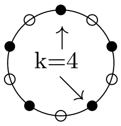
The first 30 primes are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113$. Note in the table that X is used to cross off multiples of 7. The multiples of 2, 3, 5 happen to lie on simpler lines that cut through the table.

39. 14.
$56 = 2^3 7^1$, $42 = 2^1 3^1 7^1$, and $2^1 7^1 = 14$.

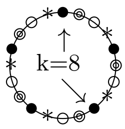
41. 18.
$-108 = (-1)2^2 3^3$, $-90 = (-1)2^1 3^2 5^1$, and $2^1 3^2 = 18$.

43. 15.
$2475 = 3^2 5^2 11^1$, $-780 = (-1)2^2 3^1 5^1 13^1$, and $3^1 5^1 = 15$.

45. (a) $2 = \gcd(10, 4)$.



(b) $4 = \gcd(20, 8)$.



(c) Say d is the answer. Then, both n and k need to be multiples of d. Since we want to pick d as large as possible, we get $d = \gcd(n, k)$.

47. *Proof.* Suppose $d_1, d_2 \in \mathbb{Z}$ both satisfy conditions (i),(ii), and (iii). By conditions (i) and (ii) for d_2 and condition (iii) for d_1 with $c = d_2$, we see that $d_2 \leq d_1$. A similar argument with d_1 and d_2 switched gives $d_1 \leq d_2$. Hence, $d_2 = d_1$. $\square$

49. *Proof.* Let $d = \gcd(m, n)$. So $d \mid m$ and $d \mid n$. Also, $d \mid (-m)$ and $d \mid n$. Suppose $c \mid (-m)$ and $c \mid n$. So $c \mid m$ and $c \mid n$. Thus, $c \mid d$. Hence, $d = \gcd(-m, n)$. $\square$

51. *Sketch.* $\gcd(m, -n) = \gcd(-n, m) = \gcd(n, m) = \gcd(m, n)$. $\square$
The first and third equalities follow from Exercise 50, and the second equality follows from Exercise 49.

53. *Proof.* Let p and q be distinct primes. Suppose $d \in \mathbb{Z}^+$ with $d \mid p$ and $d \mid q$. Since $d \mid p$, we have $d = 1$ or $d = p$. If $d = p$, then $p \mid q$, giving $p = 1$ or $p = q$. Hence, $d \neq p$. Therefore, $d = 1$. Thus, $\gcd(p, q) = 1$. $\square$

55. *Sketch.* Write $1 = 2(n) + 1(1 - 2n)$, and mimic the argument in the proof of Lemma 3.3. $\square$

57. 168.
Observe that $168 = 56 \cdot 3 = 42 \cdot 4$. Note that $42 \nmid 56$ and $42 \nmid (2 \cdot 56)$.

59. 540.
Observe that $540 = (-108)(-5) = (-90)(-6)$. Note that $-90 \nmid (-108)k$ for $k = \pm1, \pm2, \pm3, \pm4$.

61. (a) $16 = \mathrm{lcm}(4, 16)$.
(b) $30 = \mathrm{lcm}(6, 15)$.
(c) Say l is the answer. Both a and b need to divide l, and l should be chosen as small as possible. So $l = \mathrm{lcm}(a, b)$.

63. *Proof.* Let $l = \mathrm{lcm}(n, m)$. We show that $\mathrm{lcm}(m, n) = l$, according to Definition 3.7. Note that $l > 0$, $n \mid l$, and $m \mid l$. Also, if $k \in \mathbb{Z}^+$, $m \mid k$, and $n \mid k$, then $l \leq k$. Hence, $\mathrm{lcm}(m, n) = l = \mathrm{lcm}(n, m)$. $\square$

Section 3.2

1. $3 = 11 - 4(2)$.
Note that $11 - 4n = 2$ iff $n = \frac{9}{4} \notin \mathbb{Z}$. Note that $11 - 4n = 1$ iff $n = \frac{5}{2} \notin \mathbb{Z}$.

3. $4 = 12(2) + 20(-1)$.
Note that $12x + 20y = 4(3x + 5y)$ is divisible by 4. So 1, 2, and 3 are not in S.

5. $2^{17} - 1$ is prime, $2^{19} - 1$ is prime, $2^{23} - 1 = 47 \cdot 178481$, and $2^{29} - 1 = 233 \cdot 1103 \cdot 2089$.
`> isprime(2^17 - 1);`   `true`, and
`> isprime(2^19 - 1);`   `true`,
but $2^{23} - 1 = 47 \cdot 178481$, and $2^{29} - 1 = 233 \cdot 1103 \cdot 2089$.

7. *Proof.* Suppose not. So every prime has fewer than 10^8 digits. There are only $10^{(10^8)}$ natural numbers with at most 10^8 digits. So there could be at most $10^{(10^8)}$ primes. However, there are infinitely many primes. This is a contradiction. $\square$

9. (a) *Sketch.* $b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + \cdots + b + 1)$. Since $b \geq 3$, both factors are larger than 1. $\square$
That is, $(b - 1) \geq 2$ and $b^{n-1} + b^{n-2} + \cdots + b + 1 \geq 3 + 1 = 4$.
(b) *Proof.* Suppose $n \in \mathbb{Z}^+$ is not prime. So $n = rs$ for some integers $r, s \geq 2$. Since $2^r \geq 3$, it follows from part (a) that $2^n - 1 = (2^r)^s - 1$ is not prime. $\square$

11. (a) 10 remainder 7,
since $127 = 12(10) + 7$ and $0 \leq 7 < 12$.
(b) 14 remainder 6,
since $216 = 15(14) + 6$ and $0 \leq 6 < 15$.

13. (a) $45 = 7(6) + 3$ and $0 \leq 3 < 7$.
(b) $-37 = 4(-10) + 3$ and $0 \leq 3 < 4$.

15. 7 and 3.
Note that $73 = 10(7) + 3$.

17. (a) 5 and 2. Note that $67 = 13(5) + 2$.
(b) -6 and 11. Note that $-67 = 13(-6) + 11$.

19. 165 div 18 = 9 full rows. 165 mod 18 = 3 extra seats.

21. (a) 100111.

k	n	a_k
-1	39	
0	19	1
1	9	1
2	4	1
3	2	0
4	1	0
5	0	1

(b) 127.

k	n	a_k
-1	87	
0	10	7
1	1	2
2	0	1

23. n and 0.
Note that $n^2 = n(n) + 0$.

25. Because $\mathbb{Z}$ does not have a smallest element
and $\mathbb{Z}$ is a nonempty subset of $\mathbb{Z}$.

27. *Proof.* Let $a \in \mathbb{Z}$, and let S be a subset of $\mathbb{Z}$ such that $\forall\ x \in S,\ x \geq a$. Let $T = \{t\ :\ t = s - a$ for some $s \in S\}$. So $T \subseteq \mathbb{N}$. By the Well-Ordering Principle, T has a smallest element, say τ. Let $\sigma = \tau + a$. Observe that σ is the smallest element of S. (Think about it.) □
If m were an element of S smaller than σ, then $m - a$ would be an element of T smaller than τ. That is, if $m < \sigma$, then $m - a < \sigma - a = \tau$. However, there are no elements of T smaller than τ.

29. *Sketch.*

n	$n^3 - n + 2$	$(n^3 - n + 2) \bmod 6$
$6k$	$216k^3 - 6k + 2$	2
$6k + 1$	$216k^3 + 108k^2 + 12k + 2$	2
$6k + 2$	$216k^3 + 216k^2 + 66k + 8$	2
$6k + 3$	$216k^3 + 324k^2 + 156k + 26$	2
$6k + 4$	$216k^3 + 432k^2 + 282k + 62$	2
$6k + 5$	$216k^3 + 540k^2 + 444k + 122$	2

□

That is, we consider each case $n = 6k + r$ for $r = 0, 1, 2, 3, 4, 5$. In each case, we see that $n^3 - n + 2 = 6q + 2$ for some q. Specifically,

n	$n^3 - n + 2$
$6k$	$6(36k^3 - k) + 2$
$6k + 1$	$6(36k^3 + 18k^2 + 2k) + 2$
$6k + 2$	$6(36k^3 + 36k^2 + 11k + 1) + 2$
$6k + 3$	$6(36k^3 + 54k^2 + 26k + 4) + 2$
$6k + 4$	$6(36k^3 + 72k^2 + 47k + 10) + 2$
$6k + 5$	$6(36k^3 + 90k^2 + 74k + 20) + 2$

Since $(n^3 - n + 2) \bmod 6 \neq 0$, it follows that $6 \nmid (n^3 - n + 2)$.

31. *Proof.* Suppose $n \in \mathbb{Z}$ and $3 \nmid n$. So $n = 3q + r$ for some $q \in \mathbb{Z}$ and $r = 1$ or 2. *Case 1*: $r = 1$. Since $n^2 = (3q + 1)^2 = 3(3q^2 + 2q) + 1$, we see that $n^2 \bmod 3 = 1$. *Case 2*: $r = 2$. Since $n^2 = (3q + 2)^2 = 3(3q^2 + 4q + 1) + 1$, we see that $n^2 \bmod 3 = 1$. In both cases, $n^2 \bmod 3 = 1$. □

33. *Sketch.* If $n = 5k + 1$, then $n^4 - 1 = 5(125k^4 + 100k^3 + 30k^2 + 4k)$.
If $n = 5k + 2$, then $n^4 - 1 = 5(125k^4 + 200k^3 + 120k^2 + 32k + 3)$.
If $n = 5k + 3$, then $n^4 - 1 = 5(125k^4 + 300k^3 + 270k^2 + 108k + 16)$.
If $n = 5k + 4$, then $n^4 - 1 = 5(125k^4 + 400k^3 + 480k^2 + 256k + 51)$. □
Note that n is not divisible by 5 if and only if $n = 5k + r$ for some $k \in \mathbb{Z}$ and $r = 1, 2, 3, 4$. (That is, $r = 0$ is excluded.) In each case, we see that $n^4 - 1 = 5q$ for some $q \in \mathbb{Z}$.

35. (a) 4, since $4 \leq 4.4 < 5$. (b) -5, since $-5 \leq -4.4 < -4$.
(c) 9, since $8 < 8.6 \leq 9$. (d) -8, since $-9 < -8.6 \leq -8$.

37. (a) -5, since $-6 < -5 \leq -5$. (b) 3, since $3 \leq \pi < 4$.
(c) 5, since $5 \leq \frac{17}{3} < 6$. (d) 11, since $10 < 4e \leq 11$.

39. (a) $\lceil \frac{500}{32} \rceil = \lceil 15.625 \rceil = 16$.
(b) $\lceil \frac{c}{m} \rceil$. Say b is the answer. Hence, we need $mb \geq c$. So $b \geq \frac{m}{c}$ and $b \in \mathbb{Z}$. The smallest possible value for b is thus $b = \lceil \frac{c}{m} \rceil$.

41. *Sketch.* Note that
(i) $\lfloor x \rfloor + n \in \mathbb{Z}$,
(ii) $\lfloor x \rfloor + n \leq x + n$,
(iii) $x + n < 1 + \lfloor x \rfloor + n$. $\square$
By Theorem 3.8, $\lfloor y \rfloor = k$ if and only if (i) $k \in \mathbb{Z}$,
(ii) $k \leq y$, and
(iii) $y < k + 1$.
We apply this with $y = x+n$ and $k = \lfloor x \rfloor + n$ to get $\lfloor x+n \rfloor = \lfloor y \rfloor = k = \lfloor x \rfloor + n$.

43. *Proof.* Suppose $n \in \mathbb{Z}$. *Case 1*: $n = 2k$ for some $k \in \mathbb{Z}$. Observe that $k \in \mathbb{Z}$, $\frac{n}{2} \leq k$, and $k - 1 < \frac{n}{2}$. Hence, $\lceil \frac{n}{2} \rceil = k = \frac{n}{2}$. *Case 2*: $n = 2k + 1$ for some $k \in \mathbb{Z}$. Observe that $k + 1 \in \mathbb{Z}$, $\frac{n}{2} \leq k + 1$, and $(k + 1) - 1 < \frac{n}{2}$. Hence, $\lceil \frac{n}{2} \rceil = k + 1 = \frac{n+1}{2}$. $\square$

45. *Proof.* Let $n \in \mathbb{Z}$. *Case 1*: $n = 3k$ for some $k \in \mathbb{Z}$. Note that $k = \frac{n}{3}$. We then have $k \in \mathbb{Z}$, $k \leq \frac{n}{3}$, and $\frac{n}{3} < k + 1$. It follows from Theorem 3.8 that $\lfloor \frac{n}{3} \rfloor = k = \frac{n}{3}$. *Case 2*: $n = 3k + 1$ for some $k \in \mathbb{Z}$. Note that $k = \frac{n-1}{3}$. We then have $k \in \mathbb{Z}$, $k \leq \frac{n}{3}$, and $\frac{n}{3} < k + 1$. It follows from Theorem 3.8 that $\lfloor \frac{n}{3} \rfloor = k = \frac{n-1}{3}$. *Case 3*: $n = 3k + 2$ for some $k \in \mathbb{Z}$. Note that $k = \frac{n-2}{3}$. We then have $k \in \mathbb{Z}$, $k \leq \frac{n}{3}$, and $\frac{n}{3} < k + 1$. It follows from Theorem 3.8 that $\lfloor \frac{n}{3} \rfloor = k = \frac{n-2}{3}$. $\square$

47. *Proof.* Let $x, y \in \mathbb{R}$. Since $\lfloor x \rfloor \leq x$ and $\lfloor y \rfloor \leq y$, we have $\lfloor x \rfloor + \lfloor y \rfloor \leq x + y$. Since $\lfloor x \rfloor + \lfloor y \rfloor \in \mathbb{Z}$ and $\lfloor x + y \rfloor$ is the largest integer n such that $n \leq x + y$, it follows that $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$. $\square$

49. Counterexample: Let $x = \frac{1}{2}$.
Observe that $\lfloor 2x \rfloor = \lfloor 1 \rfloor = 1$, $2\lfloor x \rfloor = 2\lfloor \frac{1}{2} \rfloor = 2(0) = 0$, and $1 \neq 0$.

51. *Proof.* Let $x \in \mathbb{R}$. $(\rightarrow)$ Suppose $x \notin \mathbb{Z}$. Then $\lfloor x \rfloor \neq x$, and it must be that $\lfloor x \rfloor < x$. Since $x \leq \lceil x \rceil$, we get $\lfloor x \rfloor \neq \lceil x \rceil$. $(\leftarrow)$ Suppose $x \in \mathbb{Z}$. It follows that $\lfloor x \rfloor = x = \lceil x \rceil$. $\square$

53. *Sketch.* Certainly $\lfloor x \rfloor \in \mathbb{Z}$ and $\lfloor x \rfloor \leq \lfloor x \rfloor < \lfloor x \rfloor + 1$. $\square$
By Theorem 3.8, $\lfloor y \rfloor = k$ if and only if $k \in \mathbb{Z}$ and $k \leq y < k+1$. We apply this with $k = y = \lfloor x \rfloor$.

55. If n is odd, then $\lfloor \frac{n+1}{2} \rfloor = \frac{n+1}{2} = \lceil \frac{n}{2} \rceil$.
If n is even, then $\lfloor \frac{n+1}{2} \rfloor = \frac{n}{2} = \lceil \frac{n}{2} \rceil$.
We simply apply Theorems 3.9 and 3.10.

57. $\text{round}(x) = \lfloor x + \frac{1}{2} \rfloor$.
Say $x = n + f$, where $n = \lfloor x \rfloor$ and $f = x - \lfloor x \rfloor \in [0, 1)$. If $f < \frac{1}{2}$, then $f + \frac{1}{2} < 1$ and $\text{round}(x) = n = \lfloor n + (f + \frac{1}{2}) \rfloor = \lfloor x + \frac{1}{2} \rfloor$. If $f \geq \frac{1}{2}$, then $f + \frac{1}{2} \geq 1$ and $\text{round}(x) = n + 1 = \lfloor n + 1 + (f + \frac{1}{2} - 1) \rfloor = \lfloor x + \frac{1}{2} \rfloor$.

59. The statement is equivalent to the fact that 1 is the smallest positive integer. *Proof.* Suppose there is a smaller positive integer than 1. So the set $S = \{n : n \in \mathbb{Z}^+ \text{ and } n < 1\}$ is nonempty. By the Well-Ordering Principle, S must have a smallest element, say s. Since $0 < s < 1$, it follows that $0 < s^2 < s < 1$. Since $s^2 \in \mathbb{Z}$, this is a contradiction. □

61. *Proof.* Suppose $n_1, n_2 \in \mathbb{Z}$ with $n_1 \leq x < n_1 + 1$ and $n_2 \leq x < n_2 + 1$. Without loss of generality, say $n_1 \geq n_2$. Adding $n_2 \leq x < n_2 + 1$ to $-n_1 - 1 < -x \leq -1_2$, we get $n_2 - n_1 - 1 < 0 < n_2 - n_1 + 1$. Adding $n_1 - n_2$ to this inequality gives $-1 < n_1 - n_2 < 1$. So $0 \leq n_1 - n_2 < 1$. From Exercise 59 it follows that $n_1 - n_2 = 0$. That is, $n_1 = n_2$. □

63. *Proof.* Let $S = \{s : n = 2^r s \text{ where } r \in \mathbb{N} \text{ and } s \in \mathbb{Z}^+\}$. Since $n = 2^0 n$, it follows that $n \in S$ and thus S is nonempty. By the Well-Ordering Principle, S has a smallest element. Call it b. Since $b \in S$, there is some $a \in \mathbb{N}$ such that $n = 2^a b$. If b were even (so $\frac{b}{2} \in \mathbb{Z}$), then $n = 2^{a+1}\frac{b}{2}$, whence $\frac{b}{2}$ would be a smaller element of S than b. Therefore, b must be odd. □

65. Let m and n be integers that are not both zero. We must show that there exists an integer d such that (i) $d > 0$, (ii) $d \mid m$ and $d \mid n$, and
(iii) $\forall\, c \in \mathbb{Z}^+,\ \text{if } c \mid m \text{ and } c \mid n,\ \text{then } c \leq d.$
Let $d = \max\{a : a > 0, a \mid m, \text{ and } a \mid n\}$.
Sketch. Clearly $1 > 0$, $1 \mid m$, and $1 \mid n$. So $S = \{a : a > 0, a \mid m, \text{ and } a \mid n\}$ is nonempty. By Theorem 3.1, no element of S is bigger than n. By the Generalized Maximum Principle, S must have a largest element. That is the value d that we need. □
Conditions (i) and (ii) hold since $d \in S$. Condition (iii) holds since d is the smallest element of S.

67. 4.
Use c for the value of #. We have $3(0+5+0+7+1+6)+(3+0+0+4+2+c) = 66 + c$. Note that $10 \mid (66 + 4)$.

69. No.
$3(0 + 1 + 0 + 8 + 1 + 0) + (6 + 0 + 0 + 1 + 6 + 7) = 50$ *is* divisible by 10.

71. 3.
Use m for the value of #.
We have $10(0)+9(4)+8(4)+7(6)+6(m)+5(1)+4(0)+3(7)+2(8)+6 = 158+6m$.
Note that $11 \mid (158 + 6 \cdot 3)$.

73. No. That is the ISBN for *How the Grinch Stole Christmas*.
$10(0) + 9(3) + 8(9) + 7(4) + 6(8) + 5(0) + 4(0) + 3(7) + 2(9) + 6 = 220$ *is* divisible by 11.

75. *Sketch.* Let a and b be the consecutive digits. Note that $(3a+b)-(3b+a) = 2(a-b)$ is divisible by 10 if and only if $a-b$ is divisible by 5. $\square$

77. (a) 0110101.
The message is 0110. So $b_5 = (0+1) \bmod 2 = 1$, $b_6 = (1+1) \bmod 2 = 0$, and $b_7 = (1+0) \bmod 2 = 1$.
(b)

Message	Code Word
0000	0000000
0001	0001001
0010	0010011
0011	0011010
0100	0100110
0101	0101111
0110	0110101
0111	0111100
1000	1000100
1001	1001101
1010	1010111
1011	1011110
1100	1100010
1101	1101011
1110	1110001
1111	1111000

(c) 2.
See the second row of the table.
(d) Female, A^+.
See the eighth row of the table.
(e) 1010011 is one digit away from both 0010011 and 1010111.
Also 0010010 is one digit away from both 0010011 and 0011010.

79. "LQ KZMAMHUIAP".
We use $y = (x+8) \bmod 27$. E.g., D = 4 encrypts to $(4+8) \bmod 27 = 12 =$ L.

81. "SELL IMCLONE".
We use $x = (y-15) \bmod 27$. E.g., G = 7 decrypts to $(7-15) \bmod 27 = 19 =$ S.

Section 3.3

1. $x = 2, y = -5$.
Note that $5 = 65(2) + 25(-5)$.

3. $x = 3, y = -4$.
Note that $15(3) + 11(-4) = 1$.

5. 12.
$\gcd(24, 12) = \gcd(12, 0) = 12$.

7. 22.
$\gcd(110, 44) = \gcd(44, 22) = \gcd(22, 0) = 22$.

9. 8.
$\gcd(296, 112) = \gcd(112, 72) = \gcd(72, 40) = \gcd(40, 32) = \gcd(32, 8) = \gcd(8, 0) = 8$.

11. 1.

n	m
63	25
25	13
13	12
12	1
1	0

13. $2 = 14(-1) + 8(2)$.

$$\begin{array}{rcll} \gcd(14,8) & = & \gcd(8,6) & \text{since } 14 = 8 + 6 \\ & = & \gcd(6,2) & \text{since } 8 = 6 + 2 \\ & = & \gcd(2,0) & \text{since } 6 = (2)3 + 0 \\ & = & 2 & \text{by Example 3.8.} \end{array}$$

So $2 = 8 - 6 = 8 - (14 - 8) = -14 + (2)8 = 14(-1) + 8(2)$.

15. $5 = 50(-2) + 35(3)$.

$$\begin{array}{rcll} \gcd(50,35) & = & \gcd(35,15) & \text{since } 50 = 35 + 15 \\ & = & \gcd(15,5) & \text{since } 35 = (15)2 + 5 \\ & = & \gcd(5,0) & \text{since } 15 = (5)3 + 0 \\ & = & 5 & \text{by Example 3.8.} \end{array}$$

So $5 = 35 - (15)2 = 35 - (50 - 35)2 = 50(-2) + 35(3)$.

17. $3 = 81(3) + 60(-4)$.

$$\begin{array}{rcll} \gcd(81,60) & = & \gcd(60,21) & \text{since } 81 = 60 + 21 \\ & = & \gcd(21,18) & \text{since } 60 = (21)2 + 18 \\ & = & \gcd(18,3) & \text{since } 21 = 18 + 3 \\ & = & \gcd(3,0) & \text{since } 18 = (3)6 + 0 \\ & = & 3 & \text{by Example 3.8.} \end{array}$$

So $3 = 21 - 18 = 21 - (60 - (21)2) = -60 + (21)3 = -60 + (81 - 60)3 = 81(3) + 60(-4)$.

19. $x = -5$ and $y = 23$.

$$\begin{aligned}
\gcd(55,12) &= \gcd(12,7) && \text{since } 55 = (12)4+7\\
&= \gcd(7,5) && \text{since } 12 = 7+5\\
&= \gcd(5,2) && \text{since } 7 = 5+2\\
&= \gcd(2,1) && \text{since } 5 = (2)2+1\\
&= \gcd(1,0) && \text{since } 2 = (1)2+0\\
&= 1 && \text{by Example 3.8.}
\end{aligned}$$

So $1 = 5-(2)2 = 5-(7-5)2 = -2(7)+3(5) = -2(7)+3(12-7) = 3(12)-5(7) = 3(12)-5(55-(12)4) = 55(-5)+12(23)$.

21. No. For $m = 2, n = 3$, we can use $x = 2, y = -1$ or $x = -1, y = 1$.
See Exercise 23.

23. *Proof.* Let x_0, y_0 be any fixed pair that gives $\gcd(m,n) = mx_0 + ny_0$. Observe that, $\forall\ k \in \mathbb{Z}$, $\gcd(m,n) = mx_0 + ny_0 = m(x_0 + kn) + n(y_0 - km)$. Therefore, $x = x_0 + kn$ and $y = y_0 - km$ gives a general solution to $\gcd(m,n) = mx + ny$. $\square$

25. No. $6 \mid (2 \cdot 3)$ but $6 \nmid 2$ and $6 \nmid 3$.

27. Corollary 3.19: Let m, n, and p be integers with $n > 0$ and p prime. If $p \mid m^n$, then $p \mid m$.
Proof. Let m, n, and p be integers with $n > 0$ and p prime. Suppose $p \mid m^n$. That is, $p \mid \underbrace{m \cdot m \cdot \cdots \cdot m}_{n \text{ times}}$. By Corollary 3.18, $p \mid m$ (for one of the m's). $\square$

29. *Sketch.* Let p be prime. It follows from Corollary 3.19 that $p \mid a \leftrightarrow p \mid a^m$, and $p \mid b \leftrightarrow p \mid b^n$. $\square$
Let $d = \gcd(a,b)$ and $c = \gcd(a^m, b^n)$. So $d \mid a$, $d \mid b$, $c \mid a^m$, and $c \mid a^n$. We have $x, y \in \mathbb{Z}$ such that $c = a^m x + b^n y = a(a^{m-1}x) + b(b^{n-1}y)$. We have $u, v \in \mathbb{Z}$ such that $d = au + bv$. $(\rightarrow)$ Suppose $p \mid d$. So $p \mid a$ and $p \mid b$. It follows that p divides $a(a^{m-1}x) + b(b^{n-1}y) = c$. $(\leftarrow)$ Suppose $p \mid c$. So $p \mid a^m$ and $p \mid a^n$. Hence, $p \mid a$, and $p \mid b$. It follows that p divides $au + bv = d$.

31. *Proof.* Let $d = \gcd(m,n)$. So $d \mid m$ and $d \mid n$. Note that $d \mid m$ and $d \mid (n-m)$. If $c \mid m$ and $c \mid (n-m)$, then $c \mid m$ and $c \mid n$, whence $c \leq d$. So $\gcd(m, n-m) = d = \gcd(m,n)$. $\square$

33. *Sketch.* Argue that
$\min\{mu_1 + nv_1 \ :\ mu_1 + nv_1 > 0\} = \min\{nu_2 + mv_2 \ :\ nu_2 + mv_2 > 0\}$
by using $(u_2, v_2) = (v_1, u_1)$. $\square$
Let $S_1 = \{mu_1 + nv_1 \ :\ mu_1 + nv_1 > 0\}$ and $S_2 = \{nu_2 + mv_2 \ :\ nu_2 + mv_2 > 0\}$. We in fact show that $S_1 = S_2$. $(\subseteq)$ Suppose $x \in S_1$. So $x = mu_1 + nv_1 > 0$ for some $u_1, v_1 \in \mathbb{Z}$. Since $x = nv_1 + mu_1 > 0$ and $v_1, u_1 \in \mathbb{Z}$, we see that $x \in S_2$. $(\supseteq)$ Similar.

35. *Proof.* From the given characterization, we see that

$$\gcd(k,0) = \min\{ku + 0v \ : \ ku + 0v > 0\} = \min\{ku \ : \ ku > 0\} = k \cdot 1 = k.$$

□

37. *Proof.* Suppose $c \in \mathbb{Z}$, $c \mid m$, and $c \mid n$. So $m = ca$ and $n = cb$ for some $a, b \in \mathbb{Z}$. By Theorem 3.13, there are $x, y \in \mathbb{Z}$ such that $\gcd(m,n) = mx + ny$. Since $\gcd(m,n) = cax + cby = c(ax + by)$, we see that $c \mid \gcd(m,n)$. □

39. $(5n+3)(7) + (7n+4)(-5) = 1$. So apply Corollary 3.14.

41. (a) $ad - bc = 3(2) - 5(1) = 1$.
(b) If the $ad - bc = 1$, then Corollary 3.14 tells us that a and b are relatively prime and that c and d are relatively prime. Similarly, consider the columns.
(c) No. The counterexample $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ shows that the converse does not hold.
Note that $\gcd(1,2) = \gcd(3,1) = 1$, but the determinant is $1(1) - 2(3) = -5 \neq 1$.

Section 3.4

1. $5\frac{1}{2} = \frac{11}{2}$ and $11, 2 \in \mathbb{Z}$ with $2 \neq 0$.

3. $-13\frac{2}{5} = \frac{-67}{5}$ and $-67, 5 \in \mathbb{Z}$ with $5 \neq 0$.

5. $5.821 = \frac{5821}{1000}$ and $5821, 1000 \in \mathbb{Z}$ with $1000 \neq 0$.

7. $3.\overline{14} = \frac{311}{99}$ and $311, 99 \in \mathbb{Z}$ with $99 \neq 0$.
Let $x = 3.\overline{14}$.
So $100x = 314.\overline{14}$.
So $99x = 100x - x = 314 - 3 = 311$.
So $x = \frac{311}{99}$.

9. $-4.3\overline{21} = \frac{-713}{165}$ and $-713, 165 \in \mathbb{Z}$ with $165 \neq 0$.
Let $x = -4.3\overline{21}$.
So $10x = -43.\overline{21}$.
So $1000x = -4321.\overline{21}$.
So $990x = 1000x - 10x = -4321 + 43 = -4278$.
So $x = \frac{-4278}{990} = \frac{-713}{165}$.

11. $12.75\overline{8} = \frac{11483}{900}$ and $11483, 900 \in \mathbb{Z}$ with $900 \neq 0$.

13. *Proof.* Since $r \in \mathbb{Q}$, $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Observe that $nr = \frac{na}{b}$ and $na, b \in \mathbb{Z}$ with $b \neq 0$. Thus, $nr \in \mathbb{Q}$. □

15. (a) *Proof.* Since $s \in \mathbb{Q}$, $s = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Observe that

$-s = \frac{-a}{b}$ and $-a, b \in \mathbb{Z}$ with $b \neq 0$. Thus, $-s \in \mathbb{Q}$. $\square$
(b) *Proof.* Since $r, s \in \mathbb{Q}$, $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$. Observe that $r - s = \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$. Since $ad - bd, bd \in \mathbb{Z}$ with $bd \neq 0$, we see that $r - s \in \mathbb{Q}$. $\square$

17. *Proof.* Suppose $n \geq 0$. We can write $r = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ with $b \neq 0$. Observe that $r^n = \frac{a^n}{b^n}$ and $a^n, b^n \in \mathbb{Z}$ with $b^n \neq 0$. So $r^n \in \mathbb{Q}$. $\square$

19. $\frac{5}{3}$.
$\frac{65}{39} = \frac{5 \cdot 13}{3 \cdot 13} = \frac{5}{3}$, $\gcd(5, 3) = 1$, and $3 > 0$.

21. $\frac{-57}{8}$.
$\frac{-513}{72} = \frac{-57 \cdot 9}{8 \cdot 9} = \frac{-57}{8}$, $\gcd(-57, 8) = 1$, and $8 > 0$.

23. $\frac{157}{50}$.
$3.14 = \frac{314}{100} = \frac{157 \cdot 2}{50 \cdot 2} = \frac{157}{50}$, $\gcd(157, 50) = 1$, and $50 > 0$.

25. $-\frac{378}{1000} = \frac{-189}{500}$.

27. 0.48.

$$
\begin{array}{rrrrrrl}
 & & & . & 4 & 8 & \\
\hline
2\ 5\ ) & 1 & 2. & 0 & 0 & & \\
- & 1 & 0 & 0 & & & \\
\hline
 & & 2 & 0 & 0 & & \text{remainder } 20 \\
 & - & 2 & 0 & 0 & & \\
\hline
 & & & & 0 & & \text{remainder } 0 \leftarrow \text{END}
\end{array}
$$

29. $0.\overline{428571}$.

$$
\begin{array}{rrrrrrrrl}
 & . & 4 & 2 & 8 & 5 & 7 & 1 & \\
\hline
7\ ) & 3. & 0 & 0 & & & & & \text{(remainder 3)} \\
- & 2 & 8 & & & & & & \\
\hline
 & & 2 & 0 & & & & & \text{remainder 2} \\
 & - & 1 & 4 & & & & & \\
\hline
 & & & 6 & 0 & & & & \text{remainder 6} \\
 & & - & 5 & 6 & & & & \\
\hline
 & & & & 4 & 0 & & & \text{remainder 4} \\
 & & & - & 3 & 5 & & & \\
\hline
 & & & & & 5 & 0 & & \text{remainder 5} \\
 & & & & - & 4 & 9 & & \\
\hline
 & & & & & & 1 & 0 & \text{remainder 1} \\
 & & & & & - & & 7 & \\
\hline
 & & & & & & & 3 & \text{remainder 3}
\end{array}
$$

31. $0.5\overline{3}$.

$$\begin{array}{rrrrrl}
 & & . & 5 & \overline{3} & \\
\hline
1\ 5\ ) & & 8. & 0 & 0 & \\
 & - & 7 & 5 & & \\
\cline{3-4}
 & & & 5 & 0 & \text{remainder } 5\\
 & - & & 4 & 5 & \\
\cline{4-5}
 & & & & 5 & \text{remainder } 5 \leftarrow \text{AGAIN}
\end{array}$$

33. Yes.
$\frac{3}{6} = 0.5$ and $3 \mid 6$.

35. No.
Take $a = c = 1, b = d = 2$, and get $\frac{ad+bc}{bd} = \frac{4}{4}$, while $\frac{1}{1}$ is in lowest terms.

37. When $p \nmid n$.
Otherwise $\gcd(p, n) = p > 1$.

39. Yes. A fraction $\frac{a}{b}$ in lowest terms has a finite binary decimal expansion iff b is a power of 2.

41. *Proof.* Since $k > 0$, $2k + 1 > 0$. Since $(3k + 1)(-2) + (2k + 1)(3) = 1$, we have $\gcd(3k + 1, 2k + 1) = 1$. $\square$

43. $\sqrt{2}$. It is not in $\mathbb{Q}$.

45. *Proof.* Suppose not. So we have $a, b \in \mathbb{Z}$ such that $\sqrt{3} = \frac{a}{b}$ is in lowest terms. So $b\sqrt{3} = a$. So $b^2 3 = a^2$. So $3 \mid a^2$. By Corollary 3.19, $3 \mid a$. Write $a = 3c$ for some $c \in \mathbb{Z}$. So $b^2 3 = a^2 = 9c^2$. So $b^2 = 3c^2$. So $3 \mid b^2$. By Corollary 3.19, $3 \mid b$. So $\gcd(a, b) \geq 3$. This contradicts the assumption that $\frac{a}{b}$ is in lowest terms. $\square$

47. *Proof.* Suppose not. So we have $a, b \in \mathbb{Z}$ such that $\sqrt{13} = \frac{a}{b}$ is in lowest terms. So $b\sqrt{13} = a$. So $b^2 13 = a^2$. So $13 \mid a^2$. By Corollary 3.19, $13 \mid a$. Write $a = 13c$ for some $c \in \mathbb{Z}$. So $b^2 13 = a^2 = 13^2 c^2$. So $b^2 = 13c^2$. So $13 \mid b^2$. By Corollary 3.19, $13 \mid b$. So $\gcd(a, b) \geq 13$. This contradicts the assumption that $\frac{a}{b}$ is in lowest terms. $\square$

49. *Proof.* Suppose not. So we have $a, b \in \mathbb{Z}$ such that $\sqrt[3]{2} = \frac{a}{b}$ is in lowest terms. So $b\sqrt[3]{2} = a$. So $b^3 2 = a^3$. So $2 \mid a^3$. By Corollary 3.19, $2 \mid a$. Write $a = 2c$ for some $c \in \mathbb{Z}$. So $b^3 2 = a^3 = 2^3 c^3$. So $b^3 = 2^2 c^3$. So $2 \mid b^3$. By Corollary 3.19, $2 \mid b$. So $\gcd(a, b) \geq 2$. This contradicts the assumption that $\frac{a}{b}$ is in lowest terms. $\square$

51. *Proof.* Suppose not. So we have $a, b \in \mathbb{Z}$ such that $\sqrt[3]{7} = \frac{a}{b}$ is in lowest

terms. So $b\sqrt[3]{7} = a$. So $b^3 7 = a^3$. So $7 \mid a^3$. By Corollary 3.19, $7 \mid a$. Write $a = 7c$ for some $c \in \mathbb{Z}$. So $b^3 7 = a^3 = 7^3 c^3$. So $b^3 = 7^2 c^3$. So $7 \mid b^3$. By Corollary 3.19, $7 \mid b$. So $\gcd(a, b) \geq 7$. This contradicts the assumption that $\frac{a}{b}$ is in lowest terms. □

53. *Proof.* Suppose not. So we can write $\log_2 3 = \frac{a}{b}$, for some $a, b \in \mathbb{Z}$ with $a, b > 0$. So $2^{\frac{a}{b}} = 3$. So $2^a = 3^b$. By the Fundamental Theorem of Arithmetic (uniqueness), $a = b = 0$. This is a contradiction. □

55. *Proof.* Suppose not. So we can write $\log_3 7 = \frac{a}{b}$, for some $a, b \in \mathbb{Z}$ with $a, b > 0$. So $3^{\frac{a}{b}} = 7$. So $3^a = 7^b$. By the Fundamental Theorem of Arithmetic (uniqueness), $a = b = 0$. This is a contradiction. □

57. No, $\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}} = 2 \in \mathbb{Z} \subseteq \mathbb{Q}$.
Observe that $(\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}})^2 = \frac{(\sqrt{2}+\sqrt{6})^2}{2+\sqrt{3}} = \frac{8+4\sqrt{3}}{2+\sqrt{3}} = 4$. Since $\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}}$ is certainly positive, it must be that $\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}} = 2$.

59. No.
$\sqrt{2} + (-\sqrt{2}) = 0$.

61. *Proof.* Suppose $r = \frac{1+\sqrt{5}}{2}$ is rational. So $\sqrt{5} = 2r - 1$. However, $2r - 1$ is rational, and $\sqrt{5}$ is irrational. This is a contradiction. □

63. *Proof.* Suppose not. So $r = \frac{15+7\sqrt{5}}{4}$ is rational. So $\sqrt{5} = \frac{4r-15}{7}$. However, $\frac{4r-15}{7}$ is rational, and $\sqrt{5}$ is irrational. This is a contradiction. □
We cannot have $\sqrt{5} = \frac{4r-15}{7}$ with the left-hand side irrational and the right-hand side rational. However, r being rational forces the right-hand side to be rational.

65. *Proof.* Suppose not. So $r = \frac{7-\sqrt{2}}{3+\sqrt{2}}$ is rational. So $\sqrt{2} = \frac{7-3r}{r+1}$. It is easy to check that $r + 1 \neq 0$. So $\frac{7-3r}{r+1}$ is rational. However, $\sqrt{2}$ is irrational. This is a contradiction. □
If $-1 = r = \frac{7-\sqrt{2}}{3+\sqrt{2}}$, then $-3 - \sqrt{2} = 7 - \sqrt{2}$, which is impossible.

67. $-\frac{1}{3}$ and $\frac{3}{2}$.
$6x^4 - 7x^3 + 3x^2 - 7x - 3 = (3x + 1)(2x - 3)(x^2 + 1)$.

69. None.
$x^4 - x^3 + 5x^2 - 6x - 6 = (x^2 - x - 1)(x^2 + 6)$.

71. *Sketch.* Observe that $\sqrt{10}$ is a root of $f(x) = x^2 - 10$. However, by the Rational Roots Theorem, $f(x)$ has no rational roots. □
The only possible rational roots of $f(x) = x^2 - 10$ are ± 1 and ± 10. However,

$f(\pm 1) = -9 \neq 0$ and $f(\pm 10) = 90 \neq 0$. Any roots of $f(x) = x^2 - 10$ must therefore be irrational.

73. *Sketch.* Observe that $\sqrt{6} + \sqrt{2}$ is a root of $f(x) = x^4 - 16x^2 + 16$. However, by the Rational Roots Theorem, $f(x)$ has no rational roots. $\square$
Let $x = \sqrt{6} + \sqrt{2}$. So $x^2 = 8 + 2\sqrt{12}$. So $x^2 - 8 = 2\sqrt{12}$. So $x^4 - 16x^2 + 64 = (2\sqrt{12})^2 = 48$. That is, $x^4 - 16x^2 + 16 = 0$. The only possible rational roots of $f(x) = x^4 - 16x^2 + 16$ are ± 1, ± 2, ± 4, ± 8, and ± 16. However, $f(\pm 1) = 1$, $f(\pm 2) = -32$, $f(\pm 4) = 16$, $f(\pm 8) = 3088$, and $f(\pm 16) = 61456$. Any roots of $f(x) = x^4 - 16x^2 + 16$ must therefore be irrational.

75. *Sketch.* Observe that $\sqrt{3 - 2\sqrt{2}}$ is a root of $f(x) = x^4 - 6x^2 + 1$. However, by the Rational Roots Theorem, $f(x)$ has no rational roots. $\square$
Let $x = \sqrt{3 - 2\sqrt{2}}$. So $x^2 = 3 - 2\sqrt{2}$. So $x^2 - 3 = -2\sqrt{2}$. So $x^4 - 6x^2 + 9 = (-2\sqrt{2})^2 = 8$. That is, $x^4 - 6x^2 + 1 = 0$. The only possible rational roots of $f(x) = x^4 - 6x^2 + 1$ are ± 1. However, $f(\pm 1) = -4 \neq 0$. Any roots of $f(x) = x^4 - 6x^2 + 1$ must therefore be irrational.

77. *Sketch.* Observe that $\frac{\sqrt[4]{3}}{\sqrt{2}}$ is a root of $f(x) = 4x^4 - 3$. However, by the Rational Roots Theorem, $f(x)$ has no rational roots. $\square$
Let $x = \frac{\sqrt[4]{3}}{\sqrt{2}}$. So $x^4 = \frac{3}{4}$. Hence, $4x^4 - 3 = 0$. The only possible rational roots of $f(x) = 4x^4 - 3$ are ± 1, $\pm \frac{1}{2}$, $\pm \frac{1}{4}$, ± 3, $\pm \frac{3}{2}$, and $\pm \frac{3}{4}$. However, $f(\pm 1) = 1$, $f(\pm \frac{1}{2}) = -\frac{11}{4}$, $f(\pm \frac{1}{4}) = -\frac{191}{64}$, $f(\pm 3) = 321$, $f(\pm \frac{3}{2}) = \frac{69}{4}$, and $f(\pm \frac{3}{4}) = -\frac{111}{64}$. Any roots of $f(x) = 4x^4 - 3$ must therefore be irrational.

79. *Proof.* Suppose $r = \frac{\pi + 1}{2}$ is rational. So $\pi = 2r - 1$. However, $2r - 1$ is rational, and π is irrational. This is a contradiction. $\square$

81. We prove the contrapositive.
Proof. Suppose $\sqrt{x}$ is rational. So $\sqrt{x} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Thus, $x = (\sqrt{x})^2 = \frac{a^2}{b^2}$, and $a^2, b^2 \in \mathbb{Z}$ with $b^2 \neq 0$. Hence x is rational. $\square$

83. *Proof.* Suppose $r \in \mathbb{Q}$. So $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Observe that r is a root of $f(x) = bx - a$ and is hence algebraic. $\square$

85. *Sketch.* Let $g(x) = r_n x^n + r_{n-1} x^{n-1} + \cdots + r_1 x + r_0$ be a polynomial with rational coefficients. For each $0 \leq i \leq n$, write $r_i = \frac{a_i}{b_i}$, where $a_i, b_i \in \mathbb{Z}$ with $b_i \neq 0$. Define $f(x) = g(x) \prod_{i=1}^{n} b_i$. Observe that $f(x)$ is a polynomial with integer coefficients, and f and g have exactly the same roots. Since the roots of f are algebraic, so are the roots of g. $\square$
Notice that multiplying $g(x)$ by $\prod_{i=1}^{n} b_i$ clears all of the denominators from the coefficients, yielding integer coefficients. Since $\prod_{i=1}^{n} b_i \neq 0$, it follows that $g(x) \prod_{i=1}^{n} b_i = 0$ if and only if $g(x) = 0$. Since $f(x) = 0$ if and only if $g(x) = 0$, we see that f and g have the same roots. By definition, the roots of f are

algebraic.

87. No. $\sqrt{2}$ is algebraic since it is a root of $x^2 - 2$.

89. *Proof.* Suppose not. So $2e$ is algebraic. Thus $2e$ is a root of some polynomial $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$, where $n \in \mathbb{Z}^+, c_n, c_{n-1}, \ldots, c_1, c_0 \in \mathbb{Z}$. That is, $0 = f(2e) = c_n 2^n e^n + c_{n-1}2^{n-1}e^{n-1} + \cdots + c_1 2e + c_0$. Define $g(x) = c_n 2^n x^n + c_{n-1}2^{n-1}x^{n-1} + \cdots + c_1 2x + c_0$. Since $c_n 2^n, c_{n-1}2^{n-1}, \ldots, c_1 2, c_0 \in \mathbb{Z}$, and $g(e) = 0$, we see that e is algebraic. This is a contradiction. □

Section 3.5

1. True.
$10 \mid (55 - 15)$.

3. False.
$6 \nmid (-7 - 21)$.

5. Thursday.
1/8/1987 is 284 days before 10/19/1987 and $-284 \bmod 7 = 3$. Note that Monday + 3 = Thursday.

7. 9 P.M.
$279 \bmod 24 = 15$ and 9 P.M. is 15 hours after 6 A.M.

9. Theorem: (a) $a \equiv a \pmod{n}$.
(b) If $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$.
(c) If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.
(a) *Sketch.* $a - a = 0$ and $n \mid 0$. □
(b) Since $b - a = -(a - b)$, if $n \mid (a - b)$, then $n \mid (b - a)$. (c) *Proof.* Suppose $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$. So $n \mid (a - b)$ and $n \mid (b - c)$. That is, $a - b = nj$ and $b - c = nk$ for some $j, k \in \mathbb{Z}$. Observe that $a - c = (a - b) + (b - c) = nj + nk = n(j + k)$. So $n \mid (a - c)$. That is, $a \equiv c \pmod{n}$. □

11. *Proof.* Suppose $a_1 \equiv a_2 \pmod{n}$. So $n \mid a_1 - a_2$. That is, $a_1 - a_2 = nk$ for some $k \in \mathbb{Z}$. Hence, $-a_1 - (-a_2) = -(a_1 - a_2) = -nk = n(-k)$. Since $-k \in \mathbb{Z}$, we see that $-a_1 \equiv -a_2 \pmod{n}$. □

13. *Proof.* Suppose $a \equiv b \pmod{n}$. So $a - b = nk$ for some $k \in \mathbb{Z}$. So $a^2 - b^2 = (a + b)(a - b) = (a + b)nk = n(a + b)k$. Since $(a + b)k \in \mathbb{Z}$, we see that $a^2 \equiv b^2 \pmod{n}$. □

15. *Proof.* Lemma 3.29 tells us that $[n_1 \bmod d] \equiv n_1 \pmod{d}$ and that $[n_2 \bmod d] \equiv n_2 \pmod{d}$. ($\rightarrow$) Suppose $n_1 \bmod d = n_2 \bmod d$. Then, $[n_2 \bmod d] \equiv n_1 \pmod{d}$. It now follows from Theorem 3.26 that $n_1 \equiv [n_2 \bmod d] \equiv n_2 \pmod{d}$. ($\leftarrow$) Suppose $n_1 \equiv n_2 \pmod{d}$. So $[n_1 \bmod d] \equiv n_2 \pmod{d}$.

Since $0 \leq n_1 \bmod d < d$, it follows from the uniqueness assertion in Lemma 3.29 that $n_1 \bmod d = n_2 \bmod d$. $\square$

17. $8763 + 536 \equiv 13 + 11 \equiv 24 \pmod{25}$.
$8763 \equiv 13 \pmod{25}$ and $536 \equiv 11 \pmod{25}$. So $8763 + 536 \equiv 13 + 11 \equiv 24 \pmod{25}$.

19. 4.
Note that $10^{5379} \equiv 0 \pmod{10}$ and $14 \equiv 4 \pmod{10}$.

21. 16.
$25^{17} \equiv (25^2)^8 25 \equiv (-2)^8 25 \equiv 256 \cdot 25 \equiv 9 \cdot 6 \equiv 54 \equiv 16 \pmod{19}$.

23. 1.
$20^{50} \equiv (-1)^{50} \equiv 1 \pmod{3}$.

25. 1.
$13^{200} \equiv (-1)^{200} \equiv 1 \pmod{7}$.

27. *Proof.* Let $n \in \mathbb{Z}$. If $n \equiv 0 \pmod{3}$, then $n^3 - n - 1 \equiv 0 - 0 - 1 \equiv -1 \equiv 2 \pmod{3}$. If $n \equiv 1 \pmod{3}$, then $n^3 - n - 1 \equiv 1 - 1 - 1 \equiv -1 \equiv 2 \pmod{3}$. If $n \equiv 2 \pmod{3}$, then $n^3 - n - 1 \equiv 8 - 2 - 1 \equiv 5 \equiv 2 \pmod{3}$. In each case, $n^3 - n - 1 \equiv 2 \pmod{3}$. $\square$

29. *Sketch.* If $n \equiv 1, 2,$ or $4 \pmod{7}$, then $n^3 \equiv 1 \pmod{7}$. If $n \equiv 3, 5,$ or $6 \pmod{7}$, then $n^3 \equiv -1 \pmod{7}$. $\square$
That is, $1^3 \equiv 1 \equiv 1 \pmod{7}$, $2^3 \equiv 8 \equiv 1 \pmod{7}$, $3^3 \equiv 27 \equiv -1 \pmod{7}$, $4^3 \equiv 64 \equiv 1 \pmod{7}$, $5^3 \equiv 125 \equiv -1 \pmod{7}$, and $6^3 \equiv 216 \equiv -1 \pmod{7}$,

31. (a) *Sketch.* We have $n \equiv 1, 3, 5,$ or $7 \pmod{8}$.
So $n^2 \equiv 1, 9, 25,$ or $49 \pmod{8}$, respectively.
That is, $n^2 \equiv 1 \pmod{8}$. $\square$
(b) Multiply both sides of $n^2 \equiv 1 \pmod{n}$ by n.

33. *Proof.* Suppose $n \equiv r \pmod{3}$. So $3 \mid (n - r)$. That is, $n - r = 3k$ for some $k \in \mathbb{Z}$. Observe that $2^n - 2^r = 2^r(2^{n-r} - 1) = 2^r(2^{3k} - 1) = 2^r(8^k - 1)$. Since $8 \equiv 1 \pmod{7}$, we have $8^k \equiv 1 \pmod{7}$. That is, $7 \mid 8^k - 1$. So $7 \mid 2^n - 2^r$. Therefore, $2^n \equiv 2^r \pmod{7}$. $\square$

35. *Proof.* Suppose $a \equiv b \pmod{m}$ and $a \equiv -b \pmod{n}$ So $m \mid (a - b)$ and $n \mid (a + b)$. Hence, $mn \mid (a + b)(a - b)$. That is, $mn \mid (a^2 - b^2)$. Therefore, $a^2 \equiv b^2 \pmod{mn}$ $\square$

37. 23.
See Exercise 19 from Section 3.3. Since $55(-5) + 12(23) = 1$, we see that $12 \cdot 23 \equiv 1 \pmod{55}$.

39. 7.
$18(7) + 25(-5) = 1$.

41. "PZSNQRURHGJUX".
We use $y = (2x+8) \bmod 27$. E.g., $D = 4$ encrypts to $(2\cdot4+8) \bmod 27 = 16 = P$.

43. "BORAT".
Note that 14 is a multiplicative inverse of 2 modulo 27. Hence, we use $x = 14(y-13) \bmod 27$. E.g., $Q = 17$ decrypts to $14(17-13) \bmod 27 = 2 = B$.

45. *Proof.* Suppose to the contrary that $x, y \in \{0, 1, \ldots, n-1\}$ and $x \equiv y \pmod{n}$. Say $x > y$. Observe that $0 \le x - y < n$. (In fact, $0 < x - y \le n - 1$.) However, $n \mid (x-y)$ and $n > x - y$ is impossible here. This is a contradiction. $\square$

47. *Sketch.* Existence of an inverse is given by Lemma 3.31. That a representative can be chosen in $\{0, 1, \ldots, n-1\}$ is given by Lemma 3.29. Its uniqueness is then guaranteed by Exercise 45. $\square$

49. 6.
Use $29 = 16+8+4+1$, and $13^2 \equiv 15$, $13^4 \equiv 71$, $13^8 \equiv 36$, $13^{16} \equiv 64 \pmod{77}$.

51. 79.
Use $17 = 16 + 1$, and $31^2 \equiv 4$, $31^4 \equiv 16$, $31^8 \equiv 82$, $31^{16} \equiv 25 \pmod{87}$.

53. Note that $n = 5 \cdot 11 = 55$.
(a) $y = 2$, since $x^a \bmod n = 8^7 \bmod 55 = 2$.
(b) $y = 14$, since $x^a \bmod n = 49^7 \bmod 55 = 14$.
(c) Discover, since $y^c \bmod n = 12^3 \bmod 55 = 23 =$ Discover.
(d) MasterCard, since $y^c \bmod n = 35^3 \bmod 55 = 30 =$ MasterCard.

55. 16.
By Fermat's Little Theorem, $10^{16} \equiv 1 \pmod{17}$. By hand, we can see that $10^2 \equiv -2 \pmod{17}$. So $10^{1000} \equiv (10^{16})^{62}10^8 \equiv 10^8 \equiv (10^2)^4 \equiv (-2)^4 \equiv 16 \pmod{17}$.

57. Corollary: If $a, p \in \mathbb{Z}$ with p prime, then $a^p \equiv a \pmod{p}$.
Sketch. When $p \mid a$, $a^p \equiv 0 \equiv a \pmod{p}$. When $p \nmid a$, multiply both sides of $a^{p-1} \equiv 1 \pmod{p}$ by a. $\square$
We apply Fermat's Little Theorem in the case that $p \nmid a$.

59. *Sketch.* $2^{253} \equiv (2^8)^{31}2^5 \equiv 256^{31}2^5 \equiv 3^{31}2^5 \equiv (3^5)^6 3 \cdot 2^5 \equiv 243^6 3 \cdot 2^5 \equiv (-10)^6 3 \cdot 2^5 \equiv 162 \not\equiv 2 \pmod{253}$. $\square$

61. *Sketch.* Observe that $x \in \{1, \ldots, p-1\}$ and $x^2 \equiv 1 \pmod{p}$ if and only if

$x = 1$ or $p-1$. Hence, the suggested pairing off of values in the product $(p-1)!$ gives $(p-1)! = (p-1) \cdot 1^{\frac{p-3}{2}} \cdot 1 \equiv -1 \pmod{p}$. $\square$
Note that p divides $x^2 - 1 = (x-1)(x+1)$ if and only if $p \mid x-1$ or $p \mid x+1$. With $x \in \{1, \ldots, p-1\}$, this happens only if $x - 1 = 0$ or $x + 1 = p$. That is, $x = 1$ or $x = p - 1$.

63. $10d_1 + 9d_2 + 8d_3 + 7d_4 + 6d_5 + 5d_6 + 4d_7 + 3d_8 + 2d_9 + d_{10} =$
$-(d_1 + 2d_2 + 3d_3 + 4d_4 + 5d_5 + 6d_6 + 7d_7 + 8d_8 + 9d_9 + 10d_{10})+$
$11(d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8 + d_9 + d_{10})$.
We appeal to Lemma 3.29, so that we may work with congruence modulo 11. The above equation shows that $[10d_1 + 9d_2 + 8d_3 + 7d_4 + 6d_5 + 5d_6 + 4d_7 + 3d_8 + 2d_9 + d_{10}] \equiv -[d_1 + 2d_2 + 3d_3 + 4d_4 + 5d_5 + 6d_6 + 7d_7 + 8d_8 + 9d_9 + 10d_{10}] \pmod{11}$. If these are congruent to 0 modulo 11, then so are their negatives.

65. $x = 3$.
Note that $3^2 \equiv 9 \equiv -1 \pmod{10}$.

67. $x = 2, y = 5$.
Note that $2 \not\equiv 0 \pmod{10}$, $5 \not\equiv 0 \pmod{10}$, and $2 \cdot 5 \equiv 0 \pmod{10}$.

69. $[2]_3$.
Note that $8 \equiv 2 \pmod{3}$ and $0 \leq 2 < 3$.

71. $[3]_4$.
Note that $10 + 5 \equiv 15 \equiv 3 \pmod{4}$ and $0 \leq 3 < 4$.

73. $[1]_{10}$.
Note that $18 + 217 + 3146 \equiv 8 + 7 + 6 \equiv 21 \equiv 1 \pmod{10}$ and $0 \leq 1 < 10$.

75. $\{k : k \equiv a \pmod{n}\} = \{k : k \equiv b \pmod{n}\}$ if and only if $a \equiv b \pmod{n}$.
Proof. $(\rightarrow)$ Suppose $[a]_n = [b]_n$. Since $b \in [b]_n = [a_n]$, the definition of $[a]_n$ gives $b \equiv a \pmod{n}$. $(\leftarrow)$ Suppose $a \equiv b \pmod{n}$. Since it follows that $k \equiv a \pmod{n}$ if and only if $k \equiv b \pmod{n}$, we get $[a]_n = [b]_n$. $\square$

77. *Proof.* $(\subseteq)$ Suppose $k \in [a]_n + [b]_n$. So $k = s + t$ for some $s \in [a]_n$ and $t \in [b]_n$. Thus $s \equiv a \pmod{n}$ and $t \equiv b \pmod{n}$. Since $s + t \equiv a + b \pmod{n}$, it follows that $k \in [a+b]_n$. $(\supseteq)$ Suppose $k \in [a+b]_n$. So $k \equiv a + b \pmod{n}$. That is, $n \mid (k - a - b)$. Note that $k = a + (k - a)$ and $k - a \equiv b \pmod{n}$. Therefore, $k \in [a]_n + [b]_n$. $\square$

79. *Sketch.* By Exercises 77 and 78, $[a]_n - [b]_n = [a]_n + [-b]_n = [a-b]_n$. $\square$

81. *Sketch.* Any $k \equiv a \pmod{n}$ can be written as $k = 0 + k$. $\square$
This is also a special case of Exercise 77.

83. Note that $n = 10(10^{k-1}a_k + 10^{k-2}a_{k-1} + \cdots + a_1) + a_0$ and $10(10^{k-1}a_k + 10^{k-2}a_{k-1} + \cdots + a_1) \equiv 0 \pmod 5$. So $n \equiv 0 + a_0 \equiv a_0 \pmod 5$.

85. Note that $10^k a_k + 10^{k-1}a_{k-1} + \cdots + 10^0 a_0 \equiv 1^k a_k + 1^{k-1}a_{k-1} + \cdots + 1^0 a_0 \pmod 9$. Also, n is divisible by 3 iff $n \equiv 0, 3$, or $6 \pmod 9$.
Note that $10 \equiv 1 \pmod 9$. So

$$\begin{aligned} n &\equiv 10^k a_k + 10^{k-1}a_{k-1} + \cdots + 10^0 a_0 \\ &\equiv 1^k a_k + 1^{k-1}a_{k-1} + \cdots + 1^0 a_0 \\ &\equiv a_k + a_{k-1} + \cdots + a_0 \\ &\equiv m \pmod 9. \end{aligned}$$

Observe that n is divisible by 3 iff $n \equiv 0, 3$, or $6 \pmod 9$.

Review

1. *Proof.* Let m and n be even integers. So $m = 2j$ and $n = 2k$ for some $j, k \in \mathbb{Z}$. Observe that $mn = 2(2jk)$. Since $2jk \in \mathbb{Z}$, we see that mn is even. □

2. *Proof.* Suppose n is even. So $n = 2k$ for some $k \in \mathbb{Z}$. So $n^2 = 4k^2$. Hence, $4 \mid n^2$. □

3. No,
since $6 \nmid 52$.

4. *Sketch.* $(a+b)^3 - b^3 = a(a^2 + 3ab + 3b^2)$. □
$(a+b)^3 - b^3 = a^3 + 3a^2b + 3ab^2 + b^3 - b^3 = a(a^2 + 3ab + 3b^2)$ and $a^2 + 3ab + 3b^2 \in \mathbb{Z}$.

5. *Proof.* ($\rightarrow$) Suppose $a \mid b$. So $b = ak$ for some $k \in \mathbb{Z}$. Since $b = (-a)(-k)$, we see that $-a \mid b$. ($\leftarrow$) Suppose $-a \mid b$. So $b = -ak$ for some $k \in \mathbb{Z}$. Since $b = a(-k)$, we see that $a \mid b$. □

6. No, $91 = 7 \cdot 13$.
No, by definition, primes are greater than 1.

7. Yes.
$\gcd(14, 33) = 1$.

8. *Proof.* Suppose $a \mid n$ and $a \mid (n+2)$. So $n = aj$ and $n + 2 = ak$ for some $j, k \in \mathbb{Z}$. Observe that $2 = (n+2) - n = ak - aj = a(k - j)$. So $a \mid 2$. □

9. 91.
$\gcd(7 \cdot 11 \cdot 13, 5 \cdot 7 \cdot 13) = 7 \cdot 13 = 91$.

10. *Proof.* Let $n \in \mathbb{Z}$ with $n \neq 0$. *Case 1*: $n > 0$. So $n > 0$, $n \mid n$, and

$n \mid -n$. Also, if $c \mid n$ and $c \mid -n$, then, in particular, $c \mid n$, whence $c \leq n$. So $\gcd(n, -n) = n = |n|$. *Case 2*: $n < 0$. So $-n > 0$, $-n \mid n$, and $-n \mid -n$. Also, if $c \mid n$ and $c \mid -n$, then, in particular, $c \mid -n$, whence $c \leq -n$. So $\gcd(n, -n) = -n = |n|$. $\square$
In each case ($d = n$ or $d = -n$), we verify that $\gcd(n, -n) = d$ by checking the three conditions:
(i) $d > 0$,
(ii) $d \mid n$ and $d \mid -n$, and
(iii) if $c \mid n$ and $c \mid -n$, then $c \leq d$.

11. (a) $a = 2, b = 3, m = 2, n = 2$.
Note that $\gcd(2, 3) = 1 = \gcd(2^2, 3^2)$.
(b) *Proof.* Let $d = \gcd(a, b)$. Since $d \mid a$ and $d \mid b$, it follows that $d \mid a^m$ and $d \mid b^n$. Therefore $d \leq \gcd(a^m, b^n)$. $\square$
(c) $\gcd(a, b) > 1$, since if $\gcd(a, b) = 1$, then $\gcd(a^m, b^n) = 1$. At least one of $m > 1$ or $n > 1$, since $\gcd(a, b) = \gcd(a^1, b^1)$. Note that $\gcd(4, 6) < \gcd(4^1, 6^2)$. So both $m, n > 1$ is not forced.

12. 840.
$120 = 12 \cdot 10$, $84 = 12 \cdot 7$, and $12 \cdot 10 \cdot 7 = 840$.

13. *Sketch.* Let $i = \max\{j, k\}$. So $m^i > 0$, $m^j \mid m^i$, and $m^k \mid m^i$. If $c \in \mathbb{Z}^+$, $m^j \mid c$, and $m^k \mid c$, then $m^i \mid c$, whence $m^i \leq c$. $\square$
We verify that $\operatorname{lcm}(m^j, m^k) = l$ by checking the three conditions:
(i) $l > 0$,
(ii) $m^j \mid l$ and $m^k \mid l$, and
(iii) if $m^j \mid c$ and $m^k \mid c$, then $l \leq c$.

14. 5.
Observe that $5 = 10(-2) + 25(1)$. Note that $5 \mid (10x + 25y)$. Since 5 does not divide 1, 2, 3, or 4, there is no element smaller than 5.

15. False.
$2^{113} - 1 = 3391 \cdot 23279 \cdot 65993 \cdot 1868569 \cdot 1066818132868207$.

16. 12 remainder 5.
$101 = 8(12) + 5$ and $0 \leq 5 < 8$.

17. (a) 6 and 1, since $43 = 7(6) + 1$ and $0 \leq 1 < 7$.
(b) -6 and 3, since $-51 = 9(-6) + 3$ and $0 \leq 3 < 9$.

18. 2 remain, and each have 17.
$104 \bmod 6 = 2$. $104 \operatorname{div} 6 = 17$.

19. *Proof.* Write $n = 3q + r$, where $r = 0$, 1, or 2.
So $n^3 - n = 27q^3 + 9q^2r + 3qr^2 + r^3 - (3q + r) = 3(9q^3 + 3q^2r + qr^2 - q) + r^3 - r$.

If $r = 0$, then $r^3 - r = 0$. If $r = 1$, then $r^3 - r = 0$. If $r = 2$, then $r^3 - r = 3(2)$. In each case, $3 \mid (r^3 - r)$, so $3 \mid (n^3 - n)$. Thus, $(n^3 - n) \bmod 3 = 0$. $\square$
With the tools of Section 3.5, the following argument also works.
Sketch. If $n \equiv 0 \pmod 3$, then $n^3 - n \equiv 0 - 0 \equiv 0 \pmod 3$.
If $n \equiv 1 \pmod 3$, then $n^3 - n \equiv 1 - 1 \equiv 0 \pmod 3$.
If $n \equiv 2 \pmod 3$, then $n^3 - n \equiv 8 - 2 \equiv 6 \equiv 0 \pmod 3$. $\square$

20. *Proof.* Suppose n is odd. So $n = 2k + 1$ for some $k \in \mathbb{Z}$. So $n^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. Since $n^2 \bmod 4 = 1 \neq 0$, we see that $4 \nmid n^2$. $\square$
Alternative via contrapositive. *Proof.* Suppose $4 \mid n^2$. So $n^2 = 4k$ for some $k \in \mathbb{Z}$. Since $n^2 = 2(2k)$, we see that n^2 is even. Hence, n is even. That is, n is not odd. $\square$

21. (a) 6, since $6 \leq 6.6 < 7$.
(b) -7, since $-7 \leq -6.6 < -6$.

22. (a) 6, since $5 < 5.4 \leq 6$.
(b) -5, since $-6 < -5.4 \leq -5$.

23. *Proof.* Suppose $4 \mid n$. So $n = 4k$ for some $k \in \mathbb{Z}$. Thus, $\frac{n}{4} = k \in \mathbb{Z}$. Also, $\frac{n}{4} \leq \frac{n+2}{4}$ and $\frac{n+2}{4} < \frac{n}{4} + 1$. Therefore, $\lfloor \frac{n+2}{4} \rfloor = \frac{n}{4}$. $\square$

24. *Sketch.* Certainly $\lceil x \rceil \in \mathbb{Z}$ and $\lceil x \rceil - 1 < \lceil x \rceil \leq \lceil x \rceil$. $\square$
Since $\lceil x \rceil$ is an integer, this is effectively the fact that $\forall\, n \in \mathbb{Z},\ \lceil n \rceil = n$.

25. 3. No.
Let $c = \#$. So $10(0) + 9(8) + 8(2) + 7(1) + 6(8) + 5(c) + 4(4) + 3(6) + 2(1) + 4 = 183 + 5c$. Since $11 \mid (183 + 5 \cdot 3)$, we get $c = 3$. Suppose the check digit 4 was also smudged. Call its now unknown value d. So $179 + 5c + d$ is divisible by 11 both when $c = 3, d = 4$ and when $c = 2, d = 9$.

26. "MDPSZIDXSQ".
Use $y = x + 4 \bmod 27$. E.g., $I = 9$ encrypts to $y = 9 + 4 \bmod 27 = 13 = M$.

27. $\frac{n}{\gcd(b,n)}$.
Note in the given example that $9 = \frac{27}{\gcd(6,27)}$. In general, let m be the number of cycles. So $mk = n$. By using the ideas in Exercise 45(c) from Section 3.1, we see that $m = \gcd(b, n)$.

28. $x = 3$ and $y = -2$.
$35(3) + 49(-2) = 7$.

29. *Sketch.* Let $x = -11, y = 24$. $\square$
Observe that $85(-11) + 39(24) = 1$.

30. $\gcd(110, 88) = \gcd(88, 22) = \gcd(22, 0) = 22$.

31. $\gcd(810, 245) = \gcd(245, 75) = \gcd(75, 20) = \gcd(20, 15) = \gcd(15, 5) = \gcd(5, 0) = 5$.

32. $x = -1, y = 2$.
Note that $\gcd(81, 45) = 9$, $81 = 45 + 36$, and $45 = 36 + 9$. So $9 = 45 - 36 = 45 - (81 - 45) = 85(-1) + 45(2)$.

33. $x = 5, y = -16$.
Note that $\gcd(77, 24) = 1$, $77 = 3(24) + 5$, $24 = 4(5) + 4$, and $5 = 4 + 1$. So $1 = 5 - 4 = 5 - (24 - 4(5)) = 24(-1) + 5(5) = 24(-1) + (77 - 3(24))(5) = 77(5) + 24(-16)$.

34. *Proof.* Suppose $5 \mid a^m b^n$. Since 5 is prime, Corollary 3.17 tells us that $5 \mid a^m$ or $5 \mid b^n$. By Corollary 3.19, it follows that $5 \mid a$ or $5 \mid b$. □

35. *Sketch.* $6 \mid 52n$ iff $3 \mid 26n$ iff $3 \mid n$, by Euclid's Lemma. □
Note that $52n = 6k$ iff $26n = 3k$. Also, $\gcd(3, 26) = 1$.

36. No.
Take $a = 2$ and $b = 5$.

37. (a) Since $5 \mid (25x + 10y)$ for all integers x, y, only multiples of 5 can be achieved. For example, all values of the form $5k + 1$ cannot be achieved.
(b) 5¢ and 15¢.
(c) n¢, for all odd $n < 25$.

38. *Proof.* Observe that $6\frac{3}{4} = \frac{27}{4}$ and $27, 4 \in \mathbb{Z}$ with $4 \neq 0$. Thus, $6\frac{3}{4} \in \mathbb{Q}$. □

39. *Proof.* Observe that $1.\overline{414} = \frac{157}{111}$ and $157, 111 \in \mathbb{Z}$ with $111 \neq 0$. Thus, $1.\overline{414} \in \mathbb{Q}$. □
Since $(\frac{157}{111})^2 = \frac{24649}{12321} > \frac{24642}{12321} = 2$, it follows that $1.\overline{414} > \sqrt{2}$.

40. *Proof.* Observe that $1.6\overline{25} = \frac{1609}{990}$ and $1609, 990 \in \mathbb{Z}$ with $990 \neq 0$. □

41. *Proof.* Suppose $r \in \mathbb{Q}$. So $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Observe that $\frac{3r}{4} = \frac{3a}{4b}$ and $3a, 4b \in \mathbb{Z}$ with $4b \neq 0$. So $\frac{3r}{4} \in \mathbb{Q}$. □

42. *Proof.* Suppose $r \in \mathbb{Q}$. So $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Observe that $r^2 = \frac{a^2}{b^2}$ and $a^2, b^2 \in \mathbb{Z}$ with $b^2 \neq 0$. So $r^2 \in \mathbb{Q}$. □

43. *Sketch.* Suppose p is a prime such that $p \mid a^2$ and $p \mid b^2$. It follows that $p \mid a$ and $p \mid b$. So $p \mid \gcd(a, b)$. Thus, if $\gcd(a, b) = 1$, then it must also be that $\gcd(a^2, b^2) = 1$. □

44. $0.\overline{45}$.

$$\begin{array}{rrl} & .\;4\;5 & \\ 11\;\big) & 5.\;0\;0\;0 & \\ - & 4\;4 & \\ \hline & 6\;0 & \text{remainder } 6 \\ - & 5\;5 & \\ \hline & 5 & \text{remainder } 5 \leftarrow \text{REPEAT} \end{array}$$

45. *Sketch.* Write $\sqrt{7} = \frac{a}{b}$ in lowest terms. So $b\sqrt{7} = a$. So $b^2 7 = a^2$. So $7 \mid a^2$. By Corollary 3.19, $7 \mid a$. Write $a = 7c$. So $b^2 7 = a^2 = 49c^2$. So $b^2 = 7c^2$. So $7 \mid b^2$. By Corollary 3.19, $7 \mid b$. So $\gcd(a, b) \geq 7$. This is a contradiction. □

46. *Proof.* Suppose $r = \frac{5+\sqrt{7}}{3}$ is rational. So $\sqrt{7} = 3r - 5$. However, $3r - 5$ is rational, and $\sqrt{7}$ is irrational. This is a contradiction. Therefore, $r = \frac{5+\sqrt{7}}{3}$ must be irrational. □

47. *Sketch.* Write $\log_3 11 = \frac{a}{b}$ with $a, b \in \mathbb{Z}^+$. So $3^{\frac{a}{b}} = 11$. So $3^a = 11^b$. By the Fundamental Theorem of Arithmetic, this is impossible. □

48. *Sketch.* (a) Suppose $r = \frac{e^2-4}{3} \in \mathbb{Q}$. However, we get $e^2 = 3r + 4 \in \mathbb{Q}$.
(b) Write $\ln 2 = \frac{a}{b}$ for $a, b \in \mathbb{Z}^+$. So $e^{\frac{a}{b}} = 2$. However, $e^a = 2^b \in \mathbb{Z}$. □

49. *Sketch.* Observe that $\sqrt{3+\sqrt{2}}$ is a root of $f(x) = x^4 - 6x^2 + 7$. By the Rational Roots Theorem, f has no rational roots. □
The only possibilities $\pm 1, \pm 7$ are not roots.

50. *Sketch.* Observe that $\frac{\sqrt[3]{2}}{\sqrt{5}}$ is a root of $f(x) = 125x^6 - 4$. By the Rational Roots Theorem, f has no rational roots. □
The only possibilities $\pm\frac{2^a}{3^b}$, for $0 \leq a \leq 3$ and $0 \leq b \leq 2$, are not roots.

51. *Sketch.* Observe that $\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$ is a root of $f(x) = 256x^8 - 512x^6 + 320x^4 - 64x^2 + 2$. By the Rational Roots Theorem, f has no rational roots. □
The only possibilities $\pm\frac{1}{2^a}$, for $0 \leq a \leq 7$, are not roots.

52. No, it equals 4.
Call it x, and observe that $x^2 = 16$.

53. Yes, they are the same as the roots of $15x^2 - 8x + 12$.
That is, multiplying the given polynomial by 12 clears the denominators and leaves integer coefficients.

54. Wednesday,
since $(28 + 31 + 30 + 31 + 30 + 31 + 31 + 30 + 27) \bmod 7 = 3$ and Sunday + 3 = Wednesday.

55. *Proof.* Suppose $a \equiv b \pmod{n}$. So $n \mid (a-b)$. That is, $a-b = nk$ for some $k \in \mathbb{Z}$. So $ac - bc = (a-b)c = nkc$. Since $n \mid (ac-bc)$, it follows that $ac \equiv bc \pmod{n}$. □

56. (a) 7.
Note that $11 \equiv 2 \pmod{9}$ and $11^{10} \equiv 2^{10} \equiv 1024 \equiv 7 \pmod{9}$.
(b) 11.
Note that $23 \equiv -1 \pmod{12}$ and $23^{4321} \equiv (-1)^{4321} \equiv -1 \equiv 11 \pmod{12}$.

57. *Proof.* Suppose n is odd. So $n \equiv 1$ or $3 \pmod{4}$. If $n \equiv 1 \pmod{4}$, then $n^2 \equiv 1^2 \equiv 1 \pmod{4}$. If $n \equiv 3 \pmod{4}$, then $n^2 \equiv 3^2 \equiv 9 \equiv 1 \pmod{4}$. In both cases, $n^2 \equiv 1 \pmod{4}$. □

58. *Sketch.* If $n \equiv 1 \pmod{3}$, then $n^2 \equiv 1^2 \equiv 1 \pmod{3}$.
If $n \equiv 2 \pmod{3}$, then $n^2 \equiv 2^2 \equiv 4 \equiv 1 \pmod{3}$. □

59. "RSA".
Note that $c = 7$ is a multiplicative inverse of $a = 4$ modulo 27.
Use $x = 7(y-1) \bmod 27$.
E.g., $S = 19$ decrypts to $7(19-1) \bmod 27 = 126 \bmod 27 = 18 = R$.

60. -9,
since $11(-9) + 50(2) = 1$.

61. 172.
Use $49 = 32 + 16 + 1$, and $19^2 \equiv -30$, $19^4 \equiv 118$, $19^8 \equiv 239$, $19^{16} \equiv 35$, $19^{32} \equiv 52 \pmod{391}$.

62. (a) 32. (b) "The package has been received." is the message that was sent.
Note that $n = 7 \cdot 13 = 91$ and $m = \mathrm{lcm}(6, 12) = 6$. Moreover, $c = 5$ is a multiplicative inverse of $a = 17$ modulo 6.

63. 4.
By Fermat's Little Theorem, $9^{10} \equiv 1 \pmod{11}$. So $9^{5432} \equiv 9^2 \equiv 4 \pmod{11}$.

64. $[2]_5$,
since $7 \equiv 2 \pmod{5}$.

65. $[1]_3$,
since $8 + 2 \equiv 10 \equiv 1 \pmod{3}$.

66. $[4]_7$,
since $17 - 208 + 1343 \equiv 3 - 5 + 6 \equiv 4 \pmod{7}$.

67. *Proof.* Since $a \equiv a \pmod{n}$, we have $a \in [a]_n$. □

2.4 Chapter 4

Section 4.1

1. 3628800.
$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3628800$.

3. 21.
$\frac{7!}{5!2!} = \frac{7 \cdot 6}{2} = 21$.

5. 126.
$\frac{9!}{4!5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2} = 126$.

7. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$.

9. False.
It fails for $n = 2$, since $(2^2)! = 24$ and $(2!)^2 = 4$.

11. False.
It fails for $n = 4$, since $2! \neq 12 = \frac{4!}{2}$.

13. $4, 2, 0, -2$.
$4 - 2(0) = 4$, $4 - 2(1) = 2$, $4 - 2(2) = 0$, $4 - 2(3) = -2$.

15. $6, 12, 40, 180$.
$\frac{3!}{3-2} = 6$, $\frac{4!}{4-2} = 12$, $\frac{5!}{5-2} = 40$, $\frac{6!}{6-2} = 180$.

17. $7, 9, 11, 13$.
$3 + 2(2) = 7$, $3 + 2(3) = 9$, $3 + 2(4) = 11$, $3 + 2(5) = 13$.

19. 47.
$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$.

21. $\forall\, n \geq 1,\ t_n = 2n$.
This is an arithmetic sequence with common difference $c = 2$ and $s_0 = 2$. So $\forall\, n \geq 0,\ s_n = 2 + 2n$ works in addition to the above formula.

23. $\forall\, n \geq 0,\ s_n = 3 \cdot 2^n$.
This is a geometric sequence with multiplying factor $r = 2$ and $s_0 = 3$.

25. $\forall\, n \geq 0,\ s_n = (-1)^n(2n + 1)$.
This is an alternating sequence. Thus the factor $(-1)^n$ appears in the formula. If we remove the signs, then the sequence becomes $1, 3, 5, 7, 9, \ldots$. This is an arithmetic sequence and can also be seen to be the sequence of odd positive integers.

27. $\forall\, n \geq 1,\ s_n = \frac{1}{n}$.
The terms are fractions with numerator one. The sequence of denominators is $1, 2, 3, 4, 5, \ldots$.

29. (a) $6000(1.03) = 6180$ after 1 period. $6180(1.03) = 6365.40$ after 2 periods.
(b) $1000(1 + i)^2$. No, since $i = .02$ yields \$40.40 in interest and $i = .01$ yields \$20.10.
(c) $\forall\, n \geq 0,\ s_n = P(1 + i)^n$. A geometric sequence.
Note that $s_0 = P$, $s_1 = P + Pi = P(1 + i)$, $s_2 = s_1(1 + i) = P(1 + i)(1 + i) = P(1 + i)^2$, etc. This is a geometric sequence with multiplying factor $r = i$ and $s_0 = P$.

31. $\forall\, n \geq 0,\ t_n = 10^{n-2}$.
Let $m = n - 1$. So $n = m + 1$. Thus, $s_n = 10^{n-3} = 10^{m+1-3} = 10^{m-2} = t_m$.

33. $\forall\, n \geq 0,\ t_n = 7 + 2n$.
Let $m = n-2$. So $n = m+2$. Thus, $s_n = 3+2n = 3+2(m+2) = 7+2m = t_m$.

35. $\forall\, n \geq 0,\ t_n = (-1)^n \frac{n}{n+2}$.
Let $m = n - 2$. So $n = m + 2$. Thus, $s_n = (-1)^n \frac{n-2}{n} = (-1)^{m+2} \frac{m+2-2}{m+2} = (-1)^m(-1)^2 \frac{m}{m+2} = (-1)^m \frac{m}{m+2} = t_m$.

37. $4, 10, 28, 82$.
$s_1 = 4$, $s_2 = 3s_1 - 2 = 3(4) - 2 = 10$, $s_3 = 3s_2 - 2 = 3(10) - 2 = 28$, $s_4 = 3s_3 - 2 = 3(28) - 2 = 82$.

39. $5, 3, 1, -1$.
$s_2 = 5$, $s_3 = s_2 - 2 = 5 - 2 = 3$, $s_4 = s_3 - 2 = 3 - 2 = 1$, $s_4 = s_3 - 2 = 1 - 2 = -1$.

41. $-1, -4, -19, -94$. Use $s_1 = -\frac{1}{4}$.
In general, $s_{m+1} = 5s_m + 1 \geq s_m$ iff $4s_m \geq -1$ iff $s_m \geq \frac{-1}{4}$.

43. $t_1 = 2$, and $\forall\, n \geq 2,\ t_n = 2 + t_{n-1}$.
This is an arithmetic sequence with common difference $c = 2$ and $s_0 = 2$. So $t_0 = 2$, and $\forall\, n \geq 1,\ t_n = 2 + t_{n-1}$ also works.

45. $s_0 = 3$, and $\forall\, n \geq 1,\ s_n = 2s_{n-1}$.
This is a geometric sequence with multiplying factor $r = 2$ and $s_0 = 3$.

47. $s_0 = 1$, and $\forall\, n \geq 1,\ s_n = s_{n-1} + (-1)^n 4n$.
Consider the differences between terms: $-3 - 1 = -4$, $5 - (-3) = 8$, $-7 - 5 = -12$, $9 - (-7) = 16$, etc. So the differences form an alternating sequence of multiples of 4. That is, $s_n - s_{n-1} = (-1)^n 4n$.

49. $s_1 = 1$, and $\forall\, n \geq 2,\ s_n = \frac{s_{n-1}}{1+s_{n-1}}$.

51. (a) $s_2 = 1000(1.05)+1000 = 2050$, $s_3 = s_2(1+i)+D = 2050(1.05)+1000 = 3152.50$. (b) $s_{11} = s_{10}(1.04) + 100 = 1348.63$. (c) $s_0 = 0$, and $\forall\, n \geq 1,\ s_n = (1+i)s_{n-1} + D$. Notice the pattern in the previous parts.

53. $s_{k+1} = 3s_{(k+1)-1} - 2 = 3s_k - 2$.

55. $s_{k+1} = s_{(k+1)-1} - 2 = s_k - 2$.

57. $s_2 = 5(1) - 3(-1) = 8$,
$s_3 = 5(-1) - 3(8) = -29$,
$s_{k+1} = 5s_{(k+1)-2} - 3s_{(k+1)-1} = 5s_{k-1} - 3s_k$.

59. $\forall\, n \geq 2,\ s_n = s_{n-1} - 2$.
Let $m = n + 1$. So $\forall\, m \geq 2,\ s_m = s_{m-1} - 2$.

61. $\forall\, n \geq 2,\ s_n = 5s_{n-2} - 3s_{n-1}$.
Let $m = n + 2$. So $\forall\, m \geq 2,\ s_m = 5s_{m-2} - 3s_{m-1}$.

63. (a) In *Mathematica*, use

```
In[1]:= AppRt2[n_] := 1 + 1/(1 + AppRt2[n - 1])
In[2]:= AppRt2[0] := 1
```

(b) 1.41421. (c) The 12th.
You should play with this in *Mathematica* or some other mathematical software.

Section 4.2

1. $\frac{65}{24} \approx 2.708$. Here, $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = \frac{65}{24}$.

3. $\frac{25 \cdot 26 \cdot 51}{6} = 5525$.

5. $\sum_{i=1}^{10} i^3 = 3025$. Apply Theorem 4.2(d) with $n = 10$, since $10^3 = 1000$.

7. $\sum_{i=1}^{10} 2^i = 2047$.
Note $1 = 2^0$ and $1024 = 2^{10}$. Apply Theorem 4.3 with $r = 2$ and $n = 10$.

9. $\sum_{i=1}^{9} (-2)^i = -342$.
Theorem 4.3 with $r = -2$ and $n = 9$ gives $1-2+4-8+16-\cdots-512 = -341$.
From this we must subtract the extraneous $1 = (-2)^0$.

11. $\sum_{i=2}^{n} 3i^2 = \frac{n(n+1)(2n+1)}{2} - 3$.
The last term $3n^2$ gives us a clue. Note that $3i^2$ works for the general term and that the first term is when $i = 2$.

13. $\sum_{i=1}^{n} 4^i = \frac{4}{3}(4^n - 1)$. Note that $4 = 4^1$, so our sum starts with $i = 1$. Since the formula in Theorem 4.3 requires starting at $i = 0$, we must subtract the $i = 0$ term from that sum. That is, $\frac{4^{n+1}-1}{4-1} - 4^0 = \frac{4}{3}(4^n - 1)$.

15. $\sum_{i=2}^{n}(-3)^i = \frac{9-(-3)^{n+1}}{4}$.
Theorem 4.3 with $r = -3$ gives $\frac{(-3)^{n+1}-1}{-3-1}$. From this we must subtract the extraneous $(-3)^0 + (-3)^1$.

17. (a) $s_2 = D(1+i) + D$, $s_3 = s_2(1+i) + D = [D(1+i) + D](1+i) + D = D(1+i)^2 + D(1+i) + D$. $s_4 = s_3(1+i) + [D(1+i)^2 + D(1+i) + D](1+i) + D = D(1+i)^3 + D(1+i)^2 + D(1+i) + D$. (b) $100(1.01)^3 + 100(1.01)^2 + 100(1.01) + 100 = 406.04$. (c) $s_n = D(1+i)^{n-1} + D(1+i)^{n-2} + \cdots + D(1+i) + D = \sum_{j=0}^{n-1} D(1+i)^j = D\sum_{j=0}^{n-1}(1+i)^j = D\frac{(1+i)^n-1}{(1+i)-1} = D\frac{(1+i)^n-1}{i}$. (d) $10000 = D\frac{(1.01)^{12}-1}{.01}$ gives $D = 1000\frac{.01}{(1.01)^{12}-1} = 788.49$. (e) Since $F = D\frac{(1+i)^n-1}{i}$, deposit $D = \frac{iF}{(1+i)^N-1}$.

19. $4\sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 4[\frac{n(n+1)}{2}]^2 - 6\frac{n(n+1)}{2} - n =$
$n^4 + 2n^3 - 2n^2 - 4n = n(n+2)(n^2 - 2)$.

21. Let $j = i - 1$. So $\sum_{j=0}^{n-1} j^2 = \sum_{j=1}^{n-1} j^2 = \frac{(n-1)n[2(n-1)+1]}{6} = \frac{n(n-1)(2n-1)}{6} = \frac{2n^3-3n^2+n}{6}$.

23. $\frac{3-(\frac{1}{3})^n}{2}$. Here, $\frac{1-(\frac{1}{3})^{n+1}}{1-\frac{1}{3}} = \frac{3}{3} \cdot \frac{1-(\frac{1}{3})^{n+1}}{1-\frac{1}{3}} = \frac{3-(\frac{1}{3})^n}{2}$.

25. $2^{101} - 2^{10}$. Note $1024 = 2^{10}$.
We have $\sum_{i=10}^{100} 2^i = \sum_{i=0}^{100} 2^i - \sum_{i=0}^{9} 2^i = \frac{2^{101}-1}{2-1} - \frac{2^{10}-1}{2-1} = 2^{101} - 2^{10}$.

27. $\frac{5}{3}(4^{65} - 16)$.
$\sum_{i=0}^{64} 5 \cdot 4^i - \sum_{i=0}^{1} 5 \cdot 4^i = 5\frac{4^{65}-1}{4-1} - (5 + 20) = \frac{5}{3}(4^{65} - 16)$.

29. $2(1 - \frac{1}{3^{100}})$.
$\sum_{i=1}^{100} 4\frac{1}{3^i} = 4\sum_{i=1}^{100}(\frac{1}{3})^i = 4[(\sum_{i=0}^{100}(\frac{1}{3})^i) - 1] = 4[\frac{(\frac{1}{3})^{101}-1}{\frac{1}{3}-1} - 1] = 2(1 - \frac{1}{3^{100}})$.

31. $\sum_{i=1}^{n}(4i - 3) = 4\sum_{i=1}^{n} i - 3\sum_{i=1}^{n} 1 = 4\frac{n(n+1)}{2} - 3n = 2n(n+1) - 3n = 2n^2 - n = n(2n-1)$.

33. $\sum_{i=1}^{n}(3i^2 - i) = 3\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i = 3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} =$
$\frac{n(n+1)(2n+1)-n(n+1)}{2} = \frac{n(n+1)}{2}(2n + 1 - 1) = n^2(n+1)$.

35. $\sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2} = n(2n+1)$.
We substitute $m = 2n$ into the formula $\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$ to get $\sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2} = n(2n+1)$.

37. $3+5+7+\cdots+(2n+1) = \sum_{i=1}^{n}(2i+1) = 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = 2\frac{n(n+1)}{2} + n = n(n+1) + n = n(n+2)$.

39. $\prod_{i=1}^{n} k$.
That is,

$$\underbrace{k \cdot k \cdots k}_{n \text{ times}} = \prod_{i=1}^{n} k.$$

41. $2^{\frac{n(n+1)}{2}}$.
That is,

$$2^1 \cdot 2^2 \cdot 2^3 \cdots 2^n = 2^{1+2+3+\cdots+n} = 2^{\sum_{i=1}^{n} i} = 2^{\frac{n(n+1)}{2}}.$$

43. $L(x) = \sum_{i=1}^{n} \left(y_i \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \right)$.

45. Since $2S = \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}} = n(n+1)$, it follows that $S = \frac{n(n+1)}{2}$.
Organize the sum of the two equations as follows.

$$\begin{array}{ccccccccccc} S & = & 1 & + & 2 & + & \cdots & + & (n-1) & + & n \\ S & = & n & + & (n-1) & + & \cdots & + & 2 & + & 1 \\ \hline 2S & = & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1) \end{array}$$

47. $\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$.
By Theorem 4.4,

$$\sum_{i=1}^{n} i^5 = \frac{(n+1)((n+1)^5 - 1) - \sum_{j=1}^{4} \left[\binom{6}{j} \sum_{i=1}^{n} i^j \right]}{6}.$$

Note that

$$\sum_{j=1}^{4} \left[\binom{6}{j} \sum_{i=1}^{n} i^j \right] = \binom{6}{1} \sum_{i=1}^{n} i + \binom{6}{2} \sum_{i=1}^{n} i^2 + \binom{6}{3} \sum_{i=1}^{n} i^3 + \binom{6}{4} \sum_{i=1}^{n} i^4.$$

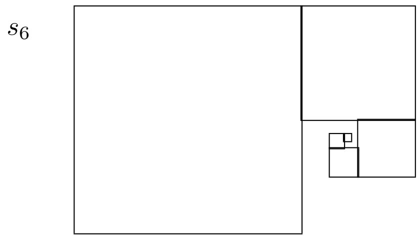
By Example 4.17,

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Also see the formulas in Theorem 4.2. After substitutions we get

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}.$$

49. (a)



(b) The area of s_4 is $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} = \frac{85}{64} = 1.32815$. The area of s_5 is $\frac{85}{64} + \frac{1}{256} = \frac{341}{256} = 1.33203125 > 1.33$.
(c) $a_n = 1 + \frac{1}{4} + (\frac{1}{4})^2 + \cdots + (\frac{1}{4})^{n-1} = \sum_{i=0}^{n-1} (\frac{1}{4})^i$.
(d) $a_n = \frac{1-(\frac{1}{4})^n}{1-\frac{1}{4}} = \frac{4}{3}(1 - \frac{1}{4^n})$.

Section 4.3

1. *Proof. Base case*: $(n = 3)$. Note that $3^2 + 1 \geq 3(3)$. *Inductive step*: Suppose $k \geq 3$ and $k^2 + 1 \geq 3k$. (Goal: $(k+1)^2 + 1 \geq 3(k+1)$.) Observe that $(k+1)^2+1 = k^2+2k+1+1 = (k^2+1)+(2k+1) \geq 3k+(2k+1) \geq 3k+3 = 3(k+1)$. $\square$

3. *Proof. Base case*: $(n = 3)$. Note that $3^2 \geq 2(3) + 1$. *Inductive step*: Suppose $k \geq 3$ and $k^2 \geq 2k + 1$. (Goal: $(k+1)^2 \geq 2(k+1)+1 = 2k+3$.) Observe that $(k+1)^2 = k^2 + 2k + 1 = k^2 + (2k+1) \geq 2k + 1 + (2k+1) = 2k + (2k+2) \geq 2k + 3 = 2(k+1) + 1$. $\square$

5. *Proof. Base case*: $(n = 4)$. Note that $2^4 \geq 4^2$. *Inductive step*: Suppose $k \geq 4$ and $2^k \geq k^2$. (Goal: $2^{k+1} \geq (k+1)^2$.) Observe that $2^{k+1} = 2 \cdot 2^k \geq 2 \cdot k^2 = k^2 + (k^2) \geq k^2 + (2k+1) = (k+1)^2$. The last inequality follows from Exercise 3. $\square$

7. *Proof. Base case*: $(n = 4)$. Note that $4! \geq 4^2$. *Inductive step*: Suppose $k \geq 4$ and $k! \geq k^2$. (Goal: $(k+1)! \geq (k+1)^2$.) Observe that $(k+1)! = (k+1) \cdot k! \geq (k+1) \cdot k^2 \geq (k+1) \cdot (k+1) = (k+1)^2$. The last inequality holds since $k^2 \geq k+1$ for $k \geq 4$. $\square$

9. *Proof. Base case*: $(n = 4)$. Note that $4! > 2^4$. *Inductive step*: Suppose $k \geq 4$ and $k! > 2^k$. (Goal: $(k+1)! > 2^{k+1}$.) Observe that $(k+1)! = (k+1) \cdot k! > (k+1) \cdot 2^k \geq 2 \cdot 2^k = 2^{k+1}$. $\square$

11. *Proof. Base case*: $(n = 0)$. Note that $3 \mid (4^0 - 1)$. *Inductive step*: Suppose $k \geq 0$ and $3 \mid (4^k - 1)$. So, $4^k - 1 = 3c$ for some $c \in \mathbb{Z}$. (Goal: $3 \mid (4^{k+1} - 1)$.) Observe that $4^{k+1} - 1 = 4 \cdot 4^k - 1 = (3+1)4^k - 1 = 3 \cdot 4^k + (4^k - 1) = 3 \cdot 4^k + 3c = 3(4^k + c)$. Thus, $3 \mid (4^{k+1} - 1)$. $\square$

13. *Proof. Base case*: $(n = 0)$. Note that $4 \mid (6^0 - 2^0)$. *Inductive step*: Suppose $k \geq 0$ and $4 \mid (6^k - 2^k)$. So, $6^k - 2^k = 4c$ for some $c \in \mathbb{Z}$. (Goal: $4 \mid (6^{k+1} - 2^{k+1})$.) Observe that $6^{k+1} - 2^{k+1} = 6 \cdot 6^k - 2 \cdot 2^k = 4 \cdot 6^k + 2(6^k - 2^k) = 4 \cdot 6^k + 2(4c) = 4(6^k + 2c)$. Thus, $4 \mid (6^{k+1} - 2^{k+1})$. $\square$

15. *Proof. Base case*: $(n = 0)$. Note that $6 \mid (0^3 - 0)$. *Inductive step*: Suppose $k \geq 0$ and $6 \mid (k^3 - k)$. So, $k^3 - k = 6c$ for some $c \in \mathbb{Z}$.
(Goal: $6 \mid ((k+1)^3 - (k+1))$.) Observe that $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = (k^3 - k) + 3k^2 + 3k = 6c + 3(k^2 + k) = 6(c + \frac{k^2+k}{2})$. Since k^2 and k have the same parity, it follows that $k^2 + k$ is even, whence $\frac{k^2+k}{2} \in \mathbb{Z}$. Thus, $6 \mid ((k+1)^3 - (k+1))$. $\square$

17. *Proof. Base case*: $(n = 1)$. Note that $3^1 + 1 = 4$. *Inductive step*: Suppose $k \geq 1$ and $s_k = 3^k + 1$. (Goal: $s_{k+1} = 3^{k+1} + 1$.) Observe that $s_{k+1} = 3s_k - 2 = 3(3^k + 1) - 2 = 3^{k+1} + 1$. $\square$

19. *Proof. Base case*: $(n = 2)$. Note that $9 - 2(2) = 5$. *Inductive step*: Suppose $k \geq 2$ and $s_k = 9 - 2k$. (Goal: $s_{k+1} = 9 - 2(k+1)$.) Observe that $s_{k+1} = s_k - 2 = 9 - 2k - 2 = 9 - 2(k+1)$. $\square$

21. (a) $s_0 = 0 = D\frac{(1+i)^0-1}{i}$, $s_1 = D = D\frac{(1+i)^1-1}{i}$, $s_2 = (1+i)D + D = (i+2)D = D\frac{(1+i)^2-1}{i}$.
(b) $(1+i)(D\frac{(1+i)^{n-1}-1}{i}) + D = D(\frac{(1+i)^n-(1+i)}{i} + \frac{i}{i}) = D\frac{(1+i)^n-1}{i}$.
(c) *Proof. Base case*: $(n = 0)$. Note that $D\frac{(1+i)^0-1}{i} = 0$. *Inductive step*: Suppose $k \geq 0$ and $s_k = D\frac{(1+i)^k-1}{i}$. Observe that $s_{k+1} = (1+i)s_k + D = (1+i)[D\frac{(1+i)^k-1}{i}] + D = D[(1+i)\frac{(1+i)^k-1}{i} + \frac{i}{i}] = D\frac{(1+i)^{k+1}-1}{i}$. $\square$
(d) $200\frac{(1.0075)^{24}-1}{.0075} = \$5237.69$.

23. (a) $A \cap (B_1 \cup B_2 \cup B_3) = A \cap ((B_1 \cup B_2) \cup B_3) = (A \cap (B_1 \cup B_2)) \cup (A \cap B_3) = ((A \cap B_1) \cup (A \cap B_2)) \cup (A \cap B_3) = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)$.
(b) *Proof. Base case*: $(n = 1)$. Let B_1 be any set. Note that $A \cap (B_1) = (A \cap B_1)$. *Inductive step*: Suppose $k \geq 1$ and, for all sets $B_1, B_2, \ldots, B_k$, $A \cap (B_1 \cup B_2 \cup \cdots \cup B_k) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_k)$. Let $B_1, B_2, \ldots, B_{k+1}$ be any sets. Observe that $A \cap (B_1 \cup B_2 \cup \cdots \cup B_{k+1}) = A \cap ((B_1 \cup B_2 \cup \cdots \cup B_k) \cup B_{k+1}) = (A \cap (B_1 \cup B_2 \cup \cdots \cup B_k)) \cup (A \cap B_{k+1}) = ((A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_k)) \cup (A \cap B_{k+1}) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_{k+1})$. $\square$
(c) Reverse $\cap$ and $\cup$ in part (b).

25. (a) $\neg(p_1 \vee p_2 \vee p_3) \equiv \neg((p_1 \vee p_2) \vee p_3) \equiv \neg(p_1 \vee p_2) \wedge \neg p_3 \equiv (\neg p_1 \wedge \neg p_2) \wedge \neg p_3 \equiv \neg p_1 \wedge \neg p_2 \wedge \neg p_3$.
(b) *Proof. Base case*: $(n = 1)$. Let p_1 be any statement form. Note that $\neg(p_1) \equiv \neg p_1$. *Inductive step*: Suppose $k \geq 1$ and, for all statement forms

$p_1, p_2, \dots, p_k$, $\neg(p_1 \vee p_2 \vee \cdots \vee p_k) \equiv \neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k$. Let $p_1, p_2, \dots, p_{k+1}$ be any statement forms. Observe that
$\neg(p_1 \vee p_2 \vee \cdots \vee p_{k+1}) \equiv \neg((p_1 \vee p_2 \vee \cdots \vee p_k) \vee p_{k+1}) \equiv \neg(p_1 \vee p_2 \vee \cdots \vee p_k) \wedge \neg p_{k+1}$
$\equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k) \wedge \neg p_{k+1} \equiv \neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_{k+1}$. $\square$
(c) Reverse $\wedge$ and $\vee$ in part (b).

27. (a) *Proof. Base case*: $(m = 1)$. Obvious. *Inductive step*: Suppose $k \geq 1$ and,
$\forall\ a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k \in \mathbb{Z}$, if $a_1 \equiv b_1 \pmod n$, $a_2 \equiv b_2 \pmod n$, $\dots$, $a_k \equiv b_k \pmod n$, then $\sum_{i=1}^{k} a_i \equiv \sum_{i=1}^{k} b_i \pmod n$.
Let $a_1, a_2, \dots, a_{k+1}, b_1, b_2, \dots, b_{k+1}$ be any integers. Suppose $a_1 \equiv b_1 \pmod n$, $a_2 \equiv b_2 \pmod n$, $\dots$, $a_{k+1} \equiv b_{k+1} \pmod n$. By the induction hypothesis, $\sum_{i=1}^{k} a_i \equiv \sum_{i=1}^{k} b_i \pmod n$. By Theorem 3.27(i), we therefore have $(\sum_{i=1}^{k} a_i) + a_{k+1} \equiv (\sum_{i=1}^{k} b_i) + b_{k+1} \pmod n$. That is, $\sum_{i=1}^{k+1} a_i \equiv \sum_{i=1}^{k+1} b_i \pmod n$. $\square$
(b) *Proof. Base case*: $(m = 1)$. Obvious. *Inductive step*: Suppose $k \geq 1$ and,
$\forall\ a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k \in \mathbb{Z}$, if $a_1 \equiv b_1 \pmod n$, $a_2 \equiv b_2 \pmod n$, $\dots$, $a_k \equiv b_k \pmod n$, then $\prod_{i=1}^{k} a_i \equiv \prod_{i=1}^{k} b_i \pmod n$.
Let $a_1, a_2, \dots, a_{k+1}, b_1, b_2, \dots, b_{k+1}$ be any integers. Suppose $a_1 \equiv b_1 \pmod n$, $a_2 \equiv b_2 \pmod n$, $\dots$, $a_{k+1} \equiv b_{k+1} \pmod n$. By the induction hypothesis, $\prod_{i=1}^{k} a_i \equiv \prod_{i=1}^{k} b_i \pmod n$. We therefore have $(\prod_{i=1}^{k} a_i) \cdot a_{k+1} \equiv (\prod_{i=1}^{k} b_i) \cdot b_{k+1} \pmod n$. That is, $\prod_{i=1}^{k+1} a_i \equiv \prod_{i=1}^{k+1} b_i \pmod n$. $\square$

29. *Proof. Base case*: $(|S| = 1)$. If $S = \{s_1\}$, then $\max(S) = s_1$. *Inductive step*: Suppose $k \geq 1$, and any set S with $|S| = k$ has a maximal element. (Goal: Any set S with $|S| = k + 1$ has a maximal element.) Suppose $s_1, s_2, \dots, s_{k+1}$ are distinct real numbers and $S = \{s_1, s_2, \dots, s_{k+1}\}$. By the induction hypothesis, the set $\{s_1, s_2, \dots, s_k\}$ has a maximal element, say s_j. Observe that $\{s_j, s_{k+1}\}$ has a maximal element $m = \begin{cases} s_j & \text{if } s_j > s_{k+1} \\ s_{k+1} & \text{otherwise.} \end{cases}$ Note that m is the maximum element of S. $\square$

31. *Proof. Base case*: $(n = 1)$. Obvious. *Inductive step*: Suppose $k \geq 1$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. Observe that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$. $\square$

33. *Proof. Base case*: $(n = 1)$. Obvious. *Inductive step*: Suppose $k \geq 1$ and $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^k = \begin{bmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{bmatrix}$. Observe that $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{bmatrix} = \begin{bmatrix} 1 & 2^{k+1} - 1 \\ 0 & 2^{k+1} \end{bmatrix}$. $\square$

35. *Proof. Base case*: $(n = 1)$. Obvious. *Inductive step*: Suppose $k \geq 1$ and

$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$. Observe that $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{k+1} =$ $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^k = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} =$ $\begin{bmatrix} \cos(\theta+k\theta) & \sin(\theta+k\theta) \\ -\sin(\theta+k\theta) & \cos(\theta+k\theta) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$. □

37. (a) $\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$.
(b) $\sin 4\theta = 2\sin 2\theta\cos 2\theta = 2\cdot 2\sin\theta\cos\theta\cos 2\theta = 4\sin\theta\cos\theta\cos 2\theta$.
(c)*Proof. Base case*: $(n = 1)$. By the double angle identity, $\sin 2\theta = 2\sin\theta\cos\theta$. *Inductive step*: Suppose $k \geq 1$ and $\sin 2^k\theta = 2^k\sin\theta\prod_{i=0}^{k-1}\cos 2^i\theta$. Now, $\sin 2^{k+1}\theta = \sin(2(2^k\theta)) = 2\sin 2^k\theta\cos 2^k\theta = 2(2^k\sin\theta\prod_{i=0}^{k-1}\cos 2^i\theta)\cos 2^k\theta = 2^{k+1}\sin\theta\prod_{i=0}^{k}\cos 2^i\theta$. □

39. $\forall\, n \geq 0, n \geq 1$.
Obviously, $0 \geq 1$ does not hold. However, suppose $k \geq 0$ and $k \geq 1$. Then $k+1 \geq k \geq 1$. So the inductive step holds.

41. *Proof.* Suppose $\forall\, n \geq a,\ P(n)$. Suppose $k \geq a$ and $P(k)$ holds. Since $k+1 \geq a$, it follows that $P(k+1)$ also holds. □

43. *Proof. Base case*: $(n = 0)$. $8^0 = 1$. *Inductive step*: Suppose $k \geq 0$ and $8^k \equiv 1 \pmod 7$. Observe that $8^{k+1} \equiv 8\cdot 8^k \equiv 1\cdot 1 \equiv 1 \pmod 7$. □

45. (a) $C_0 = 1$,
$C_1 = \frac{2(2\cdot1-1)}{1+1}C_0 = 1$,
$C_2 = \frac{2(2\cdot2-1)}{2+1}C_1 = 2$,
$C_3 = \frac{2(2\cdot3-1)}{3+1}C_2 = 5$,
$C_4 = \frac{2(2\cdot4-1)}{4+1}C_3 = 14$.
(b) *Base case*: $(n = 0)$. $C_0 = 1 = \frac{1}{0+1}\binom{0}{0}$. *Inductive step*: Suppose $k \geq 0$ and $C_k = \frac{1}{k+1}\binom{2k}{k}$. Observe that $C_{k+1} = \frac{2(2(k+1)-1)}{k+2}C_k = \frac{2(2k+1)}{k+2}\cdot\frac{1}{k+1}\binom{2k}{k} = \frac{2(2k+1)(2k)!}{(k+2)(k+1)k!k!} = \frac{2(k+1)(2k+1)(2k)!}{(k+2)(k+1)k!(k+1)k!} = \frac{(2k+2)(2k+1)(2k)!}{(k+2)(k+1)!(k+1)!} = \frac{1}{k+2}\cdot\frac{(2k+2)!}{(k+1)!(k+1)!} = \frac{1}{k+2}\binom{2(k+1)}{k+1}$. □
(c) $n = 0$: a.
$n = 1$: ab.
$n = 2$: $(ab)c$ or $a(bc)$.
$n = 3$: $(ab)(cd)$, $((ab)c)d$, $(a(bc))d$, $a((bc)d)$, or $a(b(cd))$.
$n = 4$: $((ab)c)(de)$, $(a(bc))(de)$, $(ab)((cd)e)$, $(ab)(c(de))$, $a((bc)(de))$, $a(((bc)d)e)$, $a((b(cd))e)$, $a(b((cd)e))$, $a(b(c(de)))$, $((ab)(cd))e$, $(((ab)c)d)e$, $((a(bc))d)e$, $(a((bc)d))e$, $(a(b(cd)))e$.
That is, $(abc)(de)$ in two ways, $(ab)(cde)$ in two ways, $a(bcde)$ in five ways, $(abcd)e$ in five ways.

Section 4.4

1. *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{1} 0 = 0$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k} 0 = 0$. (Goal: $\sum_{i=1}^{k+1} 0 = 0$.)
Observe that $\sum_{i=1}^{k+1} 0 = (\sum_{i=1}^{k} 0) + 0 = 0 + 0 = 0$. □

3. (a) *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. (Goal: $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.)
Observe that $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$. □
(b) $\sum_{i=1}^{n} 2i = 2\sum_{i=1}^{n} i = 2\frac{n(n+1)}{2} = n(n+1)$.
(c) $\frac{n}{2}(\frac{n}{2} + 1) = \frac{n(n+2)}{4}$. (d) $\frac{n-1}{2}(\frac{n-1}{2} + 1) = \frac{n^2-1}{4}$.

5. *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{1}(3i^2 - i) = 2 = 1^2(1+1)$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k}(3i^2 - i) = k^2(k+1)$.
(Goal: $\sum_{i=1}^{k+1}(3i^2 - i) = (k+1)^2(k+2)$.) Observe that $\sum_{i=1}^{k+1}(3i^2 - i) = \sum_{i=1}^{k}(3i^2 - i) + (3(k+1)^2 - (k+1)) = k^2(k+1) + 3(k+1)^2 - (k+1) = (k+1)[k^2 + 3(k+1) - 1] = (k+1)^2(k+2)$. That is, $\sum_{i=1}^{k+1}(3i^2 - i) = (k+1)^2(k+2)$.
□

7. *Proof. Base case*: $(n = 1)$. Note $\sum_{i=1}^{1}(2i)^3 = 8 = 2(1)(4)$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k}(2i)^3 = 2k^2(k+1)^2$. (Goal: $\sum_{i=1}^{k+1}(2i)^3 = 2(k+1)^2(k+2)^2$.) Observe that $\sum_{i=1}^{k+1}(2i)^3 = \sum_{i=1}^{k}(2i)^3 + (2(k+1))^3 = 2k^2(k+1)^2 + (2(k+1))^3 = 2(k+1)^2(k+2)^2$. □

9. *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{1}(4i - 3) = 1 = 1(2(1) - 1)$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k}(4i - 3) = k(2k - 1)$.
(Goal: $\sum_{i=1}^{k+1}(4i - 3) = (k+1)(2(k+1) - 1)$.) Observe that $\sum_{i=1}^{k+1}(4i - 3) = \sum_{i=1}^{k}(4i - 3) + (4(k+1) - 3) = k(2k - 1) + (4k + 1) = 2k^2 + 3k + 1 = (k+1)(2(k+1) - 1)$. That is, $\sum_{i=1}^{k+1}(4i - 3) = (k+1)(2(k+1) - 1)$. □

11. We prove $\sum_{i=1}^{n}(2i + 1) = n(n+2)$.
Proof. Base case: $(n = 1)$. Note that $\sum_{i=1}^{1}(2i + 1) = 3 = 1(1+2)$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k}(2i + 1) = k(k+2)$. (Goal: $\sum_{i=1}^{k+1}(2i + 1) = (k+1)(k+3)$.) Observe that $\sum_{i=1}^{k+1}(2i+1) = \sum_{i=1}^{k}(2i+1) + (2(k+1) + 1) = k(k+2) + 2(k+1) + 1 = k^2 + 4k + 3 = (k+1)(k+3)$. That is, $\sum_{i=1}^{k+1}(2i+1) = (k+1)(k+3)$. □

13. We prove $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.
Proof. Base case: $(n = 0)$. Note that $\sum_{i=0}^{0} 2^i = 1 = 2^{0+1} - 1$. *Inductive step*: Suppose $k \geq 0$ and $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$. (Goal: $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$.) Observe that $\sum_{i=0}^{k+1} 2^i = \sum_{i=0}^{k} 2^i + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$. That is, $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$. □

15. *Proof. Base case*: $(n = 2)$. Note that $\sum_{i=2}^{2} i2^i = 8 = (2-1)2^{2+1}$. *Inductive step*: Suppose $k \geq 0$ and $\sum_{i=2}^{k} i2^i = (k-1)2^{k+1}$. (Goal: $\sum_{i=2}^{k+1} i2^i = k2^{k+2}$.) Observe that $\sum_{i=2}^{k+1} i2^i = \sum_{i=2}^{k} i2^i + (k+1)2^{k+1} = (k-1)2^{k+1} + (k+1)2^{k+1} = 2k2^{k+1} = k2^{k+2}$. That is, $\sum_{i=2}^{k+1} i2^i = k2^{k+2}$. $\square$

17. *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{1} i^2 2^i = 2 = (1^2 - 2\cdot 1 + 3)2^{1+1} - 6$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k} i^2 2^i = (k^2 - 2k + 3)2^{k+1} - 6$. Observe that $\sum_{i=1}^{k+1} i^2 2^i = \sum_{i=1}^{k} i^2 2^i + (k+1)^2 2^{k+1} = (k^2 - 2k + 3)2^{k+1} - 6 + (k+1)^2 2^{k+1} = [k^2 - 2k + 3 + (k+1)^2]2^{k+1} - 6 = ((k+1)^2 - 2(k+1) + 3)2^{k+2} - 6$. $\square$

19. *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{1} (i \cdot i!) = 1 = (1+1)! - 1$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k} (i \cdot i!) = (k+1)! - 1$. Observe that $\sum_{i=1}^{k+1} (i \cdot i!) = \sum_{i=1}^{k} (i \cdot i!) + (k+1)\cdot(k+1)! = (k+1)! - 1 + (k+1)\cdot(k+1)! = [1 + (k+1)](k+1)! - 1 = (k+2)! - 1$. $\square$

21. *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{2} i = 3 = 1(2\cdot 1 + 1)$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{2k} i = k(2k+1)$. Observe that $\sum_{i=1}^{2(k+1)} i = \sum_{i=1}^{2k} i + (2k+1) + (2k+2) = k(2k+1) + (2k+1) + (2k+2) = (k+1)(2(k+1)+1)$. $\square$

Notice how the proof is affected by the last index in the sum being $i = 2n$. In the inductive step, effectively two terms are split off: the $i = 2k+1$ term and the $i = 2k+2$ term.

23. *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{2} = \frac{1}{1+1}$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$. (Goal: $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$.) Observe that $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$. $\square$

25. *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{1} \frac{1}{2^i} = \frac{1}{2} = 1 - \frac{1}{2^1}$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k} \frac{1}{2^i} = 1 - \frac{1}{2^k}$. Observe that $\sum_{i=1}^{k+1} \frac{1}{2^i} = (1 - \frac{1}{2^k}) + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$. $\square$

27. *Proof. Base case*: $(n = 1)$. Note that $\prod_{i=1}^{1} \frac{i}{i+2} = \frac{1}{3} = \frac{2}{(1+1)(1+2)}$. *Inductive step*: Suppose $k \geq 1$ and $\prod_{i=1}^{k} \frac{i}{i+2} = \frac{2}{(k+1)(k+2)}$. (Goal: $\prod_{i=1}^{k+1} \frac{i}{i+2} = \frac{2}{(k+2)(k+3)}$.) Observe that $\prod_{i=1}^{k+1} \frac{i}{i+2} = (\prod_{i=1}^{k} \frac{i}{i+2})(\frac{k+1}{k+3}) = (\frac{2}{(k+1)(k+2)})(\frac{k+1}{k+3}) = \frac{2}{(k+2)(k+3)}$. $\square$

29. *Proof. Base case*: $(n = 1)$. Note that $\prod_{i=1}^{1} r^{2i} = r^2 = r^{1(1+1)}$. *Inductive step*: Suppose $k \geq 1$ and $\prod_{i=1}^{k} r^{2i} = r^{k(k+1)}$. (Goal: $\prod_{i=1}^{k+1} r^{2i} = r^{(k+1)(k+2)}$.) Observe that $\prod_{i=1}^{k+1} r^{2i} = (\prod_{i=1}^{k} r^{2i})(r^{2(k+1)}) = (r^{k(k+1)})(r^{2(k+1)}) = r^{(k+1)(k+2)}$. $\square$

31. *Proof. Base case*: $(n = 0)$. Obvious. *Inductive step*: Suppose $k \geq 0$ and $x^{2^k} - y^{2^k} = (x - y)\prod_{i=0}^{k-1}(x^{2^i} + y^{2^i})$.
Observe that $x^{2^{k+1}} - y^{2^{k+1}} = (x^{2^k} - y^{2^k})(x^{2^k} + y^{2^k}) =$
$((x - y)\prod_{i=0}^{k-1}(x^{2^i} + y^{2^i}))(x^{2^k} + y^{2^k}) = (x - y)\prod_{i=0}^{k}(x^{2^i} + y^{2^i})$. □

33. *Proof. Base case*: $(n = 0)$. We have $s_1 \geq 2s_0 \geq s_0$.
Inductive step: Suppose $k \geq 0$ and $s_{k+1} \geq \sum_{i=0}^{k} s_i$.
Observe that $s_{k+2} \geq 2s_{k+1} = s_{k+1} + s_{k+1} \geq s_{k+1} + \sum_{i=0}^{k} s_i = \sum_{i=0}^{k+1} s_i$. □

35. *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{1} \frac{1}{i^2} = 1 = \frac{3}{2} - \frac{1}{2}$.
Inductive step: Suppose $k \geq 1$ and $\sum_{i=1}^{k} \frac{1}{i^2} \geq \frac{3}{2} - \frac{1}{k+1}$.
(Goal: $\sum_{i=1}^{k+1} \frac{1}{i^2} \geq \frac{3}{2} - \frac{1}{k+2}$.) Observe that
$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \geq \frac{3}{2} - \frac{1}{k+1} + \frac{1}{(k+1)^2} \geq \frac{3}{2} - \frac{1}{k+2}$. □
Note that $k(k+2) = k^2 + 2k \leq k^2 + 2k + 1 = (k+1)^2$. So $\frac{k}{(k+1)^2} \leq \frac{1}{k+2}$, and hence $-\frac{1}{k+1} + \frac{1}{(k+1)^2} = \frac{-(k+1)+1}{(k+1)^2} = -\frac{k}{(k+1)^2} \geq -\frac{1}{k+2}$.

37. *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{1} \frac{1}{i^2} = 1 = 2 - 1$.
Inductive step: Suppose $k \geq 1$ and $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$.
(Goal: $\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k+1}$.) Observe that
$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$. □
Since $k(k+2) = k^2 + 2k \leq k^2 + 2k + 1 = (k+1)^2$, we have $\frac{1}{(k+1)^2} + \frac{1}{k+1} = \frac{k+2}{(k+1)^2} \leq \frac{1}{k}$. Hence $-\frac{1}{k} + \frac{1}{(k+1)^2} \leq -\frac{1}{k+1}$.

39. Theorem: $\sum_{i=a}^{b}(s_i \pm t_i) = \sum_{i=a}^{b} s_i \pm \sum_{i=a}^{b} t_i$.
Since both sides are 0 when $b < a$, it suffices to consider $b \geq a$. We can consider a fixed, and so our proof is by induction on b. The base case $b = a$ is also easy to check.
Sketch. We focus on $+$ since $-$ is handled similarly.
Suppose $\sum_{i=a}^{b}(s_i + t_i) = \sum_{i=a}^{b} s_i + \sum_{i=a}^{b} t_i$. Then, $\sum_{i=a}^{b+1}(s_i + t_i) =$
$\sum_{i=a}^{b}(s_i+t_i)+s_{b+1}+t_{b+1} = \sum_{i=a}^{b} s_i+\sum_{i=a}^{b} t_i+s_{b+1}+t_{b+1} = \sum_{i=a}^{b+1} s_i+\sum_{i=a}^{b+1} t_i$.
□

Section 4.5

1. *Proof. Base cases*: $(n = 0, 1)$. Note that $0 = 2^0 - 1$ and $1 = 2^1 - 1$. *Inductive step*: Suppose $k \geq 1$ and that, for each $0 \leq i \leq k$, $s_i = 2^i - 1$.
(Goal: $s_{k+1} = 2^{k+1} - 1$.) Observe that $s_{k+1} = 3s_k - 2s_{k-1} =$
$3(2^k - 1) - 2(2^{k-1} - 1) = 3 \cdot 2^k - 3 - 2^k + 2 = 2 \cdot 2^k - 1 = 2^{k+1} - 1$. □

3. *Proof. Base cases*: $(n = 0, 1)$. Note that $2 = 1 + 1$ and $11 = 4 + 7$.
Inductive step: Suppose $k \geq 1$ and that, for each $0 \leq i \leq k$, $s_i = 4^i + 7^i$.

(Goal: $s_{k+1} = 4^{k+1} + 7^{k+1}$.) Observe that $s_{k+1} = 11s_k - 28s_{k-1} = 11(4^k+7^k)-28(4^{k-1}+7^{k-1}) = 11(4^k)+11(7^k)-7(4^k)-4(7^k) = 4(4^k)+7(7^k) = 4^{k+1} + 7^{k+1}$. □

5. *Proof. Base cases*: $(n = 0, 1)$. Note that $1 = 2^0$ and $2 = 2^1$.
Inductive step: Suppose $k \geq 1$ and that, for each $0 \leq i \leq k$, $s_i = 2^i$.
(Goal: $s_{k+1} = 2^{k+1}$.) Observe that $s_{k+1} = 4s_{k-1} = 4 \cdot 2^{k-1} = 2^{k+1}$. □

7. *Proof. Base cases*: $(n = 0, 1, 2)$. Note that $-1 = 5^0-3^0-2^0$, $0 = 5^1-3^1-2^1$, and $12 = 5^2 - 3^2 - 2^2$. *Inductive step*: Suppose $k \geq 2$ and that, for each $0 \leq i \leq k$, $s_i = 5^i - 3^i - 2^i$. (Goal: $s_{k+1} = 5^{k+1} - 3^{k+1} - 2^{k+1}$.) Observe that $s_{k+1} = 10s_k - 31s_{k-1} + 30s_{k-2} =$
$10(5^k - 3^k - 2^k) - 31(5^{k-1} - 3^{k-1} - 2^{k-1}) + 30(5^{k-2} - 3^{k-2} - 2^{k-2}) =$
$250\cdot5^{k-2}-90\cdot3^{k-2}-40\cdot2^{k-2}-155\cdot5^{k-2}+93\cdot3^{k-2}+62\cdot2^{k-2}+30\cdot5^{k-2}-30\cdot3^{k-2}-30\cdot2^{k-2} =$
$125 \cdot 5^{k-2} - 27 \cdot 3^{k-2} - 8 \cdot 2^{k-2} = 5^{k+1} - 3^{k+1} - 2^{k+1}$. □

9. *Proof. Base cases*: $(n = 0, 1)$. Note that 1 and 3 are odd.
Inductive step: Suppose $k \geq 1$ and that, for each $0 \leq i \leq k$, s_i is odd.
(Goal: s_{k+1} is odd.) Since s_{k-1} is odd, we have $c \in \mathbb{Z}$ such that $s_{k-1} = 2c + 1$. Observe that $s_{k+1} = 3s_{k-1} - 2s_k = 3(2c + 1) - 2s_k = 6c - 2s_k + 2 + 1 = 2(3c - s_k + 1) + 1$. Since $3c - s_k + 1 \in \mathbb{Z}$, we see that s_{k+1} is odd. □

11. (a) $s_2 = -6(-1) + 5(0) = 6$,
$s_3 = -6(0) + 5(6) = 30$,
$s_4 = -6(6) + 5(30) = 114$.
(b)*Proof. Base cases*: $(n = 0, 1)$. Note that $-1 = 2\cdot3^0-3\cdot2^0$ and $0 = 2\cdot3^1-3\cdot2^1$.
Inductive step: Suppose $k \geq 1$ and that, for each $0 \leq i \leq k$, $s_i = 2 \cdot 3^i - 3 \cdot 2^i$.
(Goal: $s_{k+1} = 2 \cdot 3^{k+1} - 3 \cdot 2^{k+1}$.) Observe that $s_{k+1} = -6s_{k-1} + 5s_k = -6(2 \cdot 3^{k-1} - 3 \cdot 2^{k-1}) + 5(2 \cdot 3^k - 3 \cdot 2^k) = -4 \cdot 3^k + 9 \cdot 2^k + 10 \cdot 3^k - 15 \cdot 2^k = 6 \cdot 3^k - 6 \cdot 2^k = 2 \cdot 3^{k+1} - 3 \cdot 2^{k+1}$. □
(c) Since $s_{n+1} - s_n = (2 \cdot 3^{n+1} - 3 \cdot 2^{n+1}) - (2 \cdot 3^n - 3 \cdot 2^n) = 2 \cdot 3 \cdot 3^n - 3 \cdot 2 \cdot 2^n - 2 \cdot 3^n + 3 \cdot 2^n = 4 \cdot 3^n - 3 \cdot 2^n \geq 3(3^n - 2^n) \geq 0$, we have $s_{n+1} \geq s_n$ for all $n \geq 0$.

13. (a) $s_2 = 2(1) + 1 = 3$,
$s_3 = 2(3) + 1 = 7$,
$s_4 = 2(7) + 3 = 17$.
(b) *Proof. Base cases*: $(n = 0, 1)$. Note that $1 = \frac{1}{2}((1 + \sqrt{2})^0 + (1 - \sqrt{2})^0)$ and $1 = \frac{1}{2}((1+\sqrt{2})^1 + (1-\sqrt{2})^1)$. *Inductive step*: Suppose $k \geq 1$ and that, for each $0 \leq i \leq k$, $s_i = \frac{1}{2}((1 + \sqrt{2})^i + (1 - \sqrt{2})^i)$. Observe that $s_{k+1} = 2s_k + s_{k-1} =$
$2(\frac{1}{2})[(1 + \sqrt{2})^k + (1 - \sqrt{2})^k] + \frac{1}{2}((1 + \sqrt{2})^{k-1} + (1 - \sqrt{2})^{k-1}) =$
$\frac{1}{2}((2+2\sqrt{2})(1+\sqrt{2})^{k-1}+(2-2\sqrt{2})(1-\sqrt{2})^{k-1}+(1+\sqrt{2})^{k-1}+(1-\sqrt{2})^{k-1}) =$
$\frac{1}{2}((3 + 2\sqrt{2})(1 + \sqrt{2})^{k-1} + (3 - 2\sqrt{2})(1 - \sqrt{2})^{k-1}) =$
$\frac{1}{2}((1 + \sqrt{2})^2(1 + \sqrt{2})^{k-1} + (1 - \sqrt{2})^2(1 - \sqrt{2})^{k-1}) =$
$\frac{1}{2}((1 + \sqrt{2})^{k+1} + (1 - \sqrt{2})^{k+1})$. □

(c) Here is a trace of the function call.

Encode(3, $[5, 2, 7]$)
    Encode(2, $[2, 7]$),
        Encode(1, $[7]$),
            Print 7,
        Encode(0, $[7]$),
            Print 0,
        Encode(1, $[2]$),
            Print 2,
    Encode(1, $[2, 7]$),
        Print 2,
    Encode(2, $[5, 2]$).
        Encode(1, $[2]$),
            Print 2,
        Encode(0, $[2]$),
            Print 0,
        Encode(1, $[5]$),
            Print 5.

In sequence, it prints $7, 0, 2, 2, 2, 0, 5$.

15. This can be proven with regular induction. However, the ability to refer back two previous cases instead of just one is helpful. *Sketch. Base cases*: $(n = 0, 1)$. These are easy to check. *Inductive step*: Suppose $k \geq 1$ and, for each $0 \leq m \leq k$, $\sum_{i=0}^{m}(-1)^i = \begin{cases} 1 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$.
Observe that $\sum_{i=0}^{k+1}(-1)^i = \sum_{i=0}^{k-1}(-1)^i + (-1)^k + (-1)^{k+1} = \sum_{i=0}^{k-1}(-1)^i$. Since $k+1$ and $k-1$ have the same parity, the result follows. □
The point is that $(-1)^k + (-1)^{k+1}$ is either $(-1) + (1)$ or $(1) + (-1)$. Also, the inductive hypothesis applies to $\sum_{i=0}^{k-1}(-1)^i$. Moreover, $k+1$ is even iff $k-1$ is even.

17. (a) We prove that, for all $n \geq 4$, it is possible to attain $\$n$ with \$2 bills and \$5 bills.
Proof. Base cases: $(n = 4, 5)$. We see that $\$4 = 2 \times \$2$, and $\$5 = 1 \times \$5$. *Inductive step*: Suppose $k \geq 5$ and that, for each $4 \leq i \leq k$, it is possible to attain $\$i$ with \$2 bills and \$5 bills. By the induction hypothesis, $\$(k-1) = a \times \$2 + b \times \$5$, for some $a, b \in \mathbb{N}$. Observe that $\$(k+1) = a \times \$2 + b \times \$5 + \$2 = (a+1) \times \$2 + b \times \$5$. □
(b) Increase. All odd amounts would be unachievable.

19. *Proof. Base cases*: $(n = 25, 26, 27, 28)$. We see that
25 inches $= 6 \times 4$ inches $+ 0 \times 9$ inches $+ 1$,
26 inches $= 4 \times 4$ inches $+ 1 \times 9$ inches $+ 1$,
27 inches $= 2 \times 4$ inches $+ 2 \times 9$ inches $+ 1$, and

28 inches $= 0 \times 4$ inches $+ 3 \times 9$ inches $+ 1$. *Inductive step*: Suppose $k \geq 28$ and that, for each $25 \leq i \leq k$, it is possible to attain i inches from 4-inch bricks and 9-inch bricks and a sheet of plywood 1 inch thick. By the induction hypothesis, $(k-3)$ inches $= a \times 4$ inches $+ b \times 9$ inches $+ 1$, for some $a, b \in \mathbb{N}$. Observe that $(k+1)$ inches $= (a+1) \times 4$ inches $+ b \times 9$ inches $+ 1$. $\square$

21. (a) *Sketch. Base cases*: $4(5¢) = 20¢ = 2(10¢)$ and $5(5¢) = 25¢ = 1(25¢)$. *Inductive step*: Use $(k+1)(5¢) = (k-1)(5¢) + 10¢$. $\square$
(b) 5, 6, 7, 8, 9, 15, 16, 17, 18, 19¢.
We are showing that, for each $k \geq 4$, we can achieve $(5k)¢$ (i.e., $k(5¢)$).

23. $1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$.

25. $3549 = 3 \cdot 7 \cdot 13^2$.

27. $12! = (2^2 3)(11)(2 \cdot 5)(3^2)(2^3)(7)(2 \cdot 3)(5)(2^2)(3)(2)(1) = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$.

29. (a) *Proof. Base case*: $(n = 2)$. Note that $2^2 \mid 2^2$. *Inductive step*: Suppose $k \geq 2$ and that each integer i with $2 \leq i \leq k$ has a squared prime divisor. (Goal: $k+1$ has a squared prime divisor.) Case 1: $k+1$ is prime. Clearly, $(k+1)^2 \mid (k+1)^2$. Case 2: $k+1$ is composite. Write $k+1 = rs$, where $2 \leq r \leq k$ and $2 \leq s \leq k$. So, there exists a prime p such that $p^2 \mid r^2$. That is, $r^2 = p^2 t$ for some integer t. It follows that $(k+1)^2 = r^2 s^2 = p^2 t s^2$. $\square$
(b) *Sketch.* Write $n = p_1^{e_1} \cdot p_2^{e_2} \cdot \cdots \cdot p_m^{e_m}$. So, $n^2 = (p_1^2)^{e_1} \cdot (p_2^2)^{e_2} \cdot \cdots \cdot (p_m^2)^{e_m}$. Take $p = p_1$. $\square$
Effectively, the Fundamental Theorem of Arithmetic gives us, in particular, that a prime p divides n. It then immediately follows that $p^2 \mid n^2$.

31. *Sketch.* By reordering if necessary, we may assume that $p_1 < p_2 < \cdots < p_m$. Let $d = p_1^{\min\{e_1,f_1\}} p_2^{\min\{e_2,f_2\}} \cdots p_m^{\min\{e_m,f_m\}}$, let $a = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$, and let $b = p_1^{f_1} p_2^{f_2} \cdots p_m^{f_m}$. Observe that $d \geq 1 > 0$, $d \mid a$, and $d \mid b$. Suppose $c \in \mathbb{Z}^+$ and $c \mid a$ and $c \mid b$. By the Fundamental Theorem of Arithmetic, we have a unique standard factorization $c = q_1^{g_1} q_2^{g_2} \cdots q_n^{g_n}$ for some primes $q_1, q_2, \ldots, q_n$ and natural numbers $g_1, g_2, \ldots, g_n$. Since $c \mid a$, we have $q_1 \mid a$. Since q_1 is prime, it must be that $q_1 = p_1$. Moreover, since $q_1^{g_1} \mid a$, it must be that $q_1^{g_1} \mid p_1^{e_1}$, whence $g_1 \leq e_1$. Similarly, $g_1 \leq f_1$. Hence, $g_1 \leq \min\{e_1, f_1\}$. Repeating this argument, we conclude that, for each i, we have $g_i \leq \min\{e_i, f_i\}$. Therefore, $c \mid d$, and it follows that $c \leq d$. $\square$

33. *Sketch.* Write $a = p_1^{e_1} \cdot p_2^{e_2} \cdot \cdots \cdot p_m^{e_m}$ and $b = p_1^{f_1} \cdot p_2^{f_2} \cdot \cdots \cdot p_m^{f_m}$, where $e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_m$ are *nonnegative* integers. Let $d = \gcd(a, b)$, and write $d = p_1^{c_1} \cdot p_2^{c_2} \cdot \cdots \cdot p_m^{c_m}$, where, for each $1 \leq i \leq m$, $c_i = \min\{e_i, f_i\}$. Note that $a^2 = p_1^{2e_1} \cdot p_2^{2e_2} \cdot \cdots \cdot p_m^{2e_m}$, $b^2 = p_1^{2f_1} \cdot p_2^{2f_2} \cdot \cdots \cdot p_m^{2f_m}$, and $d^2 = p_1^{2c_1} \cdot p_2^{2c_2} \cdot \cdots \cdot p_m^{2c_m}$. Since, for each $1 \leq i \leq m$, $2c_i = \min\{2e_i, 2f_i\}$, we see that $d^2 = \gcd(a^2, b^2)$. $\square$

35. (a) *Proof.* Existence: The base case $n = 1$ is obvious, so we focus on the inductive step. Suppose $k \geq 1$ and each $1 \leq i \leq k$ has a binary representation. By the Division Algorithm, $k + 1 = 2j + r$ for some $r = 0$ or 1 and some positive integer $j \leq k$. Write $j = b_m 2^m + b_{m-1}2^{m-1} + b_1 2 + b_0$. Observe that $k+1 = 2j + r = b_m 2^{m+1} + b_{m-1}2^m + b_1 2^2 + b_0 2 + r$. Thus, $k+1$ has a binary representation. Uniqueness: Suppose to the contrary that some $n > 1$ has two different binary representations. If necessary, by padding the shorter one with zeros on the left, we may assume that they have the same number of digits. Say,

$$b_m 2^m + b_{m-1}2^{m-1} + b_1 2 + b_0 = a_m 2^m + a_{m-1}2^{m-1} + a_1 2 + a_0. \tag{2.1}$$

Let j be the largest index where $b_j \neq a_j$, say $b_j = 1$ and $a_j = 0$. Since, $\sum_{i=0}^{j-1} a_i 2^i < 2^j$, equation (2.1) is impossible. Thus we have a contradiction. □
(b) *Proof.* Existence: The base case $n = 1$ is obvious, so we focus on the inductive step. Suppose $k \geq 1$ and each $1 \leq i \leq k$ has a base s representation. By the Division Algorithm, $k + 1 = sj + r$ for some $0 \leq r < s$ and some positive integer $j \leq k$. Write $j = b_m s^m + b_{m-1}s^{m-1} + b_1 s + b_0$. Observe that $k+1 = sj + r = b_m s^{m+1} + b_{m-1}s^m + b_1 s^2 + b_0 s + r$. Thus, $k+1$ has a base s representation. Uniqueness: Suppose to the contrary that some $n > 1$ has two different base s representations. If necessary, by padding the shorter one with zeros on the left, we may assume that they have the same number of digits. Say,

$$b_m s^m + b_{m-1}s^{m-1} + b_1 s + b_0 = a_m s^m + a_{m-1}s^{m-1} + a_1 s + a_0. \tag{2.2}$$

Let j be the largest index where $b_j \neq a_j$, say $b_j > a_j$. Since, $\sum_{i=0}^{j-1} a_i s^i < s^j$, equation (2.2) is impossible. Thus we have a contradiction. □

37. *Proof.* The base case is Theorem 4.2(b). Suppose $k \geq 1$ and that, for each $1 \leq j \leq k$, $n(n+1)$ is a factor of $\sum_{i=1}^{n} i^j$. By Theorem 4.4,

$$\sum_{i=1}^{n} i^{k+1} = \frac{(n+1)((n+1)^{k+1} - 1) - \sum_{j=1}^{k} \left[\binom{k+2}{j} \sum_{i=1}^{n} i^j \right]}{k+2}.$$

Since n is a factor of $(n+1)^{k+1} - 1$ and, by the inductive hypothesis, $n(n+1)$ is a factor of each $\sum_{i=1}^{n} i^j$, it follows that $n(n+1)$ is a factor of the numerator above as asserted. □
That n is a factor of $(n+1)^{k+1} - 1$ can be proven here by induction on k or can be seen by the Binomial Theorem in the next section.

39. Note that only regular induction is needed here. *Proof. Base case*: $(n = 0)$. Note that $F_0 = F_2 - 1$. *Inductive step*: Suppose $k \geq 0$ and $\sum_{i=0}^{k} F_i = F_{k+2} - 1$. Observe that $\sum_{i=0}^{k+1} F_i = (F_{k+2} - 1) + F_{k+1} = F_{k+3} - 1$. $\square$

41. Note that only regular induction is needed here.
Proof. Base case: $(n = 2)$. Note that $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}$.
Inductive step: Suppose $k \geq 2$ and $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k = \begin{bmatrix} F_k & F_{k-1} \\ F_{k-1} & F_{k-2} \end{bmatrix}$. Observe that
$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k+1} = \begin{bmatrix} F_k & F_{k-1} \\ F_{k-1} & F_{k-2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} =$
$\begin{bmatrix} F_k + F_{k-1} & F_k \\ F_{k-1} + F_{k-2} & F_{k-1} \end{bmatrix} = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}$. $\square$

43. *Proof. Base case*: $(n = 0)$. Note that $\gcd(F_0, F_1) = \gcd(1, 1) = 1$. *Inductive step*: Suppose $k \geq 0$ and $\gcd(F_k, F_{k+1}) = 1$. Suppose to the contrary that $\gcd(F_{k+1}, F_{k+2}) > 1$. So we have some integer $c > 1$ such that c divides F_{k+1} and F_{k+2}. That is, we have $a, b \in \mathbb{Z}$ such that $F_{k+1} = ca$ and $F_{k+2} = cb$. Hence, $F_k = F_{k+2} - F_{k+1} = cb - ca = c(b - a)$. Now c divides F_k and F_{k+1}, which contradicts the fact that $\gcd(F_k, F_{k+1}) = 1$. So we conclude that $\gcd(F_{k+1}, F_{k+2}) = 1$.
$\square$

45. *Proof.* Assume conditions (i) and (ii) in the hypotheses of the theorem. Suppose it is not true that $P(n)$ holds $\forall\, n \geq a$. Let S be the set of those integers $n \geq a$ for which $P(n)$ does not hold. By our assumptions, S is nonempty. Hence, by the Generalized Well-Ordering Principle, S has a smallest element, say s. Since $P(a), P(a+1), \ldots, P(b)$ all hold, it must be that $s > b$. Therefore, $s - 1 \geq b$. Since $a, \ldots, s - 1 \notin S$, it follows that $P(a), \ldots, P(s - 1)$ all hold. However, for $k = s - 1$, by condition (ii), $P(k + 1)$ must also hold. That is, $P(s)$ holds. This contradicts the fact that $s \in S$. $\square$

47. *Sketch.* Attempting to write $23 = 5t + 2c$ for $t = 0, \ldots, 4$ shows that it is impossible. Now observe that $24, \ldots, 28$ are achievable. For any number of points $k \geq 29$, we can always score $(k - 5)$ points first and then score another try. $\square$
The requirement that $t \geq c$ forces us to consider 5 base cases instead of just choosing the smaller number 2.

Section 4.6

1. $\frac{(n-1)!}{(k-1)!(n-k)!} \frac{n!}{(k+1)!(n-1-k)!} \frac{(n+1)!}{k!(n+1-k)!} = \frac{(n-1)!}{k!(n-1-k)!} \frac{n!}{(k-1)!(n+1-k)!} \frac{(n+1)!}{(k+1)!(n-k)!}$.
The numerators are the same. In the denominators, the factors on the right-hand side are the same as those on the left-hand side. They just appear in a different order.

3. $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.
$(x+y)^5=\binom{5}{0}x^5y^0+\binom{5}{1}x^4y^1+\binom{5}{2}x^3y^2+\binom{5}{3}x^2y^3+\binom{5}{4}x^1y^4+\binom{5}{5}x^0y^5=$ $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.

5. $729x^6+1458x^5y+1215x^4y^2+540x^3y^3+135x^2y^4+18xy^5+y^6$.
$(3x+y)^6=\binom{6}{0}(3x)^6y^0+\binom{6}{1}(3x)^5y^1+\binom{6}{2}(3x)^4y^2+\binom{6}{3}(3x)^3y^3+\binom{6}{4}(3x)^2y^4+\binom{6}{5}(3x)^1y^5+\binom{6}{6}(3x)^0y^6=729x^6+1458x^5y+1215x^4y^2+540x^3y^3+135x^2y^4+18xy^5+y^6$.

7. $32x^5-80x^4y+80x^3y^2-40x^2y^3+10xy^4-y^5$.
$(2x-y)^5=(2x+(-y))^5=\binom{5}{0}(2x)^5(-y)^0+\binom{5}{1}(2x)^4(-y)^1+\binom{5}{2}(2x)^3(-y)^2+\binom{5}{3}(2x)^2(-y)^3+\binom{5}{4}(2x)^1(-y)^4+\binom{5}{5}(2x)^0(-y)^5=32x^5-80x^4y+80x^3y^2-40x^2y^3+10xy^4-y^5$.

9. $x^n-nx^{n-1}+\binom{n}{2}x^{n-2}-\binom{n}{3}x^{n-3}+\cdots+(-1)^n$.
$(x-1)^n=(x+(-1))^n=\binom{n}{0}x^n(-1)^0+\binom{n}{1}x^{n-1}(-1)^1+\binom{n}{2}x^{n-2}(-1)^2+\binom{n}{3}x^{n-3}(-1)^3+\cdots+\binom{n}{n}x^0(-1)^n=x^n-nx^{n-1}+\binom{n}{2}x^{n-2}-\binom{n}{3}x^{n-3}+\cdots+(-1)^n$.

11. (a) $(x+\frac{1}{2})^n=\binom{n}{0}x^n(\frac{1}{2})^0+\binom{n}{1}x^{n-1}(\frac{1}{2})^1+\binom{n}{2}x^{n-2}(\frac{1}{2})^2+\binom{n}{3}x^{n-3}(\frac{1}{2})^3+\cdots+\binom{n}{n}x^0(\frac{1}{2})^n=x^n+\frac{1}{2}nx^{n-1}+\frac{1}{4}\binom{n}{2}x^{n-2}+\frac{1}{8}\binom{n}{3}x^{n-3}+\cdots+\frac{1}{2^n}$. (b) The relevant term is $\binom{n}{n-5}x^5(\frac{1}{2})^{n-5}=\frac{1}{2^{n-5}}\binom{n}{5}x^5$. So the coefficient of x^5 is $\frac{1}{2^{n-5}}\binom{n}{5}$.

13. $x^8+4x^6y^2+6x^4y^4+4x^2y^6+y^8$.
$(x^2+y^2)^4=\binom{4}{0}(x^2)^4(y^2)^0+\binom{4}{1}(x^2)^3(y^2)^1+\binom{4}{2}(x^2)^2(y^2)^2+\binom{4}{3}(x^2)^1(y^2)^3+\binom{4}{4}(x^2)^0(y^2)^4=x^8+4x^6y^2+6x^4y^4+4x^2y^6+y^8$.

15. $243x^{10}+405x^8y^3+270x^6y^6+90x^4y^9+15x^2y^{12}+y^{15}$.
$(3x^2+y^3)^5=\binom{5}{0}(3x^2)^5(y^3)^0+\binom{5}{1}(3x^2)^4(y^3)^1+\binom{5}{2}(3x^2)^3(y^3)^2+\binom{5}{3}(3x^2)^2(y^3)^3+\binom{5}{4}(3x^2)^1(y^3)^4+\binom{5}{5}(3x^2)^0(y^3)^5=243x^{10}+405x^8y^3+270x^6y^6+90x^4y^9+15x^2y^{12}+y^{15}$.

17. (a) $(x^2+1)^n=\binom{n}{0}(x^2)^n1^0+\binom{n}{1}(x^2)^{n-1}1^1+\binom{n}{2}(x^2)^{n-2}1^2+\binom{n}{3}(x^2)^{n-3}1^3+\cdots+\binom{n}{n-1}(x^2)^11^{n-1}+\binom{n}{n}(x^2)^01^n=x^{2n}+nx^{2n-2}+\binom{n}{2}x^{2n-4}+\binom{n}{3}x^{2n-6}+\cdots+nx^2+1$. (b) The relevant term is $\binom{n}{n-4}(x^2)^41^{n-4}=\binom{n}{4}x^8$. So the coefficient of x^8 is $\binom{n}{4}$.

19. $\binom{n}{2}n^{n-2}-\binom{n}{3}n^{n-3}+\cdots+(-1)^{n-1}n^2+(-1)^n$.
In Exercise 9, substitute $x=n$. Note that the first two terms $n^n-n\cdot n^{n-1}$ cancel.

21. (a) $(1+\frac{1}{100})^{100}\approx 2.7048$, $(1+\frac{1}{1000})^{1000}\approx 2.7169$, $(1+\frac{1}{10000})^{10000}\approx 2.7181$.
(b) $(1+\frac{1}{n})^n=\sum_{i=0}^n\binom{n}{i}1^{n-i}(\frac{1}{n})^i=\sum_{i=0}^n\frac{n!}{i!(n-i)!}\frac{1}{n^i}=\sum_{i=0}^n\frac{1}{i!}\frac{(n-1)(n-2)\cdots(n-i+1)}{n^{i-1}}$.

23. $\binom{100}{40}3^{60}2^{40}$.
Note that $60+40=100$ and $(3x+2y)^{100} = \cdots + \binom{100}{40}(3x)^{60}(2y)^{40} + \cdots = \cdots + \binom{100}{40}3^{60}2^{40}x^{60}y^{40} + \cdots$. So $\binom{100}{40}3^{60}2^{40}$ is the coefficient of $x^{60}y^{40}$.

25. $\binom{400}{10}2^{10}$.
Note that $(x^2)^{10}(y^3)^{390} = x^{20}y^{1170}$, that $10+390=400$ and, that $(2x^2+y^3)^{400} = \cdots + \binom{400}{10}(2x^2)^{10}(y^3)^{390} + \cdots = \cdots + \binom{400}{10}2^{10}x^{20}y^{1170} + \cdots$. So $\binom{400}{10}2^{10}$ is the coefficient of $x^{20}y^{1170}$.

27. 0.
Note that $(x^2)^{30-i}(y^2)^i = x^{60-2i}y^{2i}$ can never be $x^{50}y^{50}$, since there is no value of i for which both $60-2i=50$ and $2i=50$.

29. (a) $(x+x^{-1})^{10} = x^{10}+10x^8+45x^6+120x^4+210x^2+252+210x^{-2}+120x^{-4}+45x^{-6}+10x^{-8}+x^{-10}$ and $\frac{1}{2^{10}}\cdot 120 = \frac{15}{128}$.
(b) $(x^2+x^{-2})^{10} = x^{20}+10x^{16}+45x^{12}+120x^8+210x^4+252+210x^{-4}+120x^{-8}+45x^{-12}+10x^{-16}+x^{-20}$ and $\frac{1}{2^{10}}\cdot 210 = \frac{105}{512}$.
(c) $(x^3+x^{-1})^{10} = x^{30}+10x^{26}+45x^{22}+120x^{18}+210x^{14}+252x^{10}+210x^6+120x^2+45x^{-2}+10x^{-6}+x^{-10}$ and $\frac{1}{2^{10}}\cdot 0 = 0$.

31. *Proof.* $9^n = (1+8)^n = \sum_{i=0}^{n}\binom{n}{i}1^{n-i}8^i = \sum_{i=0}^{n}\binom{n}{i}8^i$. $\square$

33. *Proof.* $12^n = (10+2)^n = \sum_{i=0}^{n}\binom{n}{i}10^{n-i}2^i$. $\square$

35. *Proof.* $2^{3n} = 8^n = (5+3)^n = \sum_{i=0}^{n}\binom{n}{i}5^{n-i}3^i$. $\square$

37. *Proof.* $2^n = (3-1)^n = \sum_{i=0}^{n}\binom{n}{i}3^{n-i}(-1)^i = \sum_{i=0}^{n}(-1)^i\binom{n}{i}3^{n-i}$. $\square$

39. (a) *Proof.* $6^n = (2+4)^n = \sum_{i=0}^{n}\binom{n}{i}2^{n-i}4^i = \sum_{i=0}^{n}\binom{n}{i}2^{n-i}2^{2i} = \sum_{i=0}^{n}\binom{n}{i}2^{n+i}$. $\square$
(b) $\sum_{i=0}^{n}\binom{n}{i}2^i = 3^n$. Consider $(1+2)^n = 3^n$.

41. *Proof.* $(\frac{2}{3})^n = (1-\frac{1}{3})^n = \sum_{i=0}^{n}\binom{n}{i}1^{n-i}(-\frac{1}{3})^i = \sum_{i=0}^{n}\binom{n}{i}(-1)^i(\frac{1}{3})^i = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(\frac{1}{3})^i$. $\square$

43. *Proof.* Suppose a and b are relatively prime. By Corollary 3.14, we get $ax+by=1$, for some $x,y\in\mathbb{Z}$. So, $b^ny^n = (1-ax)^n =$
$1 - nax + \binom{n}{2}a^2x^2 - \binom{n}{3}a^3x^3 + \cdots + (-1)^na^nx^n =$
$1 + x(-na + \binom{n}{2}a^2x - \binom{n}{3}a^3x^2 + \cdots + (-1)^na^nx^{n-1})$.
With $c = -na + \binom{n}{2}a^2x - \binom{n}{3}a^3x^2 + \cdots + (-1)^na^nx^{n-1}$, we have $b^ny^n = 1 - xc$. That is, $cx + b^ny^n = 1$. By Corollary 3.14, a and b^n are relatively prime. $\square$

Review

1. $8, 64, 320, 1280, 4480$.
$2^3\binom{3}{3} = 8$,
$2^4\binom{4}{3} = 64$,
$2^5\binom{5}{3} = 320$,
$2^6\binom{6}{3} = 1280$,
$2^7\binom{7}{3} = 4480$.

2. $3, 5, 21, 437, 190965$.
$s_1 = 3$,
$s_2 = 3^2 - 4 = 5$,
$s_3 = 5^2 - 4 = 21$,
$s_4 = 21^2 - 4 = 437$,
$s_5 = 437^2 - 4 = 190965$.

3. $\forall\, n \geq 1,\ s_n = \frac{n}{2^n}$.
Note that these are fractions. The numerators form the sequence $1, 2, 3, 4, 5, \ldots$. The denominators are the powers of two $2^1, 2^2, 2^3, 2^4, 2^5, \ldots$.

4. $s_0 = -6$, and $\forall\, n \geq 1,\ s_n = s_{n-1} + 12$. This is an arithmetic sequence. The difference between consecutive terms is always 12.

5. $\forall\, n \geq 0,\ s_n = -2(-3)^n$.
This is a geometric sequence. The multiplying factor is -3.

6. (a) $500(1 + \frac{.06}{12})^2 = 505.01$.
(b) Note that 2 years is 24 months and $500(1 + \frac{.06}{12})^{24} = 563.58$.

7. $\forall\, n \geq 0,\ s_n = 8 \cdot 2^n\binom{n+3}{3}$.
Let $m = n - 3$. So $2^n\binom{n}{3} = 2^{m+3}\binom{n+3}{3} = 2^m 2^3\binom{n+3}{3} = 8 \cdot 2^m\binom{m+3}{3}$.

8. $s_k^2 - 4$. Let $n = k + 1$. So $s_{k+1} = s_{(k+1)-1}^2 - 4 = s_k^2 - 4$.

9. $\sum_{i=3}^{n} 3 \cdot 2^{i-1}$. The last term suggests that the general term might be $3 \cdot 2^{i-1}$. Note that the first term is $12 = 3 \cdot 2^{3-1}$.

10. 125250.
$\sum_{i=1}^{500} i = \frac{500(501)}{2} = 125250$.

11. $\frac{4^{11}-1}{3}$.
$\sum_{i=0}^{10} 4^i = \frac{4^{10+1}-1}{4-1} = \frac{4^{11}-1}{3}$.

12. (a) $\sum_{i=1}^{20} i = \frac{20(21)}{2} = 210$. (b) $\sum_{i=1}^{20} i^2 = \frac{20(21)(41)}{6} = 2870$.

13. 1010200.
$3\sum_{i=1}^{100} i^2 - \sum_{i=1}^{100} i + \sum_{i=1}^{100} 2 = 3\frac{100(101)(201)}{6} - \frac{100(101)}{2} + 2(100) = 1010200.$

14. $\frac{n(2n^2-9n+13)}{6}$.
$\sum_{i=1}^{n}(i^2-4i+4) = \sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 4 = \frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + 4n = \frac{n(2n^2-9n+13)}{6}$.

15. $\frac{1-(-2)^{n+1}}{3}$.
$\frac{(-2)^{n+1}-1}{-2-1} = \frac{1-(-2)^{n+1}}{1-(-2)} = \frac{1-(-2)^{n+1}}{3}$.

16. $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 = 945$.

17. (a) $s_{10} = 80000(1.05)^{12(10)} - \sum_{i=0}^{12(10)-1} 500(1.05)^i = 63,612.07$ and $s_{20} = 80000(1.05)^{12(20)} - \sum_{i=0}^{12(20)-1} 500(1.05)^i = 83,395.81$.
(b) We want $M(1.005)^{12(30)} = 500\sum_{i=0}^{359}(1.005)^i = 500\frac{1.005^{360}-1}{0.005}$. So $M =$ \$83,395.81.

18. (a) Let $j = i - 5$. So $\sum_{j=0}^{198} \frac{2}{3^{j+5}} = \frac{2}{3^5}\sum_{j=0}^{198}(\frac{1}{3})^j$.
(b) $\sum_{j=0}^{198}(\frac{1}{3})^j = \frac{2}{3^5}\frac{1-(\frac{1}{3})^{199}}{1-\frac{1}{3}} = \frac{1-(\frac{1}{3})^{199}}{3^4}$.

19. *Proof. Base case*: $(n = 9)$. Note that $9! > 4^9$. *Inductive step*: Suppose $k \geq 9$ and $k! > 4^k$. (Goal: $(k+1)! > 4^{k+1}$.) Observe that $(k+1)! = (k+1)k! > (k+1)4^k \geq 4 \cdot 4^k = 4^{k+1}$. That is, $(k+1)! > 4^{k+1}$. □

20. *Proof. Base case*: $(n = 6)$. Note that $6^2 > 4(6+2)$. *Inductive step*: Suppose $k \geq 6$ and $k^2 > 4(k+2)$. (Goal: $(k+1)^2 > 4(k+3)$.) Observe that $(k+1)^2 = k^2 + 2k + 1 > 4(k+2) + 2k + 1 = 4k + 2k + 9 > 4k + 12 = 4(k+3)$. □

21. *Proof. Base case*: $(n = 0)$. Note that $3^0 \geq 0^2 + 1$. *Inductive step*: Suppose $k \geq 0$ and $3^k \geq k^2 + 1$. (Goal: $3^{k+1} \geq (k+1)^2 + 1$.) Observe that $3^{k+1} = 3 \cdot 3^k \geq 3(k^2+1) = 3k^2 + 3 = k^2 + (2k^2+3) \geq k^2 + (2k+2) = (k+1)^2 + 1$. □

22. *Proof. Base case*: $(n = 0)$. Clearly, $3 \mid 6$. *Inductive step*: Suppose $k \geq 0$ and $3 \mid (k^3 - 4k + 6)$. So $k^3 - 4k + 6 = 3c$ for some $c \in \mathbb{Z}$. Observe that $(k+1)^3 - 4(k+1) + 6 = (k^3 - 4k + 6) + (3k^2 + 3k - 3) = 3(c + k^2 + 3k - 1)$. So $3 \mid ((k+1)^3 - 4(k+1) + 6)$. □

23. *Proof. Base case*: $(n = 0)$. Clearly, $6 \mid 0$. *Inductive step*: Suppose $k \geq 0$ and $6 \mid (7^k - 1)$. So $7^k - 1 = 6c$ for some $c \in \mathbb{Z}$. Observe that $7^{k+1} - 1 = 7 \cdot 7^k - 1 = 6 \cdot 7^k + 7^k - 1 = 6(7^k + c)$. So $6 \mid (7^{k+1} - 1)$. □

24. *Proof. Base case*: $(n = 0)$. Clearly, $3 \mid 0$. *Inductive step*: Suppose $k \geq 0$ and $3 \mid (5^k - 2^k)$. So $5^k - 2^k = 3c$ for some $c \in \mathbb{Z}$. Observe that $5^{k+1} - 2^{k+1} = 5 \cdot 5^k - 2 \cdot 2^k = 3 \cdot 5^k + 2(5^k - 2^k) = 3(5^k + 2c)$. So $3 \mid (5^{k+1} - 2^{k+1})$. □

25. (a) $s_1 = 1.005(0) + 300 = 300$,
$s_2 = 1.005(300) + 300 = 601.50$,
$s_3 = 1.005(601.50) + 300 = 904.51$.
(b) *Proof. Base case*: $(n = 0)$. Note that $0 = 60000(1.005^0 - 1)$. Suppose $k \geq 0$ and $s_k = 60000(1.005^k - 1)$. Observe that
$s_{k+1} = 1.005 s_k + 300 = (1.005)60000(1.005^k - 1) + 300 =$
$60000(1.005^{k+1}) - 60300 + 300 = 60000(1.005^{k+1} - 1)$. $\square$

26. *Proof.* The base case is the Distributive Law. Suppose $k \geq 2$ and $a(x_1 + \cdots + x_k) = ax_1 + \cdots + ax_k$. Observe that $a(x_1 + \cdots + x_k + x_{k+1}) = a((x_1 + \cdots + x_k) + x_{k+1}) = a(x_1 + \cdots + x_k) + ax_{k+1} = ax_1 + \cdots + ax_k + ax_{k+1}$.
$\square$

27. *Proof. Base case*: $(|S| = 1)$. If $S = \{s_1\}$, then $\min(S) = s_1$. *Inductive step*: Suppose $k \geq 1$ and, any set S with $|S| = k$ has a minimal element. (Goal: Any set S with $|S| = k+1$ has a minimal element.) Suppose $s_1, s_2, \ldots, s_{k+1}$ are distinct real numbers and $S = \{s_1, s_2, \ldots, s_{k+1}\}$. By the induction hypothesis, the set $\{s_1, s_2, \ldots, s_k\}$ has a minimal element, say s_j. Observe that $\{s_j, s_{k+1}\}$ has a minimal element $m = \begin{cases} s_j & \text{if } s_j < s_{k+1} \\ s_{k+1} & \text{otherwise.} \end{cases}$ Note that m is the minimum element of S. $\square$

28. *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{1}(\frac{1}{i} - \frac{1}{i+1}) = \frac{1}{2} = 1 - \frac{1}{1+1}$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k}(\frac{1}{i} - \frac{1}{i+1}) = 1 - \frac{1}{k+1}$. Observe that $\sum_{i=1}^{k+1}(\frac{1}{i} - \frac{1}{i+1}) = \sum_{i=1}^{k}(\frac{1}{i} - \frac{1}{i+1}) + (\frac{1}{k+1} - \frac{1}{k+2}) = 1 - \frac{1}{k+1} + \frac{1}{k+1} - \frac{1}{k+2} = 1 - \frac{1}{k+2}$.
$\square$

29. We prove $\forall\, n \geq 1$, $\sum_{i=1}^{n}(3i+1) = \frac{n}{2}(3n+5)$.
Proof. Base case: $(n = 1)$. Note that $4 = \frac{1}{2}(3+5)$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k}(3i+1) = \frac{k}{2}(3k+5)$. Observe that $\sum_{i=1}^{k+1}(3i+1) =$
$\sum_{i=1}^{k}(3i+1) + (3(k+1)+1) = \frac{k}{2}(3k+5) + (3k+4) = \frac{k(3k+5)+2(3k+4)}{2} = \frac{3k^2+11k+8}{2} = \frac{(k+1)(3k+8)}{2} = \frac{k+1}{2}(3(k+1)+5)$. $\square$

30. *Proof. Base case*: $(n = 1)$. Note that $3^1 = \frac{3}{2}(3^1 - 1)$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k} 3^i = \frac{3}{2}(3^k - 1)$. Now, $\sum_{i=1}^{k+1} 3^i = \sum_{i=1}^{k} 3^i + 3^{k+1} = \frac{3}{2}(3^k - 1) + 3^{k+1} = \frac{3(3^k-1)+2(3^{k+1})}{2} = \frac{3(3^{k+1})-3}{2} = \frac{3}{2}(3^{k+1} - 1)$. $\square$

31. *Proof. Base case*: $(n = 0)$. Note that $\sum_{i=0}^{0}(i+1)2^i = 1 = 0 \cdot 2^{0+1} + 1$. *Inductive step*: Suppose $k \geq 0$ and $\sum_{i=0}^{k}(i+1)2^i = k2^{k+1} + 1$. Observe that $\sum_{i=0}^{k+1}(i+1)2^i = \sum_{i=0}^{k}(i+1)2^i + (k+2)2^{k+1} = k2^{k+1} + 1 + (k+2)2^{k+1} =$
$(2k+2)2^{k+1} + 1 = (k+1)2^{k+2} + 1$. $\square$

32. *Proof. Base case*: $(n = 1)$. Note that $3 + 5 = 1(2)(4)$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k}(3i^2 + 5i) = k(k+1)(k+3)$. Now, $\sum_{i=1}^{k+1}(3i^2 + 5i) = \sum_{i=1}^{k}(3i^2+5i)+(3(k+1)^2+5(k+1)) = k(k+1)(k+3)+(k+1)(3(k+1)+5) = (k+1)((k+1)+1)((k+1)+3)$. $\square$

33. *Proof. Base case*: $(n = 1)$. Note that $\sum_{i=1}^{1} i4^i = 4 = \frac{4}{9}[4(2)+1]$. *Inductive step*: Suppose $k \geq 1$ and $\sum_{i=1}^{k} i4^i = \frac{4}{9}[4^k(3k-1)+1]$. Observe that $\sum_{i=1}^{k+1} i4^i = \sum_{i=1}^{k} i4^i + (k+1)4^{k+1} = \frac{4}{9}[4^k(3k-1)+1] + (k+1)4^{k+1} = \frac{4}{9}[4^k(3k-1)+1+9(k+1)4^k] = \frac{4}{9}[4^k(3k-1+9k+9)+1] = \frac{4}{9}[4^k(12k+8)+1] = \frac{4}{9}[4^{k+1}(3k+2)+1]$. $\square$

34. *Sketch.* The case when $n = 1$ is obvious, and the case when $n = 2$ is one of the Laws of Exponents from Appendix A. In the inductive step, we have $b^{\sum_{i=1}^{n+1} a_i} = b^{(\sum_{i=1}^{n} a_i)+a_{n+1}} = b^{(\sum_{i=1}^{n} a_i)}b^{a_{n+1}} = (\prod_{i=1}^{n} b^{a_i})b^{a_{n+1}} = \prod_{i=1}^{n+1} b^{a_i}$. $\square$

35. *Proof. Base cases*: $(n = 0, 1)$. Note that $3 \mid 6$ and $3 \mid 3$. *Inductive step*: Suppose $k \geq 1$ and that, for each $0 \leq i \leq k$, $3 \mid s_i$. (Goal: $3 \mid s_{k+1}$.) Since $3 \mid s_{k-1}$ and $3 \mid s_k$, we have $c, d \in \mathbb{Z}$ such that $s_{k-1} = 3c$ and $s_k = 3d$. Observe that $s_{k+1} = 2s_{k-1} + s_k = 2(3c) + 3d = 3(2c+d)$. Thus, $3 \mid s_{k+1}$. $\square$

36. *Proof. Base cases*: $(n = 0, 1)$. Note that $3 \mid 6$ and $3 \mid 3$. *Inductive step*: Suppose $k \geq 1$ and that, for each $0 \leq i \leq k$, $s_i = 4 \cdot 5^i + 3 \cdot 4^i$. (Goal: $s_{k+1} = 4 \cdot 5^{k+1} + 3 \cdot 4^{k+1}$.) Observe that $s_{k+1} = -20s_{k-1} + 9s_k = -20(4 \cdot 5^{k-1} + 3 \cdot 4^{k-1}) + 9(4 \cdot 5^k + 3 \cdot 4^k) = -16 \cdot 5^k - 14 \cdot 4^k + 36 \cdot 5^k + 27 \cdot 4^k = 20 \cdot 5^k + 12 \cdot 4^k = 4 \cdot 5^{k+1} + 3 \cdot 4^{k+1}$. $\square$

37. *Proof. Base cases*: $(n = 0, 1)$. Note that $5 = 2^1 + 3 \cdot 2^0$ and $16 = 2^2 + 3 \cdot 2^2$. *Inductive step*: Suppose $k \geq 0$ and that, for $0 \leq i \leq k$, $s_i = 2^{i+1} + 3 \cdot 2^{2i}$. Now, $s_{k+1} = 6s_k - 8s_{k-1} = 6(2^{k+1} + 3 \cdot 2^{2k}) - 8(2^k + 3 \cdot 2^{2(k-1)}) = 3 \cdot 2^{k+2} + 9 \cdot 2^{2k+1} - 2 \cdot 2^{k+2} - 3 \cdot 2^{2k+1} = 2^{k+2} + 6 \cdot 2^{2k+1} = 2^{(k+1)+1} + 3 \cdot 2^{2(k+1)}$. $\square$

38. (a) Four 3¢ stamps and one 8¢ stamp.
(b) *Sketch.* $14 = 2(3) + 1(8)$, $15 = 5(3)$, and $16 = 2(8)$. Further, $k + 1 = (k-2) + 1(3)$. Therefore, if, for some $k \geq 16$, we can obtain $(k-2)$¢, then an additional 3¢ stamp will yield $(k+1)$¢. $\square$
Three base cases suffice, since 3¢ is the smallest value for a single stamp.

39. $1001 = 7 \cdot 11 \cdot 13$.

40. $78408 = 2^3 \cdot 3^4 \cdot 11^2$.

41. $2^2 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 23 \cdot 43 \cdot 47$.
$\binom{50}{9} = \frac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43 \cdot 42}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 2^2 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 23 \cdot 43 \cdot 47$.

42. *Sketch.* Note that $1 = 4 - 3$. If $k \geq 1$ and $\sum_{i=1}^{k} L_i = L_{k+2} - 3$, then $\sum_{i=1}^{k+1} L_i = (\sum_{i=1}^{k} L_i) + L_{k+1} = (L_{k+2} - 3) + L_{k+1} = (L_{k+1} + L_{k+2}) - 3 = L_{k+3} - 3$. $\square$

43. *Proof. Base cases*: $(n = 0, 1)$. Note that $s_0 = 0 = 3^0(3^0 - 1)$ and $s_1 = 6 = 3^1(3^1 - 1)$. *Inductive step*: Suppose $k \geq 1$ and that, for each $0 \leq i \leq k$, $s_i = 3^i(3^i - 1)$. (Goal: $s_{k+1} = 3^{k+1}(3^{k+1} - 1)$.) Observe that
$s_{k+1} = 3(4s_k - 9s_{k-1}) = 3(4[3^k(3^k - 1)] - 9[3^{k-1}(3^{k-1} - 1)] =$
$3 \cdot 3^{k-1}(4 \cdot 3(3^k - 1) - 9(3^{k-1} - 1)) = 3^k(4 \cdot 3^{k+1} - 12 - 3^{k+1} + 9) = 3^k(3 \cdot 3^{k+1} - 3) =$
$3^{k+1}(3^{k+1} - 1)$. $\square$

44. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.
$(x + y)^4 = \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}x^0y^4 =$
$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

45. $6561x^8 - 69984x^7y + 326592x^6y^2 - 870912x^5y^3 + 1451520x^4y^4 - 1548288x^3y^5 + 1032192x^2y^6 - 393216xy^7 + 65536y^8$.
$(3x - 4y)^8 = (3x + (-4y))^8 = \binom{8}{0}(3x)^8(-4y)^0 + \binom{8}{1}(3x)^7(-4y)^1 + \binom{8}{2}(3x)^6(-4y)^2 +$
$\binom{8}{3}(3x)^5(-4y)^3 + \binom{8}{4}(3x)^4(-4y)^4 + \binom{8}{5}(3x)^3(-4y)^5 + \binom{8}{6}(3x)^2(-4y)^6 +$
$\binom{8}{7}(3x)^1(-4y)^7 + \binom{8}{8}(3x)^0(-4y)^8 = 6561x^8 - 69984x^7y + 326592x^6y^2 - 870912x^5y^3 +$
$1451520x^4y^4 - 1548288x^3y^5 + 1032192x^2y^6 - 393216xy^7 + 65536y^8$.

46. $x^{10} - 5x^8y^2 + 10x^6y^4 - 10x^4y^6 + 5x^2y^8 - y^{10}$.
$(x^2 - y^2)^5 = (x^2 + (-y^2))^5 = \binom{5}{0}(x^2)^5(-y^2)^0 + \binom{5}{1}(x^2)^4(-y^2)^1 + \binom{5}{2}(x^2)^3(-y^2)^2 +$
$\binom{5}{3}(x^2)^2(-y^2)^3 + \binom{5}{4}(x^2)^1(-y^2)^4 + \binom{5}{5}(x^2)^0(-y^2)^5 =$
$x^{10} - 5x^8y^2 + 10x^6y^4 - 10x^4y^6 + 5x^2y^8 - y^{10}$.
Be careful that $-y^2$ does not mean $(-y)^2$.

47. $\binom{100}{10}2^{90}$.
Since $(x - 2)^{10} = (x + (-2))^{100} = \cdots + \binom{100}{90}x^{10}(-2)^{90} + \cdots$, the coefficient of x^{10} is $\binom{100}{10}(-2)^{90} = \binom{100}{10}2^{90}$.

48. $-\binom{100}{25}3^{25}$.
Since $(3x - y)^{100} = (3x + (-y))^{100} = \cdots + \binom{100}{75}(3x)^{25}(-y)^{75} + \cdots$, the coefficient of $x^{25}y^{75}$ is $\binom{100}{75}3^{25}(-1)^{75} = -\binom{100}{25}3^{25}$.

49. 0. Since the exponents 30 and 40 do not add up to 80, there will be no nonzero coefficient of $x^{30}y^{40}$.

50. *Proof.* $6^n = (1 + 5)^n = \sum_{i=0}^{n} \binom{n}{i}1^{n-i}5^i = \sum_{i=0}^{n} \binom{n}{i}5^i$. $\square$

51. *Proof.* $5^n = (1 + 4)^n = \sum_{i=0}^{n} \binom{n}{i}1^{n-i}4^i = \sum_{i=0}^{n} \binom{n}{i}4^i = \sum_{i=0}^{n} \binom{n}{i}2^{2i}$. $\square$

52. *Proof.* $(-1)^n = (3 - 4)^n = \sum_{i=0}^{n} \binom{n}{i}3^{n-i}(-4)^i = \sum_{i=0}^{n}(-1)^i\binom{n}{i}3^{n-i}4^i$. $\square$

2.5 Chapter 5

Section 5.1

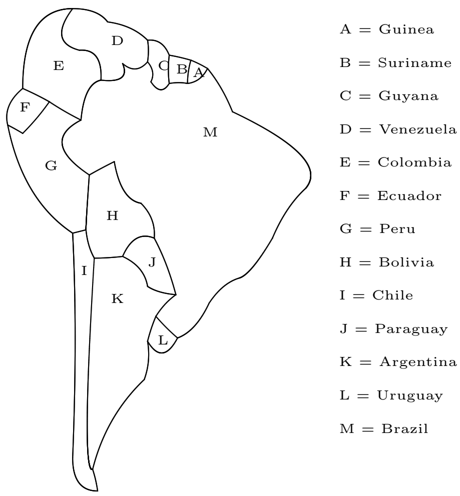
1. (a) False. $5 \nmid 1$.
(b) True. $3 \mid 6$.
(c) False. $2 \nmid 7$.

3. (a) True. $\emptyset \subseteq \mathbb{Z}$, by Theorem 1.3.
(b) False. $0 \notin X = \mathcal{P}(\mathbb{R})$, since 0 is not a subset of $\mathbb{R}$.
(c) True. $\{1, 2\} \subseteq \mathbb{R}^+$, since $2 > 1 > 0$.

5. True. $B \supseteq A$ if and only if $A \subseteq B$.

7. False. The y-axis is perpendicular to the x-axis, but the x-axis is not parallel to the y-axis.

9. (a) No. (b) Brazil, Colombia, Guyana.



11. The "is *a* son of" relation.
That is, B "is *a* son of" A if and only if A "is the father of" B.

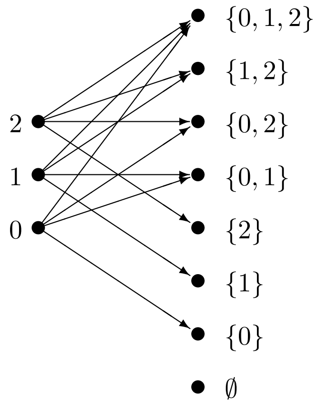
13. $\supseteq$. That is, $B \supseteq A$ if and only if $A \subseteq B$.

15. $\perp$. That is, $l_2 \perp l_1$ if and only if $l_1 \perp l_2$.

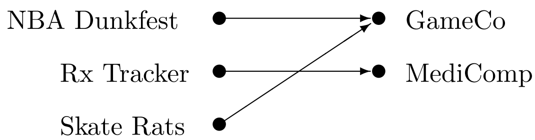
17. R itself.
That is, $y^2 + x^2 = 1$ if and only if $x^2 + y^2 = 1$.

19. (a) GameCo. (b) No. (c) Yes.

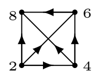
21.



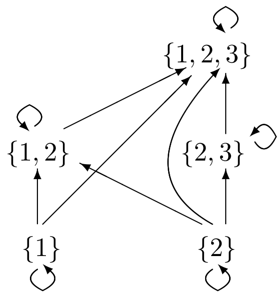
23.



25.



27.



29.

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 3 & 6 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

31.

$$\begin{array}{c} \\ \emptyset \\ \{1\} \\ \{2\} \\ \{1,2\} \end{array} \begin{array}{c} \begin{array}{cccc} \emptyset & \{1\} & \{2\} & \{1,2\} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

33.

We reflect the picture from (a) about the line $y = x$.

35.

We reflect the picture from (a) about the line $y = x$.

37 through 53.

Exercise	Reflexive	Symmetric	Antisymmetric	Transitive
37	F	T	F	F
39	T	T	F	F
41	F	T	T	T
43	F	F	T	T
45	F	F	T	F
47	T	T	F	T
49	T	F	F	F
51	T	F	F	T
53	T	F	T	T

37. Not reflexive, since $0 \not R\, 0$. Symmetric, since $\frac{1}{x} = y$ implies $\frac{1}{y} = x$. Not antisymmetric, since $\frac{1}{2} = 1/2$ and $\frac{1}{1/2} = 2$, but $2 \neq 1/2$. Not transitive, since $\frac{1}{2} = 1/2$ and $\frac{1}{1/2} = 2$, but $\frac{1}{2} \neq 2$.

39. Reflexive, since a line (being nonempty) always intersects itself. Symmetric, since l_1 intersects l_2 implies l_2 intersects l_1. Not antisymmetric, since the x-axis and y-axis intersect each other but are not equal to each other. Not transitive, since $y = 0$ intersects $x = 0$, and $x = 0$ intersects $y = 1$, but $y = 0$ and $y = 1$ are distinct lines.

41. Not reflexive, since $-1 \in \mathbb{R}$ but $\sqrt{-1} \notin \mathbb{R}$. So $-1 \not R -1$ Symmetric, since $\sqrt{x} = \sqrt{y}$ implies $\sqrt{y} = \sqrt{x}$. Antisymmetric, since $\sqrt{x} = \sqrt{y}$ and $\sqrt{y} = \sqrt{x}$ implies $x = y$ (and $x, y > 0$). Transitive, since $\sqrt{x} = \sqrt{y}$ and $\sqrt{y} = \sqrt{z}$ implies $\sqrt{x} = \sqrt{z}$.

43. Not reflexive, since no set is a *proper* subset of itself. Not symmetric, since $A \subset B$ and $B \subset A$ is impossible. Antisymmetric, since $A \subset B$ and $B \subset A$ is impossible, and an if-then statement is true when its hypothesis is false. Transitive, by the same argument as the transitivity of $\subseteq$ (Example 2.14).

45. Not reflexive, since $0+1 \neq 0$. Not symmetric, since $0+1 = 1$ and $1+1 \neq 0$. Antisymmetric, since the hypothesis $x + 1 = y$ and $y + 1 = x$ can never hold. Not transitive, since $0 + 1 = 1$ and $1 + 1 = 2$, but $0 + 1 \neq 2$.

47. Reflexive, since a and a are equal. Symmetric, since, if a and b are divisible by the same primes, then so are b and a. Not antisymmetric, since 2 and 4 are divisible by the same primes, as are 4 and 2, but $2 \neq 4$. Transitive, since, if a and b are divisible by the same primes and so are b and c, then so are a and c.

49. Reflexive, since $x \leq |x|$. Not symmetric, since $1 \leq |2|$ but $2 \not\leq |1|$. Not antisymmetric, since $1 \leq |-1|$ and $-1 \leq |1|$ but $1 \neq -1$. Not transitive, as can be seen for $x = 2$, $y = -2$, $z = 1$.

51. Reflexive, since $A \subseteq A \cup \mathbb{Z}$. Not symmetric, since $\emptyset \subseteq \mathbb{R} \cup \mathbb{Z}$ but $\mathbb{R} \not\subseteq \emptyset \cup \mathbb{Z}$. Not antisymmetric, since $\emptyset \subseteq \mathbb{Z} \cup \mathbb{Z}$ and $\mathbb{R} \not\subseteq \emptyset \cup \mathbb{Z}$ but $\emptyset \neq \mathbb{Z}$. Transitive, since, if $A \subseteq B \cup \mathbb{Z}$ and $B \subseteq C \cup \mathbb{Z}$, then $A \subseteq B \cup \mathbb{Z} \subseteq C \cup \mathbb{Z} \cup \mathbb{Z} = C \cup \mathbb{Z}$.

53. Reflexive, since $A \subseteq A$. Not symmetric, since $\emptyset \subseteq \{1\}$ but $\{1\} \not\subseteq \emptyset$. Antisymmetric, since $A \subseteq B$ and $B \subseteq A$ implies $A = B$ (see Section 2.3). Transitive, by Example 2.14.

55. Symmetric.
Two countries must share a border with each other. Counterexamples to the other properties exist in the South America example in Exercise 9.

57. *Proof.* ($\rightarrow$) Suppose R is symmetric. Since, $\forall\, x, y \in X, x\ R\ y \leftrightarrow y\ R\ x$ and $x\ R^{-1}\ y \leftrightarrow y\ R^{-1}\ x$, it follows that $R^{-1} = R$. ($\leftarrow$) Suppose $R^{-1} = R$. Suppose $x, y \in R$. Since $x\ R\ y \leftrightarrow x\ R^{-1}\ y \leftrightarrow y\ R\ x$, it follows that R is symmetric. $\square$

59. *Proof.* ($\rightarrow$) Suppose R is antisymmetric. Suppose $(x, y) \in R \cap R^{-1}$. Since $(x, y) \in R$ and $(x, y) \in R^{-1}$, we have $x\ R\ y$ and $x\ R^{-1}\ y$. That is, $x\ R\ y$ and $y\ R\ x$. Hence $x = y$, and we see that $(x, y) = (x, x) \in \Delta$. Thus, $R \cap R^{-1} \subseteq \Delta$. ($\leftarrow$) Suppose $R \cap R^{-1} \subseteq \Delta$. Suppose $x\ R\ y$ and $y\ R\ x$. Hence, $x\ R\ y$ and $x\ R^{-1}\ y$. So $(x, y) \in R$ and $(x, y) \in R^{-1}$. Since $(x, y) \in R \cap R^{-1} \subseteq \Delta$, we see that $x = y$. Thus, R is antisymmetric. $\square$

Section 5.2

1. *Proof.*
Reflexive: Let $a \in \mathbb{Z}^+$. Since $a = a \cdot 1$, we see that $a \mid a$.
That is, $a \ R \ a$.
Antisymmetric: Let $a, b \in \mathbb{Z}^+$. Suppose $a \mid b$ and $b \mid a$.
By Exercise 17 of Section 3.1, we have $a = \pm b$.
Since $a, b > 0$, it must be that $a = b$.
Transitive: Let $a, b, c \in \mathbb{Z}^+$. Suppose $a \mid b$ and $b \mid c$.
By Example 3.4 of Section 3.1, we have $a \mid c$. $\square$

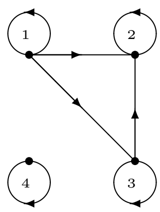
3. *Proof.* Let x, y, z be arbitrary elements of X.
Reflexive: Since $x \ R \ x$, we have $x \ R^{-1} \ x$.
Antisymmetric: Suppose $x \ R^{-1} \ y$ and $y \ R^{-1} \ x$.
That is, $y \ R \ x$ and $x \ R \ y$. Hence, $x = y$.
Transitive: Suppose $x \ R^{-1} \ y$ and $y \ R^{-1} \ z$.
That is, $y \ R \ x$ and $z \ R \ y$.
Thus, $z \ R \ x$, and we have $x \ R^{-1} \ z$. $\square$

5. Just the one in Exercise 53.
That is the only one of those relations that is reflexive, antisymmetric, and transitive. See the answers to the odd numbered exercises in Exercises 37 through 54 from Section 5.1

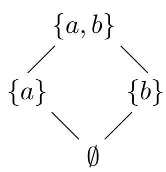
7. No. It is not antisymmetric.
Note that $\{1, 3\} \ R \ \{1, 2, 3\}$ and $\{1, 2, 3\} \ R \ \{1, 3\}$, but $\{1, 3\} \neq \{1, 2, 3\}$.

9. No. It is not reflexive.
Note that Algebra $\not R$ Algebra.

11. Yes.



13.



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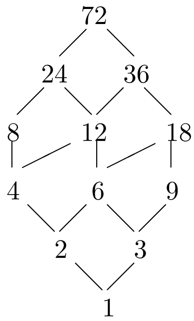
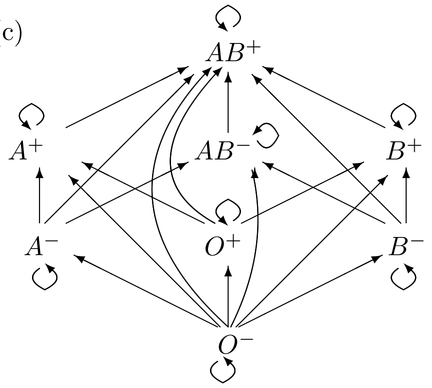
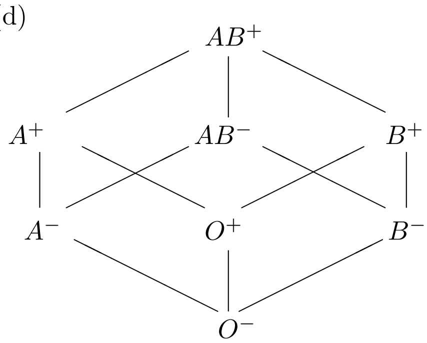


17. (a) No. (b) O^-, O^+, A^-, A^+.

(c)



(d)



In (a), note that A^+ contains the antigen Rh and A^- does not. In (b), note that O^-, O^+, A^-, A^+ are the types that contain fewer antigens than A^+.

19. *Proof.* Suppose to the contrary that a Hasse diagram contains a triangle. Since there can be no horizontal lines, the three elements involved must be at distinct heights. The line from the lowest element to the highest element follows from transitivity from the other two lines. This is a contradiction. $\square$

21. b12vitamin $\lhd$ b6vitamin.
Note that $1 < 6$. Do not be fooled by the twelve.

23. $(2, -3, 1, 0) \lhd (2, -1, 5, 3)$.
Note that $-3 < -1$.

25. $a_{n-1}a_{n-2}\cdots a_0 < b_{n-1}b_{n-2}\cdots b_0$ iff
$a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \cdots + a_0 < b_{n-1}2^{n-1} + b_{n-2}2^{n-2} + \cdots + b_0$ iff
$a_{n-1} = b_{n-1}, \ldots, a_k = b_k$ and $a_{k-1} < b_{k-1}$.
Regard $a_{n-1}a_{n-2}\cdots a_0$ and $b_{n-1}b_{n-2}\cdots b_0$ as integers represented in binary.
So we have $a_{n-1}a_{n-2}\cdots a_0 < b_{n-1}b_{n-2}\cdots b_0$ iff

$a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \cdots + a_0 < b_{n-1}2^{n-1} + b_{n-2}2^{n-2} + \cdots + b_0$. Let $k \geq 1$ be the largest index where $a_{k-1} \neq b_{k-1}$. Since, $\sum_{i=0}^{k-2} b_i 2^i < 2^{k-1}$, we must equivalently have $a_{n-1} = b_{n-1}, \ldots, a_k = b_k$ and $a_{k-1} = 0 < 1 = b_{k-1}$. This characterizes the lexicographic ordering.

27. *Proof.* Let $x, y, z \in \mathbb{R}$.
Reflexive: Of course, $\lfloor x \rfloor = \lfloor x \rfloor$.
Symmetric: Suppose $\lfloor x \rfloor = \lfloor y \rfloor$. So $\lfloor y \rfloor = \lfloor x \rfloor$.
Transitive: Suppose $\lfloor x \rfloor = \lfloor y \rfloor$ and $\lfloor y \rfloor = \lfloor z \rfloor$. Hence, $\lfloor x \rfloor = \lfloor y \rfloor = \lfloor z \rfloor$. $\square$

29. *Sketch. Reflexive*: $m_1 - n_1 = m_1 - n_1$.
Symmetric: $m_1 - n_1 = m_2 - n_2 \to m_2 - n_2 = m_1 - n_1$.
Transitive: $m_1 - n_1 = m_2 - n_2$, $m_2 - n_2 = m_3 - n_3 \to m_1 - n_1 = m_2 - n_2 = m_3 - n_3$. $\square$

31. *Proof.* Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2 \setminus \{(0,0)\}$.
Reflexive: Since $1x_1 = x_1$ and $1y_1 = y_1$, we have $(x_1, y_1)\ R\ (x_1, y_1)$.
Symmetric: Suppose $(x_1, y_1)\ R\ (x_2, y_2)$. So we have $c \neq 0$ with $cx_1 = x_2$ and $cy_1 = y_2$. Hence, $\frac{1}{c}x_2 = x_1$ and $\frac{1}{c}y_2 = y_1$. Thus, $(x_2, y_2)\ R\ (x_1, y_1)$.
Transitive: Suppose $(x_1, y_1)\ R\ (x_2, y_2)$ and $(x_2, y_2)\ R\ (x_3, y_3)$. So we have $c, d \neq 0$ such that $cx_1 = x_2$, $cy_1 = y_2$, $dx_2 = x_3$, and $dy_2 = y_3$. Since $cd \neq 0$, $cdx_1 = x_3$, and $cdy_1 = y_3$, it follows that $(x_1, y_1)\ R\ (x_3, y_3)$. $\square$

33. For example, it is not reflexive.
Note that $(1,0) \not R\ (1,0)$ since $1 + 1 \neq 0 + 0$.

35. Just the one in Exercise 47. That is the only one of those relations that is reflexive, symmetric, and transitive. See the answers to the odd numbered exercises in Exercises 37 through 54 from Section 5.1.

37. *Proof.* Suppose $x, y \in X$ and $x \in [y]$. So $x\ R\ y$. Since R is symmetric, $y\ R\ x$. Thus $y \in [x]$. The converse is handled similarly. $\square$

39. $(m_1 - n_1, 0)$ if $m_1 \geq n_1$, and $(0, n_1 - m_1)$ if $m_1 < n_1$.
Note that both coordinates must be greater than or equal to zero. If $m_1 \geq n_1$, then $m_1 - n_1 \geq 0$. Also, $(m_1, n_1)\ R\ (m_1 - n_1, 0)$ since $m_1 - n_1 = (m_1 - n_1) - 0$.

41. $(\frac{x_1}{\sqrt{x_1^2+y_1^2}}, \frac{y_1}{\sqrt{x_1^2+y_1^2}})$.
Here, we use $c = \frac{1}{\sqrt{x_1^2+y_1^2}}$. Also, note that $\left(\frac{x_1}{\sqrt{x_1^2+y_1^2}}\right)^2 + \left(\frac{y_1}{\sqrt{x_1^2+y_1^2}}\right)^2 = \frac{x_1^2}{x_1^2+y_1^2} + \frac{y_1^2}{x_1^2+y_1^2} = 1$.

43. Yes.
Each element of $\{1, 2, 3, 4, 5, 6\}$ is present in exactly one of the sets A_i.

45. No.
The element $1 \in \mathbb{R}^+$ appears in no set A_i.

47. No.
Halmos is on two subcommittees.

49. Each integer is either odd or even, and not both.
Sketch. Since no integer can be both odd and even, A_1 and A_2 are disjoint. Since each integer is either odd or even, $A_1 \cup A_2 = \mathbb{Z}$. $\square$

51. *Proof.* (Disjoint) Suppose $r_1, r_2 \in \mathbb{Q}$ with $r_1 \neq r_2$ and $A_{r_1} \cap A_{r_2} \neq \emptyset$. So we have some $(a, b) \in A_{r_1} \cap A_{r_2}$ and $r_1 = \frac{a}{b} = r_2$, a contradiction. (Union) Suppose $(a, b) \in \mathbb{Z} \times \mathbb{Z}^*$. Let $r = \frac{a}{b}$. Then, $(a, b) \in A_r \subseteq \bigcup_{r \in \mathbb{Q}} A_r$. $\square$

53. $\forall\, n \in \mathbb{Z}$, let $A_n = [n, n+1)$.
Recall that $\lfloor x \rfloor = n$ if and only if $n \in \mathbb{Z}$ and $n \leq x < n+1$ (i.e. $x \in [n, n+1)$).

55. $\forall\, b \in \mathbb{Z}$, let $A_b = \{(m, n) \ : \ m, n \in \mathbb{N} \text{ and } m - n = b\}$.
We simply group together elements (m, n) according to the difference $m - n$ that characterizes the equivalence relation.

57. $\forall\, m \in \mathbb{R}$, let $A_m = \{(x, y) \ : \ y = mx\}$.
Additionally, let $A_\infty = \{(x, y) \ : \ x = 0\}$.
Notice that, for a fixed point (x_1, y_1), the set of points of the form (cx_1, cy_1) lie on a line through the origin. Hence, each equivalence class corresponds to a line through the origin. Since the y-axis cannot be described by an equation of the form $y = mx$, we need a separate description for that line.

59. (a) $\{\text{apple}\}$, $\{\text{eat}, \text{ear}\}$, $\{\text{peace}\}$, $\{\text{car}, \text{call}\}$.
(b) $\{\text{apple}, \text{peace}\}$, $\{\text{call}\}$, $\{\text{eat}, \text{car}, \text{ear}\}$.
(c) $\{\text{apple}, \text{eat}, \text{peace}, \text{ear}\}$, $\{\text{car}, \text{call}\}$.

61. $m\, R\, n \leftrightarrow m - n$ is even.
Notice that $m - n$ is even precisely when m and n are both even or both odd.

63. $(a_1, b_1)\ R\ (a_2, b_2) \leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$.
That is, we want (a_1, b_1) and (a_2, b_2) to be in the same equivalence class precisely when $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are the same rational number.

65. (a) "has the same suffix as" or "has the same file type as."
(b) "has the same base (or file) name as."
We are adopting here the usual naming conventions of `name.suffix`.

67. Theorem: Let X be a set, R be an equivalence relation on X, and $\mathcal{A}$ be a partition of X. Then, $\mathcal{A}$ is the partition of X corresponding to R if and only if

R is the equivalence relation on X corresponding to $\mathcal{A}$.
Proof. ($\rightarrow$) Suppose $\mathcal{A}$ is the partition of X corresponding to R. Let R' be the equivalence relation on X corresponding to $\mathcal{A}$. So, $x\ R'\ y$ if and only if $\exists\ A \in \mathcal{A}$ such that $x, y \in A$ if and only if $\exists\ z \in X$ such that $x, y \in [z]_R$ if and only if $\exists\ z \in X$ such that $x\ R\ z$ and $y\ R\ z$ if and only if $x\ R\ y$. Hence $R = R'$. ($\leftarrow$) Suppose R is the equivalence relation on X corresponding to $\mathcal{A}$. So $x\ R\ y$ if and only if $\exists\ A \in \mathcal{A}$ such that $x, y \in A$. Let $\mathcal{A}'$ be the partition of X corresponding to R. So $\mathcal{A}' = \{[x]_R\ :\ x \in X\}$. Suppose $A \in \mathcal{A}$. Since $A \neq \emptyset$, we have some $y \in A$. We claim that $A = [y]_R$. ($\subseteq$) Suppose $x \in A$. Since $x, y \in A$, we have $x\ R\ y$. Hence $x \in [y]_R$. ($\supseteq$) Suppose $x \in [y]_R$. So $x\ R\ y$. Hence, we have some $A' \in \mathcal{A}$ with $x, y \in A'$. Since $y \in A$, and A' and A are either identical or disjoint, it must be that $A' = A$. So $x \in A$. Therefore, $A \in \mathcal{A}'$. We have shown that $\mathcal{A} \subseteq \mathcal{A}'$. Since $\mathcal{A}$ and $\mathcal{A}'$ are both partitions (each with union X), it must be that $\mathcal{A} = \mathcal{A}'$. $\square$

69. *Proof.* ($\subseteq$) Suppose $x \in \left(\bigcup_{A \in \mathcal{A}_1} A\right) \cup \left(\bigcup_{A \in \mathcal{A}_2} A\right)$. In the case that $x \in \bigcup_{A \in \mathcal{A}_1} A$, we have $x \in A_1$ for some $A_1 \in \mathcal{A}_1$. Since $A_1 \in \mathcal{A}_1 \cup \mathcal{A}_2$, we have $x \in \bigcup_{A \in \mathcal{A}_1 \cup \mathcal{A}_2} A$. The case in which $x \in \bigcup_{A \in \mathcal{A}_2} A$ is handled similarly. ($\supseteq$) Suppose $x \in \bigcup_{A \in \mathcal{A}_1 \cup \mathcal{A}_2} A$. So $x \in A_0$ for some $A_0 \in \mathcal{A}_1 \cup \mathcal{A}_2$. If $A_0 \in \mathcal{A}_1$, then $x \in \bigcup_{A \in \mathcal{A}_1} A$. If $A_0 \in \mathcal{A}_2$, then $x \in \bigcup_{A \in \mathcal{A}_2} A$. In any case, $x \in \left(\bigcup_{A \in \mathcal{A}_1} A\right) \cup \left(\bigcup_{A \in \mathcal{A}_2} A\right)$. $\square$

Section 5.3

1. No. The range can be a proper subset of the codomain.

3. It is not. $f(0) = \frac{1}{2} \notin \mathbb{Z}^+$.

5. It is not. $\pm\sqrt{x}$ does not specify a *unique* output value.

7. (a) It is not, since $f(1) = \frac{1}{2} \notin \mathbb{Z}$.
(b) It is. If $n \in \mathbb{Z}$, then $2n \in \mathbb{Z}$.

9. It is. $\forall\ n \in \mathbb{Z}, 2n - 1 \neq 0$.

11. (a) It is not. $\frac{1}{1} = \frac{2}{2}$, but $f(\frac{1}{1}) = 1 \neq 2 = f(\frac{2}{2})$.
(b) It is not. See part (a).
(c) It is. If $\frac{m'}{n'} = \frac{m}{n}$ then, $m' = 0$ iff $m = 0$, and $\frac{n'}{m'} = \frac{n}{m}$ when $m', m \neq 0$.

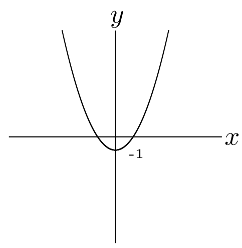
13. (a) It is not. $[0]_5 = [5]_5$ but $[0]_{10} \neq [5]_{10}$.
(b) It is. If $[a']_5 = [a]_5$ then $5 \mid (a' - a)$, and hence $10 \mid (2a' - 2a)$. So $[2a']_{10} = [2a]_{10}$.

15. Yes.
It is a constant function.

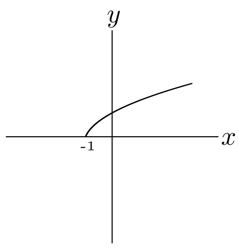
17. Domain = $\{-3, -2, \ldots, 3\}$ and range = $\{0, 1, 4, 9\}$.
Note that $\{f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3)\} = \{9, 4, 1, 0, 1, 4, 9\} = \{0, 1, 4, 9\}$.

19. Domain = $\{0, 1, \ldots, 4\}$ and range = $\{1, 2, 4, 8, 16\}$.
Note that $\{f(0), f(1), f(2), f(3), f(4)\} = \{2^0, 2^1, 2^2, 2^3, 2^4\} = \{1, 2, 4, 8, 16\}$.

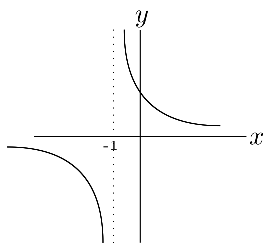
21. Domain = $\mathbb{R}$ and range = $[-1, \infty)$.



23. Domain = $[1, \infty)$ and range = $[0, \infty)$.



25. Domain = $\mathbb{R} \setminus \{-1\}$ and range = $\mathbb{R} \setminus \{0\}$.



27. $f(0) = f(4) = 4,\ f(1) = f(3) = 1,\ f(2) = 0,\ f(5) = 9,\ f(6) = 16$.
That is, $\{y\ :\ f(x) = y \text{ for some } x \in \{0, 1, 2, 3, 4, 5, 6\}\} =$
$\{f(0), f(1), f(2), f(3), f(4), f(5), f(6)\} = \{4, 1, 0, 1, 4, 9, 16\} = \{0, 1, 4, 9, 16\}$.

29. *Sketch.* $0 \le x \le 2$ iff $0 \le 3x \le 6$ iff $-2 \le 3x - 2 \le 4$. $\square$
That is, $x \in [0,2] = \text{domain}(f)$ if and only if $f(x) = 3x - 2 \in [-2,4]$.

31. *Proof.* Suppose $x \in [0,2]$. So $0 \le x \le 2$. Hence $0 \le x^2 \le 4$. That is, $x^2 \in [0,4]$. Now suppose $y \in [0,4]$. Let $x = \sqrt{y}$. Observe that $0 \le x \le 2$ and $f(x) = (\sqrt{y})^2 = y$. $\square$

33. $(g \circ f)(n) = g(f(n)) = g(n!) = (n!)^2$.

35. $(g \circ f)(x) = g(f(x)) = g(1 + 3x) = 1 - 3(1 + 3x) = -2 - 9x$.

37. $(g \circ f)(x) = \frac{1}{|x|}$.
$g \circ f : \mathbb{R} \backslash \{0\} \longrightarrow [0, \infty)$ is defined by $(g \circ f)(x) = g(f(x)) = g(\frac{1}{x^2}) = \sqrt{\frac{1}{x^2}} = \frac{1}{|x|}$.

39. (a) Yes, they both toggle the bit from the value it has to the only other possible value.
(b) 4. The constant 0, the constant 1, the identity, and the toggle map ($0 \mapsto 1$ and $1 \mapsto 0$).

41. The "is the grandfather of" relation.
That is, the father of the father is the grandfather.

43. (a) *Proof.* ($\rightarrow$) Suppose R is transitive and suppose $x\ (R \circ R)\ z$. So, there is some y such that $x\ R\ y$ and $y\ R\ z$. By transitivity, $x\ R\ z$. Hence, $R \circ R \subseteq R$.
($\leftarrow$) Suppose $R \circ R \subseteq R$ and $x\ R\ y$ and $y\ R\ z$. Since $x\ (R \circ R)\ z$ and $R \circ R \subseteq R$, it follows that $x\ R\ z$. Hence, R is transitive. $\square$
(b) *Proof.* Suppose R is reflexive and transitive, and suppose $x\ R\ y$. Since $x\ R\ x$ and $x\ R\ y$, it follows that $x\ (R \circ R)\ y$. Thus, $R \subseteq R \circ R$. Part (a) finishes the job. $\square$

45. (a) No.
(b)

Artist	Music Company
MandM	Aristotle Records
Fifty Percent	Bald Boy Records
MandM	Bald Boy Records
M.C. Escher	Aristotle Records

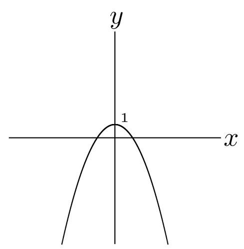
(c) Aristotle Records and Bald Boy Records.

47. (a)

Programmer	Client
Martha Lang	GameCo
Megan Johnson	MediComp
Charles Murphy	GameCo

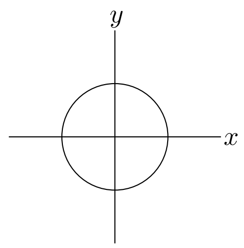
(b) Only GameCo.

49. (a)



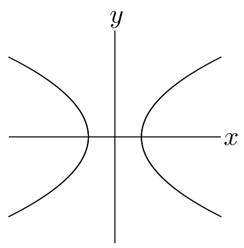
(b) Yes. Domain = $\mathbb{R}$. (c) Range = $(-\infty, 1]$.

51. (a)



(b) No. (c) None.

53.(a)



(b) No. (c) None.

55. The relation is a function iff each row has at most one 1.
If some row x were to have two or more 1's in it, then that x would relate to two or more values, and so the relation would not be a function.

57. (a) $\forall\ x \in \mathbb{R}$, $((f+g)\circ h)(x) = (f+g)(h(x)) = f(h(x)) + g(h(x)) = (f\circ h)(x) + (g\circ h)(x) = (f\circ h + g\circ h)(x)$. (b) Define $f(x) = x^2$ and $g(x) = h(x) = 1$.
So $(f\circ(g+h))(1) = f((g+h)(1)) = f(2) = 4$, and $f(g(1))+f(h(1)) = 1+1 = 2$.

59. (a) $\forall\ x \in \mathbb{R}$, $(c(f\circ g))(x) = c((f\circ g)(x)) = c(f(g(x))) = (cf)(g(x)) = ((cf)\circ g)(x)$.
(b) Let $c = 2$, $f(x) = x$, and $g(x) = 1$.
In this case, $(f\circ(cg))(x) = 4$ and $(c(f\circ g))(x) = 2$.

61. Yes.
Any constant function has this property. Also, $f(x) = |x|$.

Section 5.4

1. *Proof.* Suppose $x_1^3 + 8 = x_2^3 + 8$. So $x_1^3 = x_2^3$. Taking the cube root of both sides gives that $x_1 = x_2$. □

3. *Proof.* Suppose $n_1, n_2 \in \mathbb{Z}^-$ and $1 - n_1^2 = 1 - n_2^2$. So $n_1^2 = n_2^2$. Since $n_1, n_2 \in \mathbb{Z}^-$, we have $n_1 = n_2$. □

5. $f(0) = f(1)$, but $0 \neq 1$.

7. *Proof.* Suppose $y \in (1, \infty)$. (Goal: $y = x^2 + 1$.) Let $x = \sqrt{y-1}$. Observe that $f(x) = (\sqrt{y-1})^2 + 1 = y - 1 + 1 = y$. □

9. *Proof.* Observe that $\frac{1}{2} \in \mathbb{R}^+$. However, $f(x) = \frac{1}{2}$ is impossible, since $x^2 = -\frac{1}{2}$ has no solution in $\mathbb{R}$. □

11. *Sketch.* Suppose $y \in \mathbb{R}^+$. Let $x = \frac{-1+\sqrt{1+4y}}{2}$. Observe that $x \in \mathbb{R}^+$ and $x^2 + x = y$. □

13. (a) Let $k \in \mathbb{Z}$. Observe that $f(k, 1) = k$.
(b) $f(2, 1) = 2 = f(1, 2)$ but $(2, 1) \neq (1, 2)$.

15. *Proof.* Suppose $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$. Suppose to the contrary that $x_1 \neq x_2$. We may assume that $x_1 < x_2$. However, since f is increasing, $f(x_1) < f(x_2)$. This is a contradiction. □

17. (a) Suppose $[n'] = [n]$. Since $n' \equiv n \pmod 6$,we have $6 \mid (n' - n)$. So $6 \mid (2n' - 2n)$. That is, $2n' \equiv 2n \pmod 6$. Hence, $[2n'] = [2n]$.
(b) $f([0]) = [0] = f([3])$, but $[0] \neq [3]$.
(c) It is not the case that $\gcd(2, 6) = 1$ (as would be needed of $a = 2$ and $n = 6$ in Lemma 3.30).

19. (a) $1600081160 \bmod 625 = 535$.
(b) 0000000003216 and 0000000009461.
Note that $321 \bmod 625 = 946 \bmod 625 = 321$. Also $[3(3 + 1) + 2 + 6] \bmod 10 = [3(9 + 6) + 4 + 1] \bmod 10 = 0$.

21. (a) Suppose $x \in X$. Observe that $p(x, y_0) = x$. So $x \in \text{range}(p)$.
(b) Suppose $x_1, x_2 \in X$ and $i(x_1) = i(x_2)$. So $(x_1, y_0) = (x_2, y_0)$. Thus, $x_1 = x_2$.
(c) $\forall\, x \in X, (p \circ i)(x) = p(i(x)) = p(x, y_0) = x$.
(d) $\forall\, x \in X, y \in Y, (i \circ p)(x, y) = i(p(x, y)) = i(x) = (x, y_0)$.

23. Suppose $i(a_1) = i(a_2)$. Then $a_1 = a_2$. That is, $a_1 = i(a_1) = i(a_2) = a_2$.

25. *Proof.* Suppose $B \in \mathcal{P}([0,1])$. That is, $B \subseteq [0,1]$. Observe that $B \subseteq \mathbb{R}$ (so $B \in \mathcal{P}(\mathbb{R})$) and $f(B) = B$. □

27. f and g are onto. f and g are not one-to-one.
Note that $f((A,A)) = g((A,A)) = A$. Also, $f((\{0\},\emptyset)) = \emptyset = f((\emptyset,\emptyset))$ and $g((\{0\},\emptyset)) = \{0\} = g((\{0\},\{0\}))$

29. One-to-one.
Two guests cannot share a seat, and all the seats need not be filled.

31. Lemma: Let $f : X \longrightarrow Y$ be any function. Then, $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_Y \circ f = f$.
$\forall\, x \in X, (f \circ \mathrm{id}_X)(x) = f(\mathrm{id}_X(x)) = f(x)$
and $(\mathrm{id}_X \circ f)(x) = \mathrm{id}_X(f(x)) = f(x)$.

33. Since Exercise 1 established that f is one-to-one, it remains to show that f is onto. Suppose $y \in \mathbb{R}$. Observe that $(y-8)^{\frac{1}{3}} \in \mathbb{R}$ and $f((y-8)^{\frac{1}{3}}) = ((y-8)^{\frac{1}{3}})^3 + 8 = y - 8 + 8 = y$.

35. *Sketch.* $-2 \mapsto 0$, $-1 \mapsto 2$, $0 \mapsto 4$, $1 \mapsto 6$, $2 \mapsto 8$. □
Observe that all elements of the codomain are in the range. So f is onto. It is also clear that, if $k_1 \neq k_2$, then $f(k_1) \neq f(k_2)$. So f is one-to-one.

37. *Proof.* (Onto) Let $j \in \mathbb{Z}$ with $-n \leq j \leq n$. Hence $1 \leq j + 1 + n \leq 2n + 1$ and $f(j + 1 + n) = j$. So $j \in \mathrm{range}(f)$. (One-to-one) Suppose $f(k_1) = f(k_2)$. Since $k_1 - 1 - n = k_2 - 1 - n$, we have $k_1 = k_2$. So f is one-to-one. □

39. (a) Suppose $[k'] = [k]$. So $[mk'] = [mk]$ and $[m + k'] = [m + k]$. That is, $mk' \equiv mk \pmod n$ and $m + k' \equiv m + k \pmod n$, by Theorem 3.26.
(b) *Proof.* ($\rightarrow$) Suppose f is a bijection. Since f is onto, there is some $k \in \mathbb{Z}$ such that $f([k]) = [1]$. That is, $[1] = [mk]$. Since $1 \equiv mk \pmod n$, there is $j \in \mathbb{Z}$ such that $nj = 1 + mk$. Since $km + jn = 1$, it follows from Corollary 3.14 that $\gcd(m,k) = 1$. ($\leftarrow$) Suppose $\gcd(m,k) = 1$. So there are $j, k \in \mathbb{Z}$ such that $mk + nj = 1$. That is, $f([k]) = [mk] = [1]$. Hence, for each $y \in \mathbb{Z}$, $f([ky]) = [y]$. Thus f is onto. Now suppose $f([k_1]) = f([k_2])$. So $[mk_1] = [mk_2]$. That is, $mk_1 \equiv mk_2 \pmod n$. By the Modular Cancellation Rule (Lemma 3.30), $k_1 \equiv k_2 \pmod n$. That is, $[k_1] \equiv [k_2]$. So f is one-to-one. □
(c) For any choice of m, the function g is a bijection.

41. If every column contains at most one 1, then f is one-to-one.
Each column represents a possible output value y. If there are two 1's in a column y, then two input values are mapped to y and f is not one-to-one.

43. Theorem: Suppose $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$.
(a) If f and g are one-to-one, then $g \circ f$ is one-to-one.
(b) If f and g are onto, then $g \circ f$ is onto.
(c) If f and g are bijective, then $g \circ f$ is bijective.
Proof of (b). *Proof.* Suppose f and g are onto. Suppose $z \in Z$. We have $y \in Y$ such that $g(y) = z$. We have $x \in X$ such that $f(x) = y$. That is, $(g \circ f)(x) = g(f(x)) = g(y) = z$. Thus $g \circ f$ is onto. $\square$
Proof for (c). *Proof.* Suppose f and g are bijective. Since f and g are one-to-one, $g \circ f$ is one-to-one, by (a). Since f and g are onto, $g \circ f$ is onto, by (b). Hence, $g \circ f$ is bijective. $\square$

45. *Proof.* ($\rightarrow$) Suppose f is symmetric and $x \in X$. Let $f(x) = y$. Since $x\ f\ y$, it follows that $y\ f\ x$. That is $f(y) = x$. Now, $(f \circ f)(x) = f(f(x)) = f(y) = x$. Thus $f \circ f = \mathrm{id}_X$. ($\leftarrow$) Suppose $f \circ f = \mathrm{id}_X$ and $x\ f\ y$. That is, $f(x) = y$. Observe that $x = (f \circ f)(x) = f(f(x)) = f(y)$. Hence $y\ f\ x$. Thus f is symmetric. $\square$

47. (a) *Proof.* Suppose $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$. Thus, $(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2)$. Since $g \circ f$ is one-to-one, it follows that $x_1 = x_2$. So f is one-to-one. $\square$
(b) $X = Y = Z = (0, \infty)$, $f(x) = 1 + \sqrt{x}$, $g(x) = (x - 1)^2$.
Note that $(g \circ f)(x) = g(1 + \sqrt{x}) = (1 + \sqrt{x} - 1)^2 = x$. It follows that $g \circ f$ is one-to-one. However, $g(2) = g(0)$. So g is not one-to-one.

49. (a) Since $g \circ f$ is one-to-one, the result follows from Exercise 47(a).
(b) Since $g \circ f$ is onto, the result follows from Exercise 48(a).
(c) For $X = Y = Z = [0, \infty)$, $f(x) = 1 + \sqrt{x}$ is not onto (since $0 \notin \mathrm{range}(f)$) and $g(x) = (x - 1)^2$ is not one-to-one (since $g(2) = g(0)$).

51. (a) *Proof.* Suppose f and g are increasing. Suppose $x < y$. Since f is increasing $f(x) < f(y)$. Since g is increasing, $g(f(x)) < g(f(y))$. Hence $g \circ f$ is increasing. $\square$
(b) $f(x) = g(x) = -x$ gives $(g \circ f)(x) = x$. Both f and g are decreasing here.

53. $\forall\ x \in \mathbb{R}, g(f(x)) = g(2x + 5) = \frac{2x+5-5}{2} = x$ and $f(g(x)) = f(\frac{x-5}{2}) = 2(\frac{x-5}{2}) + 5 = x$.
That is, $g \circ f = \mathrm{id}_{\mathbb{R}}$ and $f \circ g = \mathrm{id}_{\mathbb{R}}$.

55. $\forall\ x \in \mathbb{R}, g(f(x)) = g(4 - 2x) = 2 - \frac{1}{2}(4 - 2x) = 2 - 2 + x = x$ and $f(g(x)) = f(2 - \frac{1}{2}x) = 4 - 2(2 - \frac{1}{2}x) = 4 - 4 + x = x$.
That is, $g \circ f = \mathrm{id}_{\mathbb{R}}$ and $f \circ g = \mathrm{id}_{\mathbb{R}}$.

57. $g(f(2)) = g(1) = 2$,
$g(f(3)) = g(3) = 3$,
$g(f(4)) = g(6) = 4$, and

$g(f(5)) = g(10) = 5$.
One similarly shows that $g \circ f = \mathrm{id}$.
For a small domain (and codomain), it is reasonable to directly check every possible input value. In fact, the formulas given for f and g are too awkward to deal with generally here.

59. $\forall\, r \in \mathbb{Q}^+, (f \circ f)(r) = f(\frac{1}{r}) = \frac{1}{\frac{1}{r}} = r$.
That is, $f \circ f = \mathrm{id}_{\mathbb{R}}$ (and $f \circ f = \mathrm{id}_{\mathbb{R}}$).

61. $f^{-1}(x) = \sqrt[3]{\frac{x-1}{4}}$.
Let $y = 4x^3 + 1$. So $4x^3 = y - 1$. So $x^3 = \frac{y-1}{4}$. So $x = \sqrt[3]{\frac{y-1}{4}}$. Thus, $f^{-1}(y) = \sqrt[3]{\frac{y-1}{4}}$.

63.

Phone Number	Name
555-3148	Blair, Tina
555-3992	Walsh, Carol
555-4500	Tillman, Paul
555-6301	Jennings, Robert

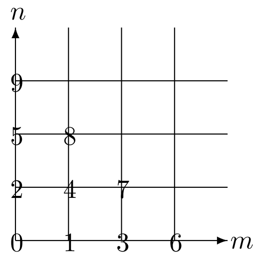
The function need not be one-to-one and need not be onto.
The function would not be one-to-one, if two people from one household are both listed with the same phone number. It is unlikely that the function is onto, since there would then be no room for new phone numbers.

65. *Proof.* Suppose $f : X \longrightarrow Y$ is a bijection. By Theorem 5.10(a), f^{-1} is a function. Also, f and f^{-1} and f are inverses of one another. By Theorem 5.10(b) (applied to $g = f$), f^{-1} is a bijection. □

67. (a) *Proof.* Suppose f_1 and f_2 are one-to-one. Suppose that $(f_1 \times f_2)((x_1', x_2')) = (f_1 \times f_2)((x_1, x_2))$. Since $(f_1(x_1'), f_2(x_2')) = (f_1(x_1), f_2(x_2))$, we have $f_1(x_1') = f_1(x_1)$ and $f_2(x_2') = f_2(x_2)$. Thus, $x_1' = x_1$ and $x_2' = x_2$. That is, $(x_1', x_2') = (x_1, x_2)$. □
(b) *Proof.* Suppose f_1 and f_2 are onto. Suppose $(y_1, y_2) \in Y_1 \times Y_2$. We have $x_1 \in X_1$ such that $f_1(x_1) = y_1$ and $x_2 \in X_2$ such that $f_2(x_2) = y_2$. That is, $(f_1 \times f_2)((x_1, x_2)) = (f_1(x_1), f_2(x_2)) = (y_1, y_2)$. □
(c) *Proof.* Suppose f_1 and f_2 are bijective. Since f_1 and f_2 are one-to-one, $f_1 \times f_2$ is one-to-one, by part (a). Since f_1 and f_2 are onto, $f_1 \times f_2$ is onto, by part (b). Thus, $f_1 \times f_2$ is bijective. □

69. (a) On the following picture of $\mathbb{N} \times \mathbb{N}$, at each point (m, n), we plot the value of $f((m, n))$. Indeed, compute several specific values of $f((m, n))$ to confirm this

pattern. Note that each point (m, n) is assigned a unique value from $\mathbb{N}$.



(b) *Sketch.* $\forall\ m \in \mathbb{N}, g(m+1) = m$. So g is onto. If $n_1 - 1 = n_2 - 1$, then $n_1 = n_2$. So g is one-to-one. $\square$
(c) This follows from part (b) and Exercise 67(c). That is, since g and g are bijective, so is $g \times g$.
(d) This follows from part (c), part (a), Exercise 65, and Theorem 5.9(c). Since g is bijective, so is g^{-1}. Since $g \times g$ and f are bijective, so is $f \circ (g \times g)$. Since $f \circ (g \times g)$ and g^{-1} are bijective, so is $g^{-1} \circ f \circ (g \times g)$.

71. (a) -2, since $2^{-2} = \frac{1}{4}$. (b) 4, since $3^4 = 81$.
(c) $\frac{1}{4}$, since $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$. (d) -1, $e^{-1} = \frac{1}{3}$. Recall that ln has base e.

73. $\log_2 3$. Let T be the tripling time. So $a2^T = p(T) = 3p(0) = 3a$. Since $2^T = 3$, it follows that $T = \log_2 3$.

75. (a) Since $b^0 = 1$. (b) Since $b^{\log_b y + \log_b z} = b^{\log_b y} b^{\log_b z} = yz$.
(c) Since $b^{a \log_b y} = (b^{\log_b y})^a = y^a$.
In each case, we simply use the characterization of logarithms given in Equation (5.4) and the basic Laws of Exponents from Appendix A.

77. If $a, b \in \mathbb{Z}^+$ and $2^a = 2^b$, then the Fundamental Theorem of Arithmetic tells us that $a = b$. That is, if $f(a) = f(b)$, then $a = b$. So f is one-to-one.

Section 5.5

1. (a) $f(\{0,1,2,3\}) = \{f(0), f(1), f(2), f(3)\} = \{1,1,2,6\} = \{1,2,6\}$.
(b) No. $f(\{3\}) = \{6\}$. The image of a set is a set.

3. $\{10, -5, -6, -5\} = \{-6, -5, 10\}$.

5. $f([-2,2]) = [-3,5]$.
$f([-2,2]) = \{t\ :\ t \in \mathbb{R}$ and $f(s) = t$ for some $s \in [-2,2]\} =$
$\{t\ :\ t \in \mathbb{R}$ and $2s + 1 = t$ for some $-2 \le s \le 2\} =$
$\{t\ :\ t \in \mathbb{R}$ and $s = \frac{t-1}{2}$ for some $-2 \le s \le 2\} = \{t\ :\ t \in \mathbb{R}$ and $-2 \le \frac{t-1}{2} \le 2\} =$
$\{t\ :\ t \in \mathbb{R}$ and $-4 \le t - 1 \le 4\} = \{t\ :\ t \in \mathbb{R}$ and $-3 \le t \le 5\} = [-3,5]$.

7. $f([1,3]) = [2,10]$.
Proof. ($\subseteq$) Suppose $t \in f([1,3])$. So $s^2+1 = t$ for some $1 \le s \le 3$. So $2 = 1^2+1 \le t \le 3^2+1 = 10$. That is, $t \in [2,10]$. ($\supseteq$) Suppose $t \in [2,10]$. So $2 \le t \le 10$. Hence, $1 \le t-1 \le 9$. Thus, $1 \le \sqrt{t-1} \le 3$. Observe that $(\sqrt{t-1})^2 + 1 = t$ and $1 \le \sqrt{t-1} \le 3$. Therefore, $t \in f([1,3])$. $\square$

9. *Proof.* ($\subseteq$) Suppose $z \in g(\text{range}(f))$. So we have $y \in \text{range}(f)$ such that $g(y) = z$. We must also have $x \in X$ such that $f(x) = y$. Since $z = g(y) = g(f(x)) = (g \circ f)(x)$, we see that $z \in \text{range}(g \circ f)$. ($\supseteq$) Suppose $z \in \text{range}(g \circ f)$. So we have $x \in X$ such that $(g \circ f)(x) = z$. Let $y = f(x)$. So $y \in \text{range}(f)$. Since $z = g(f(x)) = g(y)$, we see that $z \in g(\text{range}(f))$. $\square$

11. (a) $\{0,1,2,4\}$, since $f(0) = f(1) = 1$, $f(2) = 2$, and $f(4)$.
(b) $\{3\}$, since $f(3) = 6$ and there is no value n such that $f(n) = 10$.
(c) No. $f^{-1}(\{120\}) = \{5\}$. The inverse image of a set is a set.

13. $\{-2,-1,1,2\}$. The set of x for which $x^4 - 6 = -5$ or 10.
That is, $x^4 = 1$ or 16.

15. $f^{-1}(\{-1\}) = O$, the set of odd integers.
If n is even, then $f(n) = 1$. If n is odd, then $f(n) = -1$. Thus, $f(n) = -1$ iff n is odd.

17. The set of relatively prime pairs of positive integers.
Recall that m and n are relatively prime if and only if $\gcd(m,n) = 1$.

19. $f^{-1}([1,4]) = [-2,-1] \cup [1,2]$.
Proof. ($\supseteq$) Suppose $x \in [-2,-1] \cup [1,2]$. So $-2 \le x \le -1$ or $1 \le x \le 2$. Equivalently, $1 \le -x \le 2$ or $1 \le x \le 2$. Since $(-x)^2 = x^2$, in either case we have $x^2 \in [1,4]$. ($\subseteq$) We prove the contrapositive. Suppose $x \notin [-2,-1] \cup [1,2]$. So $x \in (-\infty,-2) \cup (-1,1) \cup (2,\infty)$. By considering each possible case, we see that $x^2 \in [0,1) \cup (4,\infty)$. That is, $f(x) = x^2 \notin [1,4]$. Hence, $x \notin f^{-1}([1,4])$. $\square$

21. *Proof.* ($\subseteq$) Suppose $(x,y) \in f(\mathbb{R})$. So, we have some $z \in \mathbb{R}$ such that $(x,y) = f(z) = (z,z)$. Thus, $y = z = x$. Hence, $g(x,y) = x - y = 0$, and we see that $(x,y) \in g^{-1}(\{0\})$. ($\supseteq$) Suppose $(x,y) \in g^{-1}(\{0\})$. Since $x - y = g(x,y) = 0$, we get $y = x$. Thus, $f(x) = (x,x) = (x,y)$. That is, $(x,y) \in f(\mathbb{R})$. $\square$

23. (a) Eagles and Huskies. Image.
(b) KSU, Northwestern, UNH, and Villanova. Inverse Image.
The function given by the table maps colleges to nicknames.

25. (a) The set E of even integers,
since $R(\{2\}) = \{n \,:\, 2\,R\,n\} = \{n \,:\, 2 \mid n\} = E$.
(b) $\{3,5,7\}$, since 3, 5, and 7 are the primes p such that $p \mid 15$ or $p \mid 35$.

27. (a) $\{0\}$, $\{1\}$, and $\{0,1\}$. Of course, each of these sets contains 0 or 1 as an element. $R(\{0,1\}) = \{A : A \subset \mathbb{Z}, A \cap \{0,1\} \neq \emptyset\}$ is the set of sets that contain 0 or 1 as an element. (b) $\{0,1,2,3,4,6\}$. An integer n is in this set if and only if $n \in \{0,1,2,3,\}$ or $n \in \{0,2,4,6\}$.

29. (a) No. Two customers ordered a wrench. (b) A wrench and pliers. Inverse image. (c) Susan Brower and Abe Roth. Image. The function given by the table maps parts to customers.

31. (a) *Proof.* Suppose $S_1 \subseteq S_2$. Now suppose $y \in f(S_1)$. So $y = f(x)$ for some $x \in S_1$. Since $S_1 \subseteq S_2$, we have $x \in S_2$ and $f(x) = y$. Thus $y \in f(S_2)$. $\square$
(b) *Proof.* Suppose $T_1 \subseteq T_2$. Now suppose $x \in f^{-1}(T_1)$. So $f(x) \in T_1$. Since $T_1 \subseteq T_2$, we have $f(x) \in T_2$. Thus $x \in f^{-1}(T_2)$. $\square$

33. (a) ($\supseteq$) By the definition of $f^{-1}(Y)$, we have $f^{-1}(Y) \subseteq X$. ($\subseteq$) Suppose $x \in X$. Since $f(x) \in Y$, we have $x \in f^{-1}(Y)$.
(b) Since Y is the codomain, this follows from the definition of $f(X)$.
(c) Let $X = Y = \mathbb{R}$ and $f(x) = x^2$. So $f(X) = [0, \infty) \neq Y$.

35. (a) *Sketch.* ($\subseteq$) Suppose $x \in f^{-1}(T_1 \cup T_2)$. So $f(x) \in T_1 \cup T_2$. For $i = 1, 2$, if $f(x) \in T_i$, then $x \in f^{-1}(T_i)$. Hence, $x \in f^{-1}(T_1) \cup f^{-1}(T_2)$. ($\supseteq$) By Exercise 31(b), for $i = 1, 2$, $f^{-1}(T_i) \subseteq f^{-1}(T_1 \cup T_2)$. Hence, $f^{-1}(T_1) \cup f^{-1}(T_2) \subseteq f^{-1}(T_1 \cup T_2)$. $\square$
(b) *Sketch.* By Exercise 31(b), for $i = 1, 2$, $f^{-1}(T_1 \cap T_2) \subseteq f^{-1}(T_i)$. So $f^{-1}(T_1 \cap T_2) \subseteq f^{-1}(T_1) \cap f^{-1}(T_2)$. Now suppose $x \in f^{-1}(T_1) \cap f^{-1}(T_2)$. So for $i = 1, 2$, $f(x) \in T_i$. Since $f(x) \in T_1 \cap T_2$, $x \in f^{-1}(T_1 \cap T_2)$. $\square$
The proofs here rest mainly on the definition of the inverse image of a set. However, some work is saved by appealing to the result in Exercise 31(b).

37. (a) *Proof.* Suppose $x \in S \cap f^{-1}(T)$. Since $x \in S$, $f(x) \in f(S)$. Since $x \in f^{-1}(T)$, $f(x) \in T$. Hence, $f(x) \in f(S) \cap T$. $\square$
(b) *Proof.* Suppose $x \in S \cap f^{-1}(T)$. So $f(x) \in f(S \cap f^{-1}(T))$. By part (a), $f(x) \in f(S) \cap T$. Hence, $x \in f^{-1}(f(S) \cap T)$. $\square$

39. (a) *Proof.* ($\rightarrow$) Suppose f is one-to-one. Suppose $F(A_1) = F(A_2)$. So $f(A_1) = f(A_2)$. We claim that $A_1 = A_2$. Suppose $x \in A_1$. So $f(x) \in f(A_1) = f(A_2)$. Hence $f(x) = f(x')$ for some $x' \in A_2$. Since f is one-to-one, $x = x' \in A_2$. Thus $A_1 \subseteq A_2$, and a symmetric argument gives that $A_2 \subseteq A_1$. Therefore, F is one-to-one. ($\leftarrow$) Suppose F is one-to-one. Suppose $f(x_1) = f(x_2)$. Since $F(\{x_1\}) = f(\{x_1\}) = f(\{x_2\}) = F(\{x_2\})$, it follows that $\{x_1\} = \{x_2\}$ and hence $x_1 = x_2$. Therefore, f is one-to-one. $\square$
(b) *Proof.* ($\rightarrow$) Suppose f is onto. Let $B \in \mathcal{P}(Y)$, and let $A = f^{-1}(B)$. Since f is onto, $F(A) = f(A) = B$. Thus F is onto. ($\leftarrow$) Suppose F is onto. Let $y \in Y$. We have some $A \subseteq X$ such that $f(A) = F(A) = \{y\}$. Note that we must have $A \neq \emptyset$. Pick $x \in A$. Then $f(x) = y$. Thus f is onto. $\square$

41. $\{0\}$.
Note that 0 is in every interval $(-r^2, r^2)$. For any other value $x \in \mathbb{R} \setminus \{0\}$, we have $x \notin (-(\sqrt{x})^2, (\sqrt{x})^2)$.

43. $[1, \infty)$. Note that 1 is the smallest element of $\mathbb{Z}^+$.
For any $x \in [1, \infty)$, we have $x \in [\lfloor x \rfloor, \lfloor x \rfloor + 2)$.

45. $[0, 1]$.
Note that $[0, 1]$ is one of the sets and $\forall\, n \in \mathbb{Z}^+,\ [0, 1] \subseteq [0, n]$.

47. $\bigcup_{x \in [3,4)} A_x$.

49. $\bigcap_{\alpha \text{ is a vowel}} A_\alpha$.

51. *Sketch.* ($\subseteq$) Suppose $y \in f(\bigcup\limits_{i \in \mathcal{I}} A_i)$. So $y = f(x)$ for some $x \in A_i$ for some $i \in \mathcal{I}$. Note $y \in f(A_i)$. ($\supseteq$) Suppose $y \in \bigcup\limits_{i \in \mathcal{I}} f(A_i)$. So for some $i \in \mathcal{I}$, $y = f(x)$ for some $x \in A_i$. Note $A_i \in \bigcup\limits_{i \in \mathcal{I}} A_i$. $\square$

53. *Sketch.* ($\subseteq$) Suppose $x \in f^{-1}(\bigcap\limits_{i \in \mathcal{I}} A_i)$. So $f(x) \in A_i$ for each $i \in \mathcal{I}$. So $x \in f^{-1}(A_i)$ for each $i \in \mathcal{I}$. ($\supseteq$) Suppose $x \in \bigcap\limits_{i \in \mathcal{I}} f^{-1}(A_i)$. So, for each $i \in \mathcal{I}$, $x \in f^{-1}(A_i)$. So $f(x) \in A_i$ for each $i \in \mathcal{I}$. $\square$

55. (a) *Sketch.* Suppose $x \in \bigcup_{i \in \mathcal{J}} A_i$. So $x \in A_{i_0}$ for some $i_0 \in \mathcal{J} \subseteq \mathcal{I}$. Thus, $x \in \bigcup\limits_{i \in \mathcal{I}} A_i$. $\square$(b) *Sketch.* Suppose $x \in \bigcap\limits_{i \in \mathcal{I}} A_i$. Since $J \subseteq I$, in particular, $\forall\, i \in \mathcal{J}, x \in A_i$. Thus, $x \in \bigcap\limits_{i \in \mathcal{J}} A_i$. $\square$
These are generalizations of the proofs that $A \subseteq A \cup B$ and $A \cap B \subseteq A$. In Section 2.2, see Exercise 35 and Example 2.13.

57. (a) *Proof.* ($\subseteq$) Suppose $x \in B \cup \bigcap\limits_{i \in \mathcal{I}} A_i$. If $x \in B$, then, $\forall\, i \in \mathcal{I}, x \in B \cup A_i$. So $x \in \bigcap\limits_{i \in \mathcal{I}}(B \cup A_i)$. If $x \in \bigcap\limits_{i \in \mathcal{I}} A_i$, then $\forall\, i \in \mathcal{I}$, $x \in A_i \subseteq B \cup A_i$. So $x \in \bigcap\limits_{i \in \mathcal{I}}(B \cup A_i)$. ($\supseteq$) Suppose $x \in \bigcap\limits_{i \in \mathcal{I}}(B \cup A_i)$. So, $\forall\, i \in \mathcal{I}, x \in B \cup A_i$. If $x \notin B$, then it must be that, $\forall\, i \in \mathcal{I}, x \in A_i$. So $x \in \bigcap\limits_{i \in \mathcal{I}} A_i$. In any case, $x \in B \cup \bigcap\limits_{i \in \mathcal{I}} A_i$. $\square$

(b) *Proof.* ($\subseteq$) Suppose $x \in B \cap \bigcup_{i\in\mathcal{I}} A_i$. So $x \in B$ and $x \in A_{i_0}$ for some $i_0 \in \mathcal{I}$. Thus $x \in B \cap A_{i_0} \subseteq \bigcup_{i\in\mathcal{I}}(B \cap A_i)$. ($\supseteq$) Suppose $x \in \bigcup_{i\in\mathcal{I}}(B \cap A_i)$. So $x \in B \cap A_{i_0}$ for some $i_0 \in \mathcal{I}$. Hence, $x \in B$ and $x \in A_{i_0} \subseteq \bigcup_{i\in\mathcal{I}} A_i$. Thus $x \in B \cap \bigcup_{i\in\mathcal{I}} A_i$. $\square$

59. It follows from Exercise 57(b) that
$(\bigcup_{i\in\mathcal{I}} A_i) \setminus B = (\bigcup_{i\in\mathcal{I}} A_i) \cap B^c = \bigcup_{i\in\mathcal{I}}(A_i \cap B^c) = \bigcup_{i\in\mathcal{I}}(A_i \setminus B)$.

61. Yes.
Use Exercises 33(a), 53, and 52.

Section 5.6

1. 76.
Note that $10 - (-65) + 1 = 76$.

3. 3.
Note that $\{\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4}\} = \{1, 4, 6, 4, 1\} = \{1, 4, 6\}$.

5. 3.
The set is $\{0, 2, -2\}$.

7. 4.
The set of clients is GameCo, MediComp, HealthCorp, PlayBox.

9. The function $f : \{0, 1, \ldots, n\} \longrightarrow \{1, 2, \ldots, n+1\}$ given by $f(k) = k+1$ is a bijection.
It has inverse $g : \{1, 2, \ldots, n+1\} \longrightarrow \{0, 1, \ldots, n\}$ given by $g(k) = k - 1$.

11. The function $f : \{n^2, n^2+1, \ldots, (n+1)^2\} \longrightarrow \{1, 2, \ldots, 2n+1\}$ given by $f(k) = k - n^2 + 1$ is a bijection.
It has inverse $g : \{1, 2, \ldots, 2n+1\} \longrightarrow \{n^2, n^2+1, \ldots, (n+1)^2\}$ given by $g(k) = k + n^2 - 1$.

13. The function $f : \mathbb{N} \longrightarrow \mathbb{Z}^-$ given by $f(n) = -n - 1$ is a bijection.
It has inverse $g : \mathbb{Z}^- \longrightarrow \mathbb{N}$ given by $g(n) = -n - 1$.

15. The function $f : \mathbb{Z}^+ \longrightarrow \{k^2 : k \in \mathbb{Z}^+\}$ given by $f(m) = m^2$ is a bijection.
It has inverse $g : \{k^2 : k \in \mathbb{Z}^+\} \longrightarrow \mathbb{Z}^+$ given by $g(m) = \sqrt{m}$. Note that $\sqrt{m} \in \mathbb{Z}^+$ for all $m \in \{k^2 : k \in \mathbb{Z}^+\}$.

17. The function $f(x) = 2(x - 3) + 1$ is a bijection from $[3, 8]$ to $[1, 11]$.
It has inverse given by $g(x) = \frac{1}{2}(x - 1) + 3$.

19. For (a), (b), and (c), the function given by $f(x) = \frac{x-a}{b-a}$ is a bijection.
It has inverse given by $g(x) = (b-a)x + a$.

21. $f(x) = \frac{b-x}{b-a}$ gives a bijection.
It has inverse given by $g(x) = (a-b)x + b$.

23. For (a) and (b), $f(x) = \frac{x-a}{b-x}$ gives a bijection.
It has inverse given by $g(x) = \frac{bx+a}{x+1}$.

25. $f(x) = \begin{cases} \frac{x-a}{2(b-a)} & \text{if } x \in [a,b), \\ \frac{1}{2} + \frac{x-c}{2(d-c)} & \text{if } x \in [c,d) \end{cases}$ gives a bijection.
It has inverse given by $g(x) = \begin{cases} 2(b-a)x + a & \text{if } x \in [0, \frac{1}{2}), \\ \frac{1}{2} + 2(d-c)(x - \frac{1}{2}) + c & \text{if } x \in [\frac{1}{2}, 1). \end{cases}$

27. Theorem: If A has the same cardinality as B, then B has the same cardinality as A.
Proof. Suppose A has the same cardinality as B. So we have a bijection $f : A \longrightarrow B$. Since $f^{-1} : B \longrightarrow A$ is also a bijection, B has the same cardinality as A. □

29. Corollary: Let A have cardinality $n \neq m$. Then, A does not have cardinality m.
Proof. We can assume that $0 \leq m < n$. Suppose to the contrary that A has cardinality m. So $m \in \mathbb{N}$, and we have a bijection $f : A \longrightarrow \{1, 2, \ldots, m\}$. In particular, f is one-to-one. By the Pigeon Hole Principle, we cannot have $n > m$. Note that $f^{-1} : \{1, 2, \ldots, m\} \longrightarrow A$ is also a bijection and is, in particular, one-to-one. Hence, by the Pigeon Hole Principle (applied to f^{-1}) we cannot have $n < m$. Therefore, it must be that $m = n$, a contradiction. We conclude that A does not have cardinality m. □

31. Apply the contrapositive of Corollary 5.13.
The contrapositive says that, if $n \neq m$, then there is no bijection $f : A \longrightarrow B$. This was argued in Exercise 29.

33. There are 2^{16} possible integers, and $70000 > 2^{16}$.
Since the set of indices has size greater than $2^{16} = 65536$, there must be two or more indices assigned the same value.

35. *Sketch.* The bijections $f : A \longrightarrow C$ and $g : B \longrightarrow D$ can be used to form a bijection $f \times g : A \times B \longrightarrow C \times D$, by Exercise 67(c) from Section 5.4. □

37. Assume $m \geq 1$ and $A = \{1, \ldots, m\}$. By induction on n, we can prove that, for any $n \geq 1$, $|A \times \{1, \ldots, n\}| = mn$.
Sketch. When $n = 1$, we have $|A \times \{1\}| = |\{(1,1), \ldots, (m,1)\}| = m = m \cdot 1$.

Suppose $k \geq 1$ and $|A \times \{1, \ldots, k\}| = mk$. Observe that $A \times \{1, \ldots, k, k+1\} = (A \times \{1, \ldots, k\}) \cup (A \times \{k+1\})$, a disjoint union. So $|A \times \{1, \ldots, k, k+1\}| = mk + m = m(k+1)$. $\square$

39. Exercise 69(c) from Section 5.4 gives a bijection $\mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$.
Hence, $\mathbb{Z}^+ \times \mathbb{Z}^+$ has the same cardinality as $\mathbb{Z}^+$. That is, $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countably infinite.

41. *Sketch.* Let C_1 and C_2 be countable sets. So we have bijections $f : C_1 \longrightarrow \mathbb{Z}^+$ and $g : C_2 \longrightarrow \mathbb{Z}^+$. By Exercise 67(c) from Section 5.4, we know that $f \times g : C_1 \times C_2 \longrightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$ is a bijection. Exercise 39 gives a bijection $h : \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$. The composite $h \circ (f \times g)$ is the desired bijection. $\square$
That is, $h \circ (f \times g)$ establishes that $C_1 \times C_2$ has the same cardinality as $\mathbb{Z}^+$.

43. $h(1) = z$ and $\forall\, k \geq 2, h(k) = g(k-1)$.
Since $z \notin G$ and g is one-to-one, it follows that h is one-to-one. Since $h(1) = z$ and range$(g) = G$, it follows that range$(h) = G \cup \{z\}$. That is, h is onto. So h is a bijection.

45. (a) We show $\forall\, n \in \mathbb{N}$, for any set B with cardinality n and any subset $A \subseteq B$, that A is finite. The proof is by induction on n.
Proof. Base case: $n = 0$. The only set with cardinality 0 is $B = \emptyset$. If $A \subseteq B$, then $A = \emptyset$. So A has cardinality 0 and is finite as well. *Inductive step*: Let $k \geq 0$ and suppose, for any set B of cardinality k and any subset $A \subseteq B$, that A is finite. Let B be a set with cardinality $k+1$, and suppose $A \subseteq B$. If $A = B$, then A has cardinality $k+1$ and is finite as well. If $A \subset B$, then we have some element $b \in B \setminus A$. Define $B' = B \setminus \{b\}$. So $A \subseteq B'$. We claim that B' has cardinality k and thus the inductive hypothesis finishes the proof. Let $f : B \longrightarrow \{1, 2, \ldots, k+1\}$ be a bijection, and define $f' : B' \longrightarrow \{1, 2, \ldots, k\}$ by
$$f'(b') = \begin{cases} f(b') & \text{if } f(b') < f(b), \\ f(b') - 1 & \text{if } f(b') > f(b). \end{cases}$$
Thus f' is a bijection showing that B' has cardinality k. $\square$
(b) This is the contrapositive of part (a).

47. (a) *Proof.* Suppose $A \subseteq \mathbb{Z}^+$, and define the function $f : \mathbb{Z}^+ \longrightarrow A$ by $f(n) = \min(A \setminus \{f(1), f(2), \ldots, f(n-1)\})$. In fact, f is an increasing function. Hence, f is one-to-one. Now suppose $a \in A \subseteq \mathbb{Z}^+$. Let m be the cardinality of $\{k \ : \ k \in A \text{ and } k \leq a\}$. In fact, $f(m) = a$. So f is onto. Therefore, f is a bijection. $\square$
(b) *Proof.* Suppose $g : A \longrightarrow \mathbb{Z}^+$ is one-to-one. Let $A' = g(A)$, and define $g' : A \longrightarrow A'$ by $g'(a) = g(a)$. Since g' is a bijection, A' is an infinite subset of $\mathbb{Z}^+$. By part (a), A' is countably infinite. Hence, A is countably infinite. $\square$

49. *Proof.* Suppose B is countable. If A is finite, then A is countable. So it suffices to assume that A is infinite. We have a bijection $f : B \longrightarrow \mathbb{Z}^+$. Let

$i : A \longrightarrow B$ be the inclusion. So $g = f \circ i$ is a one-to-one map $A \longrightarrow \mathbb{Z}^+$. By Exercise 47, A is countably infinite, hence countable. $\square$

51. *Proof.* The bijection $b : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$ given by $b(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{1-n}{2} & \text{if } n \text{ is odd,} \end{cases}$ together with the results in Exercise 69 from Section 5.4, enables us to construct a bijection $g : \mathbb{Z}^+ \longrightarrow \mathbb{Z} \times \mathbb{Z}^+$. Since $h : \mathbb{Z} \times \mathbb{Z}^+ \longrightarrow \mathbb{Q}$ defined by $h((m,n)) = \frac{m}{n}$ is onto, the composite $h \circ g$ is an onto map $\mathbb{Z}^+ \longrightarrow \mathbb{Q}$. By Exercise 48, $\mathbb{Q}$ is countably infinite. $\square$

53. *Proof.* Since B is countable, we have a bijection $g : \mathbb{Z}^+ \longrightarrow B$. For each $b \in B$, since A_b is countable, we have a bijection $f_b : \mathbb{Z}^+ \longrightarrow A_b$. We claim that $h : \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \bigcup_{b \in B} A_b$ defined by $h((m,n)) = f_{g(m)}(n)$ is onto. Suppose $a \in \bigcup_{b \in B} A_b$. So $a \in A_{b_0}$ for some $b_0 \in B$. Since g is onto, we have $m \in \mathbb{Z}^+$ such that $g(m) = b_0$. Since f_{b_0} is onto, we have $n \in \mathbb{Z}^+$ such that $f_{b_0}(n) = a$. Thus, $h((m,n)) = f_{b_0}(n) = a$. So h is onto. Since Exercise 69 from Section 5.4 guarantees a bijection $w : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$, we have an onto map
$h \circ w : \mathbb{Z}^+ \longrightarrow \bigcup_{b \in B} A_b$. By Exercise 48, $\bigcup_{b \in B} A_b$ is countable. $\square$

Review

1. (a) Yes, since $2 = 2^1$.
(b) No, since $0 \neq 2^0$.
(c) No, since $1 \neq 2^2$.

2.

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

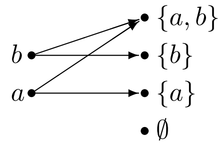
3. $x \; R^{-1} \; y$ iff $y = 2^x$.
Recall that $x \; R^{-1} \; y$ iff $y \; R \; x$.

4. The transpose of the matrix in Exercise 2.

$$\begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

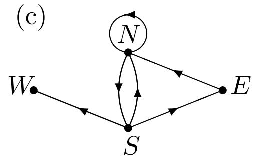
5. (a) No.
(b) Yes.
(c) Computer Science and Mathematics.

6.



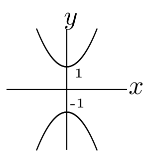
7. (a) NNSE. Let R be the given relation. First, we can move N to $(0, 1)$. Since $N\ R\ N$, we can move N to $(0, 2)$. Next, since $N\ R\ S$, we can move S back to $(0, 1)$. Since $S\ R\ E$, we can move E to $(1, 1)$.
(b) No. There is no way to move south without moving north first. Hence, our initial position cannot be lowered.

(c)



8. (a) No, since $(\sqrt{2})^2 - (\sqrt{3})^2 = -1 \neq 1$.
(b) Yes, since $(-1)^2 - 0^2 = 1$.
(c) No, since $0^2 - 1^2 = -1 \neq 1$.

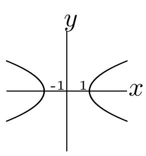
9.



This is a hyperbola, with asymptotes $y = \pm x$.

10. $x\ R^{-1}\ y$ iff $x^2 - y^2 = 1$.
That is, $x\ R^{-1}\ y$ iff $y\ R\ x$.

11. Reflect the graph from Exercise 9 about the line $y = x$.



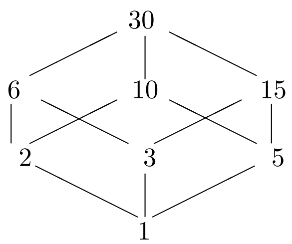
12. Not reflexive, since $1 \not R\ 1$.
Symmetric, since $x^2 + y^2 = y^2 + x^2$.
Not antisymmetric, as can be seen with $x = \frac{1}{\sqrt{2}}$ and $y = \frac{-1}{\sqrt{2}}$.
Not transitive, as can be seen with $x = 1$, $y = 0$, and $z = 1$.

13. Not reflexive, since $a \mid a$ but $a = a$.
Not symmetric, since $1 \mid 2$ and $2 \nmid 1$.
Antisymmetric, since $a \mid b$, $b \mid a$, and $a \neq b$ can never happen.
Transitive by Example 3.4 from Section 3.1.

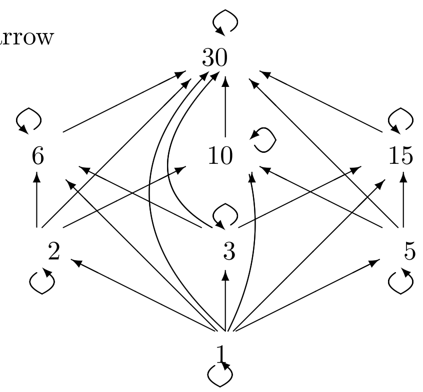
14. *Proof.* Let x, y, and z represent arbitrary elements of X. *Reflexive*: Since $x \; R_1 \; x$ and $x \; R_2 \; x$, we automatically have $x \; R \; x$.
Antisymmetric: Suppose $x \; R \; y$ and $y \; R \; x$. In particular, $x \; R_1 \; y$ and $x \; R_2 \; y$. Hence, $x = y$. *Transitive*: Suppose $x \; R \; y$ and $y \; R \; z$. That is, $x \; R_1 \; y$, $x \; R_2 \; y$, $y \; R_1 \; z$, and $y \; R_2 \; z$. Hence, $x \; R_1 \; z$ and $x \; R_2 \; z$. Thus, $x \; R \; z$. $\square$

15. $=$.
We need a relation R with all of the properties: reflexive, symmetric, antisymmetric, and transitive. If $x \; R \; y$, then $y \; R \; x$ (by symmetry), whence $x = y$ (by antisymmetry).

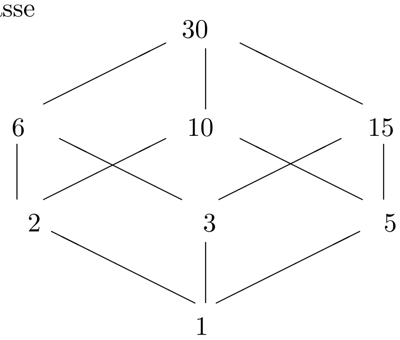
16.



arrow



Hasse



17. $(6, -1, -3, 5, 2) \preceq (6, -1, 2, -4, 7)$,
since $-3 < 2$.

18. (a) $(40, 16, 4) \prec (40, 18, 2)$ but $(40, 16, 4)$ is a better record than $(40, 18, 2)$.
(b) Use triples (W, T, L) instead.
If two teams have played the same number of games and they both have the same number of wins, then the team with fewer losses should be considered better.

19. *Proof.* Let x, y, and z represent arbitrary elements of X. *Reflexive*: Since $x^2 = x^2$, we automatically have $x \; R \; x$. *Symmetric*: Suppose $x \; R \; y$. That is, $x^2 = y^2$. Since the symmetry of equality gives $y^2 = x^2$, we have $y \; R \; x$. *Transitive*: Suppose $x \; R \; y$ and $y \; R \; z$. That is, $x^2 = y^2$ and $y^2 = z^2$. From the transitivity of equality it follows that $x^2 = z^2$. Therefore, $x \; R \; z$. $\square$

20. (a) $[x] = \{x, -x\}$, since $x^2 = y^2$ iff $x = \pm y$. Note that $\{0, -0\} = \{0\}$.
(b) $|x|$. The choices are x or $-x$, and $|x|$ is the nonnegative choice. Recall that

$$0 \leq |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

21. $\forall \, m, n \in \mathbb{Z}$, if $m \neq n$, then $(m-1, m] \cap (n-1, n] = \emptyset$.
Also, $\bigcup_{n \in \mathbb{Z}} (n-1, n] = \mathbb{R}$.
In particular, $\forall \, x \in \mathbb{R}, x \in (\lceil x \rceil - 1, \lceil x \rceil] = A_{\lceil x \rceil}$, and each x has a *unique* ceiling.

22. (a) A vertical line through $(x, 0)$.
(b) Each point (x, y) lies on a unique vertical line $\{x\} \times \mathbb{R}$.
$\forall \, x_1, x_2 \in \mathbb{R}$, if $x_1 \neq x_2$, then $(\{x_1\} \times \mathbb{R}) \cap (\{x_2\} \times \mathbb{R}) = \emptyset$ since $\{x_1\} \cap \{x_2\} = \emptyset$.
Since $\bigcup_{x \in \mathbb{R}} \{x\} = \mathbb{R}$, it follows that $\bigcup_{x \in \mathbb{R}} (\{x\} \times \mathbb{R}) = \mathbb{R}^2$.

23. No. The sets A_n are not disjoint, since $1 \in A_1 \cap A_2$.

24. $\forall \, x \in [0, \infty)$, let $A_x = \{-x, x\}$. In Exercise 20, we saw that $[x] = \{-x, x\}$. Since $[-x] = [x]$, it suffices to use $\{-x, x\}$ when $x \geq 0$.

25. $x \; R \; y$ iff $\lceil x \rceil = \lceil y \rceil$.
Note that, for each $n \in \mathbb{Z}$, we have $x \in A_n$ if and only if $n - 1 < x \leq n$. By definition, $n = \lceil x \rceil$. So $x, y \in A_n$ if and only if $\lceil x \rceil = n = \lceil y \rceil$.

26. $(x_1, y_1) \; R \; (x_2, y_2)$ iff $x_1 = x_2$.
For each $x \in \mathbb{R}$, we have $(x_1, y_1), (x_2, y_2) \in A_x = \{x\} \times \mathbb{R}$ if and only if $x_1 = x = x_2$.

27. $f(1)$ is not defined uniquely.
$f(1)$ is defined both as 1 and 2. That is, $x = 1$ fits in both pieces of the definition. However, $2 - x = 2 - 1 = 1 \neq 2 = 1 + 1 = x + 1$.

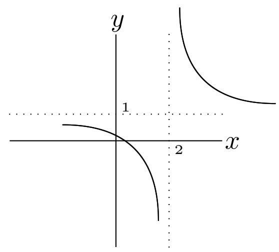
28. No. $f(-1) = \frac{1}{2} \notin \mathbb{Z}$.

29. Domain $= \{2, 3, 4, 5\}$ and range $= \{1, 3, 6, 10\}$.
Note that range$(f) = \{f(2), f(3), f(4), f(5)\} = \{\binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \binom{5}{2}\} = \{1, 3, 6, 10\}$.

30. *Proof.* $(\subseteq)$ Suppose $y \in$ range(f). So $y = 2 - x$ for some $-1 \leq x \leq 2$.

Hence, $0 \leq y = 2 - x \leq 3$. That is, $y \in [0, 3]$. ($\supseteq$) Suppose $y \in [0, 3]$. Observe that $2 - y \in [-1, 2]$ and $f(2 - y) = y$. So $y \in \text{range}(f)$. $\square$

31. Domain $= \mathbb{R} \setminus \{2\}$ and range $= \mathbb{R} \setminus \{1\}$.



32. $g \circ f : \mathbb{Z}^+ \longrightarrow \mathbb{R}$ is given by $(g \circ f)(n) = n - 2$.
That is, $(g \circ f)(n) = g(f(n)) = g(\sqrt{n-1}) = (\sqrt{n-1})^2 - 1 = n - 1 - 1 = n - 2$.
Also, note that the domain is adopted from f, and the codomain is adopted from g.

33. (a) *Of Mice and Cats* and *Raisins of Wrath*.
(b)

Publisher	Customer
Book Farm	Raul Cortez
Authority Pubs	Mary Wright
Word Factory	Mary Wright
Book Farm	David Franklin

(c) Raul Cortez and David Franklin.

34. Observe that $x \; R \; y$ and $y \; S \; z$ if and only if $z \; S^{-1} \; y$ and $y \; R^{-1} \; x$. Hence, $x \; (S \circ R) \; z$ if and only if $z \; (R^{-1} \circ S^{-1}) \; x$.
By Definition 5.16 in the exercises from Section 5.3,

$$x \; (S \circ R) \; z \quad \text{if and only if} \quad \exists \, y \in Y \text{ such that } x \; R \; y \text{ and } y \; S \; z.$$

Consequently,

$$z \; (R^{-1} \circ S^{-1}) \; x \quad \text{if and only if} \quad \exists \, y \in Y \text{ such that } z \; S^{-1} \; y \text{ and } y \; R^{-1} \; x.$$

In fact, we can apply the same value y in both instances.

35. *Proof.* Suppose $n_1, n_2 \in \mathbb{Z}$ and $f(n_1) = f(n_2)$. So $3n_1 - 2 = 3n_2 - 2$. It follows that $n_1 = n_2$. $\square$

36. $f(0) = f(1)$ but $0 \neq 1$.
Namely, $f(0) = 0^3 - 0 = 0 = 1^3 - 1 = f(1)$. So two distinct input values have the same output value.

37. $f(-1) = 0$, $f(2) = 3$, and $f(-3) = f(3) = 8$.
We have explicitly displayed that each element of the codomain $\{0, 3, 8\}$ is actually in the range.

38. $f(x) = -11$ is impossible.
$\forall\, x \in \mathbb{R}, x^2 - 10 \geq -10$. So, there is no $x \in \mathbb{R}$ for which $f(x) = -11$.

39. (a) 84.
(b) $036-77-5484$ and $036-77-5709$.
Note that 5484 mod 225 = 84 and 5709 mod 225 = 84.

40. *Proof.* (One-to-one) Suppose $f(r_1) = f(r_2)$. So $\frac{r_1}{n} = \frac{r_2}{n}$. Hence $r_1 = r_2$.
(Onto) Suppose $s \in \mathbb{Q}$. Let $r = ns$, and observe that $f(r) = s$. □
Alternatively, $s \mapsto ns$ is the inverse of f.

41. *Proof.* (One-to-one) Suppose $f(n_1) = f(n_2)$. Since the second coordinate of the output is 0 for even input and 1 for odd input, it must be that n_1 and n_2 have the same parity. In both the even and odd cases, it is easy to then see that $n_1 = n_2$. (Onto) Suppose $(m, i) \in \mathbb{Z} \times \{0, 1\}$. Let $n = 2m + i$. Observe that $f(n) = (m, i)$. □

42. *Sketch.* If $f'(x_1) = f'(x_2)$, then $f(x_1) = f(x_2)$, and hence $x_1 = x_2$. So f' is one-to-one. If $y' \in Y'$, then there is some $x \in X$ such that $f'(x) = f(x) = y'$. So f' is onto. Thus, f' is a bijective. □

43. (a) *Proof.* Suppose $f : [0, 2] \longrightarrow [0, 1]$. Let $x \in [0, 2]$. Since $f(x) \in [0, 1]$, it follows that $g(f(x)) = f(x)$. Hence $g \circ f = f$. □
(b) Define $f(x) = \frac{x}{2}$.
So $(f \circ g)(2) = f(1) = \frac{1}{2}$ and $f(2) = 1$. Thus, $f \circ g \neq f$.

44. We confirm that $\forall\, x \in \mathbb{R}, f(g(x)) = x$ and $g(f(x)) = x$.
$\forall\, x \in \mathbb{R}, (f \circ g)(x) = f(g(x)) = f(2 - 3x) = \frac{2-(2-3x)}{3} = x$ and
$(g \circ f)(x) = g(f(x)) = g(\frac{2-x}{3}) = 2 - 3(\frac{2-x}{3}) = x$.

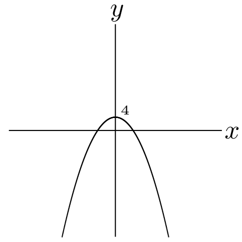
45. *Proof.* Suppose $f : X \longrightarrow Y$. Let $g_1 : Y \longrightarrow X$ and $g_2 : Y \longrightarrow X$ be inverses of f. That is, $g_1 \circ f = \mathrm{id}_X$, $f \circ g_1 = \mathrm{id}_Y$, $g_2 \circ f = \mathrm{id}_X$, and $f \circ g_2 = \mathrm{id}_Y$.
It follows that $g_2 = g_2 \circ \mathrm{id}_Y = g_2 \circ (f \circ g_1) = (g_2 \circ f) \circ g_1 = \mathrm{id}_X \circ g_1 = g_1$. □

46. 3,
since $5^3 = 125$.

47. 1,
since $e^1 = e$ and the base of ln is e.

48. (a) $f(\{-1,0,1,2\}) = \{f(-1), f(0), f(1), f(2)\} = \{-1,0,-1,4\} = \{-1,0,4\}$.
(b) $\{-1,0,1,2\}$.
Note that $f(-1) = f(1) = -1$; $f(0) = 0$; there is no n such that $f(n) = 1$; $f(2) = 4$; and there is no n such that $f(n) = 9$.

49. (a) $[0,4]$. (b) $[-\sqrt{3}, \sqrt{3}]$.
Observe that, if $-1 \le x \le 2$, then $0 \le 4 - x^2 \le 4$. And, if $1 \le 4 - x^2 \le 4$, then $-\sqrt{3} \le x \le \sqrt{3}$.

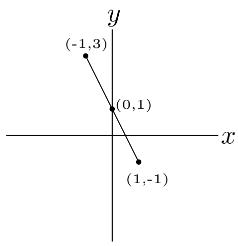


50.

S	$f(S)$
$\{1\}$	$\{-1\}$
$[0,1]$	$[-1,1]$
$(-1,0)$	$(1,3)$

T	$f^{-1}(T)$
$\{1\}$	$\{0\}$
$[1,4)$	$[-1,0]$
$(-4,-2)$	$\emptyset$

Note the graph of the function and some of its individual function values.



x	$f(x)$
-1	3
0	1
1	-1

Hence, note that there are no values of $x \in [-1,1] = \text{domain}(f)$ such that $3 < f(x) < 4$ or $-4 < f(x) < -1$. That is, $\text{range}(f) = [-1,3] \subset (-4,4)$. Consequently, $f^{-1}([1,4)) = f^{-1}([1,3])$ and $f^{-1}((-4,-2)) = \emptyset$.

51. (a) Megan Johnson, Martha Lang, and Abe Roth.
(b) Inverse image.

Student	Major
Abe Roth	Computer Science
Megan Johnson	Mathematics
Richard Kelley	Computer Science
Martha Lang	Physics
Abe Roth	Mathematics

52. (a) True. See Exercise 35 from Section 5.5.
(b) False. See Exercise 36 from Section 5.5.

53. (a) False. Let $f(x) = 0$, $S_1 = \{-1\}$, and $S_2 = \{1\}$. So $f(S_1 \triangle S_2) = f(\{-1, 1\}) = \{0\}$, but $f(S_1) \triangle f(S_2) = \{0\} \triangle \{0\} = \emptyset$.
(b) True. $f^{-1}(T_1 \triangle T_2) = f^{-1}((T_1 \setminus T_2) \cup (T_2 \setminus T_1)) =$
$f^{-1}(T_1 \setminus T_2) \cup f^{-1}(T_2 \setminus T_1) = f^{-1}(T_1 \cap T_2{}^c) \cup f^{-1}(T_2 \cap T_1{}^c) =$
$(f^{-1}(T_1) \cap f^{-1}(T_2{}^c)) \cup (f^{-1}(T_2) \cap f^{-1}(T_1{}^c)) =$
$(f^{-1}(T_1) \cap (f^{-1}(T_2))^c) \cup (f^{-1}(T_1) \cap (f^{-1}(T_2))^c) =$
$(f^{-1}(T_1) \setminus f^{-1}(T_2)) \cup (f^{-1}(T_2) \setminus f^{-1}(T_1)) = f^{-1}(T_1) \triangle f^{-1}(T_2)$.

54. $(0, 5)$. Note that $(0, 3) \cup (2, 5) = (0, 5)$ and $\forall\, r \in [0, 2], (r, r + 3) \subseteq (0, 5)$.

55. $\{0\}$. Note that $\{m \,:\, m = nk \text{ for some } n \in \mathbb{Z}\}$ is the set of multiples of k. Hence, to be in the desired intersection, an integer would have to be a multiple of every integer k. Of course, only 0 has that property.

56. $S = \bigcup_{r \in [80,115]} A_r$. Also, $S = A_{80} \cup A_{90} \cup A_{100} \cup A_{110} \cup A_{115}$.
Note that \$115,000 + \$10,000 = \$125,000.

57. *Proof.* Since $\mathcal{I} \neq \emptyset$, we have some $j \in \mathcal{I}$. ($\subseteq$) Suppose $x \in \displaystyle\bigcup_{i \in \mathcal{I}} (B \cup A_i)$. So we have some $i_0 \in \mathcal{I}$ such that $x \in B \cup A_{i_0}$. If $x \notin B$, then $x \in A_{i_0} \subseteq \displaystyle\bigcup_{i \in \mathcal{I}} A_i$. In any case, $x \in B \cup \displaystyle\bigcup_{i \in \mathcal{I}} A_i$. ($\supseteq$) Suppose $x \in B \cup \displaystyle\bigcup_{i \in \mathcal{I}} A_i$. If $x \notin B$, then $x \in A_{i_0}$ for some $i_0 \in \mathcal{I}$, and thus $x \in B \cup A_{i_0} \subseteq \displaystyle\bigcup_{i \in \mathcal{I}} (B \cup A_i)$. If $x \in B$, then $x \in B \cup A_j \subseteq \displaystyle\bigcup_{i \in \mathcal{I}} (B \cup A_i)$. $\square$

58. 3. The set is $\{4, 5, 6\}$.

59. $f : \{-100, -99, \ldots, 200\} \longrightarrow \{1, 2, \ldots, 301\}$ defined by $f(n) = n + 101$ is a bijection. Its inverse is given by $g(n) = n - 101$.

60. 3. Namely, Computer Science, Mathematics, and Physics.

61. $f : [-1, 0) \longrightarrow (1, 7]$ defined by $f(x) = 1 - 6x$ is a bijection.
Its inverse is given by $g(x) = \frac{1-x}{6}$.

62. $f : \{2k \,:\, k \in \mathbb{Z}, 0 \leq k \leq n\} \longrightarrow \{1, 2, \ldots, n+1\}$ defined by $f(m) = \frac{m}{2} + 1$ is a bijection.
Its inverse is given by $g(m) = 2m - 2$. Be sure to note that $\frac{m}{2} \in \mathbb{N}$ for each $m \in \{2k \,:\, k \in \mathbb{Z}, 0 \leq k \leq n\}$. So the definition of f is valid.

63. $f : \mathbb{Z} \longrightarrow T$ defined by $f(n) = 10n$ is a bijection.
Its inverse is given by $g(n) = \frac{10}{n}$.

64. The function $f : A \times B \longrightarrow B \times A$ defined by $(a, b) \mapsto (b, a)$ is a bijection.
The function $g : B \times A \longrightarrow A \times B$ defined by $(b, a) \mapsto (a, b)$ is its inverse.

65. False. Let $A = B = C = (0, 1)$ and $D = (0, 2)$.
Then $B \setminus A = \emptyset$ and $D \setminus C = [1, 2)$ do not have the same cardinality.

66. *Proof.* Suppose to the contrary that, for some $m \in \mathbb{N}$, there is a bijection $f : [1, 2] \longrightarrow \{1, 2, \ldots, m\}$. Let $i : \{1 + \frac{1}{n} \ : \ 1 \leq n \leq m + 1\} \longrightarrow [1, 2]$ be the inclusion of a set of cardinality $m + 1$. The composite $f \circ i$ is a one-to-one map that contradicts the Pigeon Hole Principle. $\square$

67. $f : \{3k \ : \ k \in \mathbb{Z}^+\} \longrightarrow \mathbb{Z}^+$ defined by $f(n) = \frac{n}{3}$ is a bijection.
Its inverse is given by $g(n) = 3n$.

68. *Proof.* Suppose to the contrary that $\mathbb{R}^2$ is countable. Then the subset $\mathbb{R} \times \{0\}$ is countable. Since $\mathbb{R}$ and $\mathbb{R} \times \{0\}$ have the same cardinality, $\mathbb{R}$ is countable. This is a contradiction. $\square$

2.6 Chapter 6

Section 6.1

1. $6 \cdot 8 = 48$.

3. $31 \cdot 3 \cdot 4 = 372$.

5. The possibilities are: AA, AB, AC, AD, BB, BC, BD, CC, CD, DD. So, 10. The number of choices for the second letter depends on the first letter.

7. (a) The possible outcomes are: $2H$, $2T$, $4H$, $4T$, $6H$, $6T$, $1\clubsuit$, $1\diamondsuit$, $1\heartsuit$, $1\spadesuit$, $3\clubsuit$, $3\diamondsuit$, $3\heartsuit$, $3\spadesuit$, $5\clubsuit$, $5\diamondsuit$, $5\heartsuit$, $5\spadesuit$. So, 18.
(b) No. There are 6 outcomes involving a coin and 12 involving a card. More outcomes involve a card.

9. (a) $26^3 \cdot 10^3 = 17576000$.
(b) $26^3 \cdot (10^3 - 1) = 17558424$.
(c) $(26^3 - 1) \cdot (10^3 - 1) = 17557425$.

11. $5040/3 = 1680$ days.

13. $26 \cdot 25 \cdot 5^3 = 81250$.

15. $26 \cdot 36^4 \cdot 10 = 436700160$.

17. $10^4 = 10000$.

19. $26 \cdot 25 \cdot 24 \cdot 10^4 = 156000000$.

21. $4^6 = 4096$.

23. $16^{10} = 1099511627776$.

25. $5 \cdot 4 \cdot 3 = 60$.

27. $3 \cdot 2 \cdot 52 = 312$.

29. $850 - 85 + 1 = 766$.

31. $\frac{100}{10} + 1 = 11$.

33. $\frac{7000}{28} = 250$.

35. $|\{84, 91, \ldots, 7994\}| = \frac{7994}{7} - \frac{84}{7} + 1 = 1131$.

37. $|\{204, 210, \ldots, 1998\}| = \frac{1998}{6} - \frac{204}{6} + 1 = 333$.

39. $\frac{776}{2} - \frac{18}{2} + 1 = 380$.

41. 86.
Note that 0 is included. We have $\lfloor \frac{2^8}{3} \rfloor + 1 = 86$.

43. $365(59) + 15 + 24 + 243 + 11 = 21828$.

45. $365(65) + 17 + 14 + 181 + 120 = 23957$.

47. $365(86) + 20 + 27 + 153 + 1 = 31591$.

Section 6.2

1. $4! = 24$.

3. $6! = 720$.

5. $200 \cdot 199 \cdot \cdots \cdot 191 = P(200, 10)$.

7. $20 \cdot 19 \cdot 18 \cdot 17 = 116280$.

9. $P(8, 3) = 336$.

11. $\frac{20 \cdot 19 \cdot 18 \cdot 17}{4 \cdot 3 \cdot 2 \cdot 1} = 4845$.

13. $\binom{6}{3} 25^3 = 312500$.

15. $\binom{10}{5} 15^5 = 191362500$.

17. $\binom{8}{2}\binom{10}{2}\binom{20}{2} = 239400$.

19. $\binom{8}{3}\binom{6}{2} = 840$.

21. $\binom{26}{6} = 230230$.

23. (a) $\binom{12}{5} = 792$.
(b) $\binom{3}{1}\binom{5}{2}\binom{4}{2} = 180$.

25. $\binom{8}{2}\binom{6}{2} 14^4 = 16134720$.

27. $\binom{10}{4} P(7, 6) = 1058400$.

29. (a) $P(15, 5)P(15, 5)P(15, 4)P(15, 5)P(15, 5) = P(15, 5)^4 P(15, 4)$.

(b) i. There are 5! possibilities for the column with BINGO.
So, $5!P(15,5)^3P(15,4)$.
ii. Since the free cell starts with a chip, there are 4! possibilities for the column with BINGO. So, $4!P(15,5)^4$.

31. $P(6,5) = 720$.

33. $\binom{5}{2}5^3 = 1250$.

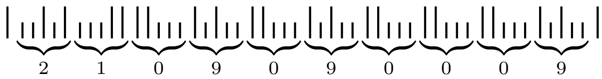
35. $6 \cdot \binom{5}{3} \cdot 5 = 300$.

37. 3.
The run could start with a 1, a 2, or a 3. So, 3.

39. $\binom{6}{4} = 15$.

41. $6 \cdot \binom{5}{2} = 60$.

43. 21090-9000.



45. (a) The first number in Pascal's triangle greater than or equal to 16 is $20 = \binom{6}{3}$. So each digit will use a total of 6 bars.
(b) 3 bars will be long.

47. (a) $P(6,4) = \$360$.
(b) $P(20,4) = \$116280$.
(c) When the horses favored to win do well, the superfecta payoff is less than \$116,280.
E.g., if the four most highly favored horses finish in the top four, then that pays off much less than if the four least highly favored horses do so. That is, long shots give a higher payoff.

Section 6.3

1. $\binom{10}{5}\binom{12}{3} + \binom{10}{6}\binom{12}{2} = 69300$.

3. $2\binom{6}{4} \cdot 9^2 = 2430$.

5. $1 + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} = 176$.

7. $\binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 22$.

9. $\binom{35}{12} - [\binom{18}{0}\binom{17}{12} + \binom{18}{1}\binom{17}{11}] = 834222844$.

11. $\binom{5}{0}+\binom{5}{2}+\binom{5}{4}=16$.

13. $(1000-333)-(499-166)=334$.

15. $166+71-23=214$.

17. $(1000+400-200)-(99+39-19)=1081$.

19. (a) $6^8=1679616$.
(b) $6^8=1679616$.
(c) $6^8+6^8-2^8=3358976$.

21. $2\binom{8}{3}\cdot 3^5-\binom{8}{3}\binom{5}{3}2^2=24976$.

23. $2\binom{13}{2}\binom{39}{3}-\binom{13}{2}^2\cdot 26=1267500$.

25. $\binom{39}{5}+\binom{13}{1}\binom{39}{4}=1645020$.

27. $144-(72+48-24)=48$.

29. $[1000-(500+200-100)]-[99-(49+19-9)]=360$.

31. (a) "MEET ME."
That is, we have the decodings: $y=A\mapsto x=M$, $y=W\mapsto x=E$,
$y=B\mapsto x=T$, $y=C\mapsto x=$' '.
(b) $y=4x+3$ and 4 is relatively prime to 27.
Note, e.g., that ' ' $=0$, $4\cdot 0+3=3$, and $C=3$.
(c) 486.
Observe that, if $kx\equiv j \pmod{d}$ for all x, then $k\equiv j\equiv 0 \pmod{d}$. This follows by plugging in first $x=1$ and then $x=0$. Hence, if $(a_1x+b_1)\equiv(a_2x+b_2) \pmod{d}$ for all x, then $a_1\equiv a_2 \pmod{d}$ and $b_1\equiv b_2 \pmod{d}$. This holds since $(a_1-a_2)x\equiv(b_2-b_1) \pmod{d}$ for all x. Consequently, we need only consider a and b values in $\{0,\ldots,26\}$. Since, there are 18 choices for a that are relatively prime to 27, there are only $18\cdot 27=486$ different linear ciphers.

33. $[1771-(885+354-177)]-[170-(85+34-17)]=641$.

35. *Theorem*: If $A_1,A_2,\ldots,A_n$ are disjoint sets, then
$|A_1\cup A_2\cup\cdots\cup A_n|=|A_1|+|A_2|+\cdots+|A_n|$.
Sketch. (By induction) The Addition Principle handles the case in which $n=2$ (also check when $n=1$). So suppose $k\geq 2$ and $A_1,\ldots,A_{k+1}$ are disjoint sets.

Observe that

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}| &= |(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}| \\ &= |A_1 \cup A_2 \cup \cdots \cup A_k| + |A_{k+1}| \\ &= |A_1| + |A_2| + \cdots + |A_k| + |A_{k+1}|.\end{aligned}$$

The first equality follows from associativity of unions. The second follows from the Addition Principle (Check that $A_1 \cup A_2 \cup \cdots \cup A_k$ and A_{k+1} are disjoint). The inductive hypothesis gives the last. $\square$

37. $5 + \binom{5}{2} = 15$.

39. $1 + 5 + (5 + \binom{5}{2}) = 21$.

41. $6\binom{5}{4} \cdot 5 + 6 = 156$.

43. $P(6,5) + 6\binom{5}{2}P(5,3) = 4320$.

45. $5 + 1 = 6$.

47. $2(1 + 5) = 12$.

49. $\binom{6}{2} = 15$.

51. $\binom{6}{4} + 6\binom{5}{2} = 75$.

Section 6.4

1. (a) $\frac{250}{1000} = .25$. (b) $\frac{119+63}{1000} = .182$.

3. $\frac{6}{36} = \frac{1}{6}$.

5. $\frac{10}{36} = \frac{5}{18}$.

7. (a) $\{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$.
These are equivalent to binary sequences of length 3.
(b) $\frac{4}{8} = \frac{1}{2}$.

9. (a) $\{00, 01, 02, 03, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33\}$.
This is equivalent to $\{0, 1, 2, 3\} \times \{0, 1, 2, 3\}$.
(b) $\frac{6}{16} = \frac{3}{8}$.

11. $\frac{4 \cdot 3!}{6^3} = \frac{1}{9}$.

13. $\frac{6 \cdot 5 \cdot 4}{6^3} = \frac{5}{9}$.

15. $\frac{\binom{20}{5}\binom{30}{7}}{\binom{50}{12}} = \frac{24279264}{93384347} \approx .2600.$

17. (a) $\frac{\binom{20}{12}+\binom{30}{12}}{\binom{50}{12}} = \frac{1332603}{1867686940} \approx .0007.$
(b) No.
It is $\frac{\binom{30}{12}}{\binom{20}{12}} \approx 687$ times more likely.
(c) $1 - \frac{1332603}{1867686940} = \frac{1866354337}{1867686940} \approx .9993.$

19. $\frac{8 \cdot 7}{18 \cdot 17} = \frac{28}{153}.$

21. $\frac{1+8+\binom{8}{2}}{2^8} = \frac{37}{256}.$

23. (a) $\frac{\binom{10}{5}}{2^{10}} = \frac{63}{256} \approx .2461.$
(b) Bet against it,
since it happens only about one fourth of the time.

25. $\frac{\binom{26}{6}}{P(26,6)} = \frac{1}{720}.$

27. $\frac{3^4+\binom{4}{2}3^4+3^4}{6^4} = \frac{1}{2}.$

29. Probability Complement Principle: If E is an event in a sample space S, then $P(E) = 1 - P(E^c)$.
Assumption: The outcomes in S are equally likely.
Proof. $P(E) = \frac{|E|}{|S|} = \frac{|S|-|E^c|}{|S|} = 1 - \frac{|E^c|}{|S|} = 1 - P(E^c)$. □

31. $\frac{3 \cdot 8!}{9!} = \frac{1}{3}.$

33. There are 365^n possible birthday values for n people. The complementary event is that no two people have the same birthday, and there are

$$365 \cdot 364 \cdot \cdots \cdot (365 - n + 1) = P(365, n)$$

possible outcomes of that type. If we let $p(n)$ be the probability that at least two of n people have the same birthday, then $p(n) = 1 - \frac{P(365,n)}{365^n}$.
(a) $p(15) \approx .253$.
(b) $p(30) \approx .706$.
(c) $n = 23$,
since $p(22) \approx .476$ and $p(23) \approx .507$.

35. $\frac{6}{36} + \frac{3}{36} - \frac{1}{36} = \frac{2}{9}.$

37. $1 - \frac{3^2}{5^2} = \frac{16}{25}.$

39. $\frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}.$

41. *Proof.* The assertion is certainly true for $k = 1$ die. So assume $k \geq 1$ and the assertion is true for k dice. Suppose we have $k + 1$ dice, and pretend that one is red and the remaining k are green. The red die is equally likely to come up even or odd. By our induction hypothesis, the sum of the green dice is equally likely to be even or odd. The total sum is even if and only if either the red die is even and the green sum is even or the red die is odd and the green sum is odd. Hence, the probability that the total sum is even is $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2}$. $\square$

43. $\frac{1}{\binom{39}{5}} = \frac{1}{575757}$.

45. $1 - \frac{\binom{36}{5}5+5\binom{36}{4}5+\binom{36}{5}}{\binom{41}{5}6} = \frac{253937}{1498796} \approx .1694$.

47. (a) $1 - \frac{14(\binom{70}{5}+5\binom{70}{4}+\binom{5}{2}\binom{70}{3})}{15\binom{75}{5}} = \frac{2933734}{43148475} \approx .068$.

(b) $\frac{\binom{70}{5}}{15\binom{75}{5}} = \frac{2017169}{43148475} \approx .0467$.

49. $\frac{3 \cdot 4!}{6^4} = \frac{1}{18}$.

51. $\frac{6\binom{4}{2} \cdot 5 \cdot 4}{6^4} = \frac{5}{9}$.

53. $\frac{7}{27}$.

$$\underbrace{2(1 - \frac{5^4}{6^4} - \frac{4 \cdot 5^3}{6^4})}_{\text{at least 2 sixes}} - \underbrace{\frac{\binom{4}{2}}{6^4}}_{\text{2 of each}} = \frac{7}{27}.$$

or

$$2(\underbrace{\binom{4}{3} \cdot 5}_{\text{3 sixes}} + \underbrace{1}_{\text{4 sixes}}) + \underbrace{(2\binom{4}{2} \cdot 5^2 - \binom{4}{2})}_{\text{2 fives or 2 sixes}} = \frac{7}{27}.$$

55. $\frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$.

57. $\frac{\frac{4}{16}}{\frac{10}{16}} = \frac{2}{5}$.

59. (a) Yes. $P(E \cap F_1) = \frac{3}{52} = P(E) \cdot P(F_1)$, since $P(E) = \frac{1}{4}$, $P(F_1) = \frac{3}{13}$.
(b) No. $P(E \cap F_2) = \frac{1}{4} \neq P(E) \cdot P(F_2)$, since $P(F_2) = \frac{1}{2}$.

61. *Sketch.* ($\rightarrow$) Suppose E and F are independent. So $P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)P(F)}{P(F)} = P(E)$. Similarly, $P(F \mid E) = P(F)$. ($\leftarrow$) Suppose $P(E \mid F) = P(E)$. So $P(E \cap F) = P(E \mid F)P(F) = P(E)P(F)$. $\square$
Note Definitions 6.5 and 6.6.

63. (a) 26%, since $p = (.1)(.5) + (.3)(.4) + (.9)(.1) = .26$.
(b) $\frac{(.3)(.4)}{.26} = \frac{6}{13} \approx .4615$.

65. $\frac{(.95)(.8)}{(.95)(.8)+(.1)(.2)} = \frac{38}{39} \approx .974.$

67. (a) $4(.5)+3(.3)+2(.1)+1(.05)+0(.05) = 3.15$ stars.
(b) $.4(.5)+.2(.3)+.5(.1)+.8(.05)+.9(.05) = 0.395$.
(c) We use Bayes' Formula.

Number of Stars	Probability of Ranking on Website
4	$\frac{.4(.5)}{.395} = \frac{40}{79}$
3	$\frac{.2(.3)}{.395} = \frac{12}{79}$
2	$\frac{.5(.1)}{.395} = \frac{10}{79}$
1	$\frac{.8(.05)}{.395} = \frac{8}{79}$
0	$\frac{.9(.05)}{.395} = \frac{9}{79}$

(d) $4(\frac{40}{79})+3(\frac{12}{79})+2(\frac{10}{79})+1(\frac{8}{79})+0(\frac{9}{79}) = \frac{224}{79} \approx 2.84$ stars.

69. Bayes' Formula: If $S = F_1 \cup \cdots \cup F_n$ is a disjoint union, then

$$P(F_k \mid E) = \frac{P(E \mid F_k)P(F_k)}{\sum_{i=1}^{n} P(E \mid F_i)P(F_i)}.$$

Proof. $P(F_k \mid E) = \frac{P(F_k \cap E)}{P(E)} = \frac{P(E|F_k)P(F_k)}{P(E)} = \frac{P(E|F_k)P(F_k)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)}$. $\square$

Section 6.5

1. $\binom{4}{2}^4 = 1296$.

3. $\binom{10}{6} = 210$.

5. $\binom{7}{4}\binom{3}{2} = 105$.

7. $\binom{4}{3}\binom{3}{1}\binom{3}{2} = 36$.

9. $\binom{4}{3}\binom{6}{3} + \binom{7}{4}\binom{3}{2} - 36 = 149$.

11. (a) $\frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{16}$.
(b) $p = \frac{1}{8}$.

13. $4 \cdot \binom{13}{5} - 40 = 5108$.

15. $13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 3744$.

17. $10 \cdot 2^5 - 10 \cdot 2 = 300$.

19. $\underbrace{(\binom{52}{5} - 1302540)}_{\text{something}} - \underbrace{9\binom{4}{2}\binom{12}{3} \cdot 4^3}_{\text{at best a pair of tens}} = 536100.$

21. $N = 10 \cdot 4 \cdot 2^5 = 1280$ and $p \approx .00001392$.

23. $N = 4\binom{26}{5} - \underbrace{1280}_{\text{See Exercise 21}} = 261840$ and $p \approx .00284725$.

25. $N = \binom{13}{2}\binom{8}{2}^2 \cdot 88 - \underbrace{\binom{13}{2} \cdot 4 \cdot 11 \cdot 2}_{\text{Two Pair Flush}} = 5374512$ and $p \approx .05844242$.

27. $N = 13\binom{8}{3}\binom{12}{2} \cdot 8^2 = 3075072$ and $p \approx .03343832$.

29. $\binom{10+3-1}{10} = \binom{12}{10} = 66$.

31. $\binom{12+4-1}{12} = \binom{15}{12} = 455$.

33. $\binom{8+3-1}{8} = \binom{10}{8} = 45$.

35. $\binom{4+6-1}{4} = \binom{9}{4} = 126$.

37. (a) $\binom{8+4-1}{8} = \binom{11}{8} = 165$.
(b) No. 1 penny, 3 nickels, and 4 dimes is worth the same as 6 pennies and 2 quarters.

39. $\frac{\binom{3}{2}\binom{2}{1}\binom{4}{1}+\binom{3}{1}\binom{2}{2}\binom{4}{1}+\binom{3}{1}\binom{2}{1}\binom{4}{2}}{\binom{9}{4}} = \frac{72}{126} = \frac{4}{7}$ or $1 - \frac{\binom{5}{4}+\binom{7}{4}+\binom{6}{4}-\binom{4}{4}}{\binom{9}{4}} = \frac{4}{7}$.

41. $\binom{5+6-1}{5} = 252$.

43. $N = 13\binom{48}{3} = 224848$ and $p \approx .00168067$.

45. $N = \underbrace{9 \cdot}_{\text{non-Ace high}} 4[\binom{47}{2} - \underbrace{46}_{\text{not new high}}] + \underbrace{4}_{\text{Ace high}}\binom{47}{2} = 41584$ and $p \approx .00031083$.

47. $N = 13\binom{4}{3}\binom{12}{4} \cdot 4^4 - [\underbrace{10 \cdot 5\binom{4}{3} \cdot 4^4}_{\text{straight}} + \underbrace{4\binom{13}{5} \cdot 5\binom{3}{2}}_{\text{flush}} - \underbrace{10 \cdot 4 \cdot 5\binom{3}{2}}_{\text{both}}] = 6461620$ and $p \approx .04829870$.

49. $N = (\binom{8}{2} + 9\binom{7}{2})(4^7 - \binom{7}{5} \cdot 4 \cdot 3^2 - \binom{7}{6} \cdot 4 \cdot 3 - 4) +$
$6(8 + 7 \cdot 9)(4^5\binom{4}{2} - \binom{4}{2} \cdot 2 \cdot 5 \cdot 3 - \binom{4}{2}^2 - \binom{4}{2}^2) + 10\binom{5}{2}(4^3\binom{4}{2}^2 - 4 \cdot 3^2) +$
$5 \cdot 10(4^4\binom{4}{3} - 4\binom{3}{2}) = 6180020$ and $p \approx .04619382$.

51. $N = (6\binom{13}{6} - 6(8 + 7 \cdot 9))(4^5\binom{4}{2} - \binom{4}{2} \cdot 2 \cdot 5 \cdot 3 - \binom{4}{2}^2 - \binom{4}{2}^2) = 58627800$ and $p \approx .43822546$.

53. Consider each of the 44 remaining cards.
$P(A \text{ wins}) \approx .3864$ and $P(B \text{ wins}) \approx .6136$.

55. If at least one more 2 is drawn, then player B wins. If no more 2's are drawn, then Player 2 wins iff no more K's or 7's are drawn. $P(A \text{ wins}) \approx .2828$ and $P(B \text{ wins}) \approx .7172$.

57. An exhaustive analysis using software gives $P(A \text{ wins}) \approx .4754$, $P(B \text{ wins}) \approx .5211$, and $P(\text{tie}) \approx .0035$.

Section 6.6

1. (a) $\frac{40!}{40} = 39!$
(b) $\frac{39!}{2}$.

3. $\frac{4!}{4\cdot 2} = 3$.

5. (a) $\frac{8!}{8} = 5040$.
(b) $2 \cdot 6! = 1440$.
(c) $6! = 720$.

7. $\frac{P(14,8)}{8\cdot 2} = 7567560$.

9. $5 \cdot \frac{4!}{4} = 30$ or $\frac{6!}{24} = 30$.

11. $\frac{20!}{20\cdot 3} = \frac{19!}{3}$.

13. $\frac{5!}{2\cdot 3} = 20$.

15. $\frac{6!}{4} = 180$.

17. $\frac{\binom{6}{3}}{2} = 10$.

19. $\underbrace{\frac{\binom{20}{5}\binom{15}{5}\binom{10}{5}\binom{5}{5}}{4!}}_{\text{teams}} \cdot \underbrace{3}_{\text{games}} = 1466593128$.

21. $\frac{\binom{21}{6}\binom{15}{5}\binom{10}{5}\binom{5}{5}}{3!} = 6844101264$.

23. $\frac{\binom{16}{2}\binom{14}{2}\binom{12}{2}\cdots\binom{2}{2}}{8!} = 2027025$.

25. $\overbrace{\frac{\binom{50}{3}\binom{47}{3}\cdots\binom{23}{3}}{10!}}^{\text{3-person scrambles}} \cdot \overbrace{\frac{\binom{20}{2}\binom{18}{2}\cdots\binom{2}{2}}{10!}}^{\text{2-person scrambles}} = \frac{\binom{50}{3}\binom{47}{3}\cdots\binom{23}{3}\binom{20}{2}\binom{18}{2}\cdots\binom{2}{2}}{(10!)^2}$.

27. $\frac{\binom{60}{6}\frac{5!}{2}\binom{54}{6}\frac{5!}{2}\cdots\cdots\binom{6}{6}\frac{5!}{2}}{10!} = \frac{\binom{60}{6}\binom{54}{6}\cdots\binom{6}{6}(\frac{5!}{2})^{10}}{10!}$.

29. $\frac{9!}{4} = 90720$.

31. $\frac{\binom{24}{2}\binom{22}{2}\cdots\binom{2}{2}}{12} = 1262305504828368000$.

33. $60 + 40 + 50 + 40 - 3(20) = 130$.

35. $20 + 10 + 30 + 20 - 6 - 10 = 64$.

37. $\frac{6!}{6} = 120$.

39. $\frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3} = 30$.

Review

1. $9! = 362880$.

2. $3^5 8^2 = 15552$.

3. $26^2 10 \cdot 9 \cdot 8 = 486720$.

4. 13 hours. In general, n loads take $\frac{1}{2}(n+1)$ hours.

5. $\lfloor \frac{1000}{7} \rfloor = \frac{994}{7} = 142$.

6. $\frac{3999}{3} - \frac{201}{3} + 1 = 1267$.

7. $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7893600$.

8. $4(365) + 1 + 8 + 31 + 31 + 29 + 31 + 30 + 31 + 5 = 1657$.

9. $6! = 720$.

10. (a) $100 \cdot 99 \cdot 98 = 970200$. (b) $100 \cdot 98 \cdot 96 = 940800$.

11. $\binom{6}{1}\binom{20}{4}\binom{22}{5} = 765529380$.

12. $\{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{a,f\}, \{b,c\}, \{b,d\}, \{b,e\}, \{b,f\}, \{c,d\}, \{c,e\}, \{c,f\}, \{d,e\}, \{d,f\}, \{e,f\}$.

13. $P(14,4) = 24024$.

14. $\binom{6}{3}\binom{4}{2} = 120$.

15. $\binom{26}{2} + \binom{13}{2} = 403$.

16. $2^8 - [1 + 8 + \binom{8}{2}] = 219.$

17. $\binom{5}{2}\binom{7}{3} + \binom{3}{2}\binom{9}{3} - \binom{5}{2}\binom{3}{2} \cdot 4 = 482.$

18. $3^8 + 3^8 - 2^8 = 12866.$

19. $101 + 72 - 14 = 159.$

20. $10000 - (5000 + 2000 - 1000) = 4000.$

21. $13\binom{4}{3}12 \cdot 4 + 13 = 2509.$

22. $\frac{2}{36} = \frac{1}{18}.$

23. $p = \frac{\binom{50}{12}}{\binom{70}{12}} = \frac{2900135}{254154182} \approx .0114.$ So, approximately yes.

24. $1 - \frac{1+6+\binom{6}{2}}{2^6} = \frac{21}{32}.$

25. $\frac{4\binom{13}{4}}{\binom{52}{4}} = \frac{44}{4165}.$

26. $\frac{\binom{5}{3}+\binom{5}{4}+\binom{5}{5}}{2^5} = \frac{1}{2}.$

27. $\frac{\binom{10}{3}\binom{6}{3}\binom{8}{3}}{\binom{24}{9}} = \frac{8400}{81719} \approx .10.$

28. $\frac{1}{\binom{53}{6}} = \frac{1}{22957480}.$

29. $1 - \frac{\binom{38}{5}+5\binom{38}{4}}{\binom{43}{5}} = \frac{\binom{5}{2}\cdot\binom{38}{3}+\binom{5}{3}\binom{38}{2}+5\cdot 38+1}{\binom{43}{5}} = \frac{4361}{45838} \approx .0951.$

30. (a) $\frac{\frac{3\cdot\binom{4}{2}}{\binom{52}{2}}}{\frac{\binom{12}{2}}{\binom{52}{2}}} = \frac{3}{11}.$

(b) No. $P(E) \cdot P(F) = \frac{13\cdot\binom{4}{2}}{\binom{52}{2}} \cdot \frac{\binom{12}{2}}{\binom{52}{2}} \neq \frac{3\cdot\binom{4}{2}}{\binom{52}{2}} = P(E \cap F).$

31. (a) 83%, since $p = (.7)(.2) + (.8)(.3) + (.9)(.5) = .83.$
(b) $\frac{(.9)(.5)}{.83} = \frac{45}{83} \approx .5422.$

32. No.
There are $\binom{10}{6} = 210$ routes and $5 \cdot 48 = 240$ days.

33. $\binom{4}{2}\binom{6}{4} = 90.$

34. $\binom{4}{3}\binom{6}{3} + \binom{4}{4}\binom{6}{2} = 95.$

35. $\binom{4}{2}\binom{6}{4} = 90.$

36. $\binom{13}{2}\binom{4}{3}\binom{4}{2} = 1872.$

37. $\binom{13}{5} \cdot 4^5 - 10 \cdot 4^5 - \binom{13}{5} \cdot 4 + 40 = 1302540.$

38. $13\binom{48}{2} = 14664.$

39. $13\binom{4}{3}\binom{12}{3}4^3 = 732160.$

40. $\binom{13}{3}\binom{4}{2}^3 + \binom{13}{2}\binom{4}{2}^2\binom{11}{2} \cdot 4^2 = 2532816.$
The first summand counts three pair.

41. $13\binom{4}{3} \cdot 12\binom{4}{2} \cdot 11 \cdot 4 + \binom{13}{2}\binom{4}{3}^2 = 165984.$

42. $\underbrace{9\cdot}_{\text{non-Ace high}} 4 \cdot 46 + \underbrace{1\cdot}_{\text{Ace high}} 4 \cdot 47 = 1844$ or $10 \cdot 4 \cdot 47 - \underbrace{9 \cdot 4}_{\text{6-card straight}} = 1844.$

43. $4\binom{13}{5} \cdot 39 + 4\binom{13}{6} - 1844 = 205792.$

44. $\underbrace{10 \cdot 5\binom{4}{2} \cdot 4^4}_{\text{pair}} + \underbrace{9 \cdot 4^5 \cdot 7 \cdot 4 + 1 \cdot 4^5 \cdot 8 \cdot 4}_{\text{non-pair}} - 1844 = 365772.$

45. $13\binom{4}{2}\binom{12}{4} \cdot 4^4 - 10 \cdot 5 \cdot 4^4\binom{4}{2} - 4\binom{13}{5} \cdot 5 \cdot 3 + 40 \cdot 5 \cdot 3 = 9730740.$

46. 6608748.
See Exercises 38 through 45 and Table 2.1.
$\binom{52}{6} - [1844 + 14664 + 165984 + 205792 + 365772 + 732160 + 2532816 + 9730740] = 6608748.$

Exercise	Hand	Number Possible	Probability (to 8 places)
42	Straight-Flush	1844	.00009058
38	Four of a Kind	14664	.00071931
41	Full House	165984	.00815305
43	Flush	205792	.01010897
44	Straight	365772	.01796653
39	Three of a Kind	732160	.03596332
40	Two Pairs	2532816	.12441062
45	One Pair	9730740	.47796893
46	Nothing	6608748	.32461927
	Total	20358520	1

Table 2.1: Likelihood of Poker Hands from 6 cards

47. $\frac{236}{495} \approx .4768$. Daniel wins under any of the following conditions on his down cards: a King, a Queen, a Jack, a 10, two Aces, two 9's, two 2's, or a 2 and a 9. So

$$P(\text{Daniel wins}) = \frac{[\binom{45}{2} - \binom{32}{2}] + 3 + 1 + 3 + 3 \cdot 2}{\binom{45}{2}} = \frac{169}{330} \approx .5121.$$

48. $\binom{100+5-1}{100} = \binom{104}{100} = 4598126$.

49. $\frac{\binom{26+5-1}{5}}{26^5} = \frac{5481}{456976} \approx .0120$.

50. (a) $\frac{20!}{20} = 19!$. (b) 20. In each case, the first spin determines the way. (c) $P(40 \text{ or more}) = \frac{13}{20}$.

51. $\frac{8!}{24} = 1680$.

52. $\frac{\binom{22}{11}}{2} = 352716$.

53. $\frac{\binom{24}{8}\binom{16}{8}\binom{8}{4}\binom{4}{4}}{2 \cdot 2} = 165646455975$.

54. $30 + 40 + 25 - 3(10) = 65$.

55. $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{4} = 630$.

56. (a) $P(20, 4) = 116280$. (b) $P(20, 3) \cdot \binom{17}{3} = 4651200$.

57. (a) $\binom{10}{5} = 252$. (b) $\frac{2^{10} - 252}{2} = 386$.

58. $729 - \frac{729}{3} = 486$.

59. $2^8 + 8 \cdot 2^7 + \binom{8}{2} \cdot 2^6 = 3072$.

60. 36!. What matters is a number's position relative to 0 on the wheel.

61. $\binom{10}{5} = 252$.

62. $(769 + 588 - 45) - (76 + 58 - 4) = 1182$.

63. $\frac{\binom{8}{2}\binom{6}{2} + 8 \cdot 6 \cdot \binom{5}{2} + 5}{\binom{19}{4}} = \frac{905}{3876}$.

64. $\frac{1}{2}$. Odd and even sums are equally likely. See Exercise 41 from Section 6.4.

65. (a) $\frac{2\binom{26}{3}}{\binom{52}{3}} = \frac{4}{17}$. (b) $\frac{13\binom{4}{3}}{\binom{52}{3}} = \frac{1}{425}$.

66. $\frac{\binom{11}{3}\binom{39}{2} + \binom{11}{4}39 + \binom{11}{5} + 3\binom{13}{5}}{\binom{50}{5}} = \frac{69729}{1059380} \approx .0658$.

2.7 Chapter 7

Section 7.1

1. $3 \cdot 2^5 - 3 = 93$.

3. Let A_{bar} contain those missing a specified type of candy bar.
$|A_{\text{snickers}}{}^c \cap A_{\text{mounds}}{}^c \cap A_{\text{butterfingers}}{}^c| =$
$\binom{24}{5} - [\binom{18}{5} + \binom{14}{5} + \binom{16}{5}] + [\binom{6}{5} + \binom{10}{5} + \binom{8}{5}] = 27880$.

5. Let A_{fruit} contain those missing a certain type of fruit.
$|A_{\text{banana}} \cup A_{\text{apple}} \cup A_{\text{orange}}| = [\binom{16}{6} + \binom{15}{6} + \binom{13}{6}] - [\binom{6}{6} + \binom{7}{6} + \binom{9}{6}] = 14637$.

7. (a) $\frac{4 \cdot 3^6 - \binom{4}{2} \cdot 2^6 + \binom{4}{3}}{4^6} = \frac{2536}{4096} = \frac{317}{512} \approx .62$.
(b) $\frac{1+6\cdot 2+\binom{6}{2}}{4^6} = \frac{7}{1024} \approx .0068$.

9. Let A_{suit} contain those missing a specified suit.
$\frac{|A_\clubsuit{}^c \cap A_\diamondsuit{}^c \cap A_\heartsuit{}^c \cap A_\spadesuit{}^c|}{\binom{52}{7}} = \frac{\binom{52}{7} - 4\binom{39}{7} + \binom{4}{2}\binom{26}{7} - \binom{4}{3}\binom{13}{7}}{\binom{52}{7}} = \frac{63713}{111860} \approx .5696$.

11. Let A_{coin} contain those missing a specified coin type.
$\frac{|A_{\text{quarter}} \cup A_{\text{dime}} \cup A_{\text{nickel}} \cup A_{\text{penny}}|}{\binom{27}{5}} =$
$\frac{[\binom{19}{5}+\binom{22}{5}+\binom{23}{5}+\binom{17}{5}]-[\binom{14}{5}+\binom{15}{5}+\binom{9}{5}+\binom{18}{5}+\binom{12}{5}+\binom{13}{5}]+[\binom{8}{5}+\binom{5}{5}+\binom{10}{5}]}{\binom{27}{5}} =$
$\frac{271}{351} \approx .7721$.

13. $300 - (\frac{300}{2} + \frac{300}{3} + \frac{300}{5}) + (\frac{300}{2\cdot 3} + \frac{300}{2\cdot 5} + \frac{300}{3\cdot 5}) - \frac{300}{2\cdot 3\cdot 5} = 300 - 310 + 100 - 10 = 80$.

15. $1100 - (\frac{1100}{2} + \frac{1100}{5} + \frac{1100}{11}) + (\frac{1100}{2\cdot 5} + \frac{1100}{2\cdot 11} + \frac{1100}{5\cdot 11}) - \frac{1100}{2\cdot 5\cdot 11} = 1100 - 870 + 180 - 10 = 400$.

17. $2100 - (\frac{2100}{2} + \frac{2100}{3} + \frac{2100}{5} + \frac{2100}{7}) + (\frac{2100}{2\cdot 3} + \frac{2100}{2\cdot 5} + \frac{2100}{2\cdot 7} + \frac{2100}{3\cdot 5} + \frac{2100}{3\cdot 7} + \frac{2100}{5\cdot 7}) - (\frac{2100}{2\cdot 3\cdot 5} + \frac{2100}{2\cdot 3\cdot 7} + \frac{2100}{2\cdot 5\cdot 7} + \frac{2100}{3\cdot 5\cdot 7}) + \frac{2100}{2\cdot 3\cdot 5\cdot 7} = 480$.

19. *Proof.* Write n as a product of powers of primes $p_1^{k_1} \cdots p_m^{k_m}$. So

$$
\begin{aligned}
\phi(n) &= \phi(p_1^{k_1})\phi(p_2^{k_2}) \cdots \phi(p_m^{k_m}) \text{ by (7.2)} \\
&= p_1^{k_1}(1 - \frac{1}{p_1})p_2^{k_2}(1 - \frac{1}{p_2}) \cdots p_m^{k_m}(1 - \frac{1}{p_m}) \text{ by (7.1)} \\
&= p_1^{k_1}p_2^{k_2} \cdots p_m^{k_m}(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_m}) \text{ by commutativity} \\
&= n\prod_{p|n}(1 - \tfrac{1}{p}) \text{ by substitution}
\end{aligned}
$$

That is, (7.3) holds. □

21. *Proof.* Let p and q be the only prime divisors of n.
So $\phi(n) = n-(\frac{n}{p}+\frac{n}{q})+\frac{n}{pq} = n-\frac{n}{p}-\frac{n}{q}+\frac{n}{pq} = n(1-\frac{1}{p}-\frac{1}{q}+\frac{1}{pq}) = n(1-\frac{1}{p})(1-\frac{1}{q})$.
□
That is, let A_p contain the numbers in $\{1,2,\ldots,n\}$ that are divisible by p, and define A_q similarly. So $\phi(n) = |{A_p}^c \cap {A_q}^c|$.

23. $10500-(5250+3500+2100)+(1750+1050+700)-350 = 2800$.

25. $(\lfloor\frac{10000}{3}\rfloor+\lfloor\frac{10000}{13}\rfloor+\lfloor\frac{10000}{23}\rfloor+\lfloor\frac{10000}{43}\rfloor)-(\lfloor\frac{10000}{3\cdot13}\rfloor+\lfloor\frac{10000}{3\cdot23}\rfloor+\lfloor\frac{10000}{3\cdot43}\rfloor+\lfloor\frac{10000}{13\cdot23}\rfloor+\lfloor\frac{10000}{13\cdot43}\rfloor+\lfloor\frac{10000}{23\cdot43}\rfloor)+(\lfloor\frac{10000}{3\cdot13\cdot23}\rfloor+\lfloor\frac{10000}{3\cdot13\cdot43}\rfloor+\lfloor\frac{10000}{3\cdot23\cdot43}\rfloor+\lfloor\frac{10000}{13\cdot23\cdot43}\rfloor)-\lfloor\frac{10000}{3\cdot13\cdot23\cdot43}\rfloor = 4768-537+19-0=4250$.

27. $\frac{53}{144} = .3680\bar{5}$ agrees with $\frac{1}{e}$ to 2 decimal places.
$1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720} = \frac{53}{144} = .3680\bar{5}$, which agrees with $\frac{1}{e}$ to 2 decimal places.

29. (a) $1-\frac{11}{30} = \frac{19}{30} \approx .633$.
(b) $\frac{1}{5!} = \frac{1}{120}$.
(c) $\frac{\binom{5}{3}+1}{120} = \frac{11}{120}$.

31. .36787944.
See the paragraph following Example 7.3.

33. Corollary: $|{A_1}^c \cap {A_2}^c \cap \cdots \cap {A_n}^c| = \sum_{i=0}^{n}(-1)^i S_i$.
$|{A_1}^c \cap {A_2}^c \cap \cdots \cap {A_n}^c| = |(A_1 \cup A_2 \cup \cdots \cup A_n)^c| = |\mathcal{U}| - |A_1 \cup A_2 \cup \cdots \cup A_n| = |\mathcal{U}| - \sum_{i=1}^{n}(-1)^{i-1}S_i = |S_0| + \sum_{i=1}^{n}(-1)^i S_i = \sum_{i=0}^{n}(-1)^i S_i$.

35. $|A_1\cap A_2\cap A_3|+|A_1\cap A_2\cap A_4|+|A_1\cap A_2\cap A_5|+|A_1\cap A_3\cap A_4|+|A_1\cap A_3\cap A_5|+|A_1\cap A_4\cap A_5|+|A_2\cap A_3\cap A_4|+|A_2\cap A_3\cap A_5|+|A_2\cap A_4\cap A_5|+|A_3\cap A_4\cap A_5|$.

37. $|A_1 \cap A_2 \cap A_3| = |\mathcal{U}| - (|{A_1}^c| + |{A_2}^c| + |{A_3}^c|)+$
$(|{A_1}^c \cap {A_2}^c| + |{A_1}^c \cap {A_3}^c| + |{A_2}^c \cap {A_3}^c|) - |{A_1}^c \cap {A_2}^c \cap {A_3}^c|$.
That is, we use the fact that ${A_1}^{cc} = A_i$.

39. $5^{10} - 5\cdot 4^{10} + \binom{5}{2}\cdot 3^{10} + \binom{5}{3}\cdot 2^{10} + \binom{5}{4}\cdot 1^{10} = 5^{10} - 4662625 = 5103000$.

41. Let $p(n) = \frac{6^n - 6\cdot 5^n + \binom{6}{2}\cdot 4^n - \binom{6}{3}\cdot 3^n + \binom{6}{4}\cdot 2^n - \binom{6}{5}}{6^n}$.
(a) $p(6) = \frac{5}{324} \approx .0153$.
(b) $p(10) = \frac{38045}{139968} \approx .2718$.
(c) Use 13 dice.
Note that $p(12) \approx .4378$ and $p(13) \approx .5139$.

43. $n^4 - 4n^3 + 6n^2 - (3n+n^2) = n^4 - 4n^3 + 5n^2 - 2n = n(n-1)^2(n-2)$.

45. $n^5 - 6n^4 + 15n^3 - (n^3 + 19n^2) + (4n^2 + 11n) - 6n + n =$
$n^5 - 6n^4 + 14n^3 - 15n^2 + 6n = n(n-1)(n-2)(n^2 - 3n + 3)$.

Section 7.2

1. $\frac{7!}{3!2!2!} = 210$.

3. $\frac{15!}{2!3!6!4!} = 6306300$.

5. $\binom{12}{4,3,5} = 27720$.

7. (a) $\binom{15}{5,3,4,3} = 12612600$. (b) $\binom{14}{4,3,4,3} + \binom{14}{5,3,3,3} = 7567560$.
(c) $\binom{14}{4,3,4,3} = 4204200$.

9. $\frac{\binom{12}{4,4,4}}{3!} = 5775$.

11. $\frac{\binom{22}{5,5,6,6}}{2^2} = 37642556952$.

13. (a) $\frac{\binom{16}{4,4,4,4}}{4!} \cdot 3^4 = 212837625$.
(b) $\binom{16}{4,4,4,4} \cdot 3^4 = 5108103000$.

15. (a) $\binom{18}{6,6,6}(5!)^3 = 29640619008000$.
(b) $\frac{\binom{18}{6,6,6}}{3!}(5!)^3 = 4940103168000$.
(c) $\frac{\binom{18}{6,6,6}}{3!} = 2858856$.

17. $(0,0,3), (0,1,2), (1,0,2), (0,2,1), (1,1,1), (2,0,1), (0,3,0), (1,2,0),$
$(2,1,0), (3,0,0)$. Note that there are $\binom{3+3-1}{3} = 10$ of them. See Remark 7.1.

19. $x^2 - 2xy + 2xz + y^2 - 2yz + z^2$.
The sum in $(x + (-y) + z)^2 = \sum_T \binom{2}{k_1,k_2,k_3} x^{k_1}(-y)^{k_2} z^{k_3}$ is indexed over
$T = \{(k_1,k_2,k_3) \ : \ k_1,k_2,k_3 \in \mathbb{N} \text{ and } k_1 + k_2 + k_3 = 2\} =$
$\{(2,0,0), (1,1,0), (1,0,1), (0,2,0), (0,1,1), (0,0,2)\}$.
So $(x + (-y) + z)^2 =$
$\binom{2}{2,0,0}x^2(-y)^0z^0 + \binom{2}{1,1,0}x^1(-y)^1z^0 + \binom{2}{1,0,1}x^1(-y)^0z^1 + \binom{2}{0,2,0}x^0(-y)^2z^0$
$+\binom{2}{0,1,1}x^0(-y)^1z^1 + \binom{2}{0,0,2}x^0(-y)^0z^2 = x^2 - 2xy + 2xz + y^2 - 2yz + z^2$.

21. $x^4 + 4x^3y + 4x^3 + 6x^2y^2 + 12x^2y + 6x^2 + 4xy^3 + 12xy^2 + 12xy + 4x + y^4 +$
$4y^3 + 6y^2 + 4y + 1$.
The sum in $(x + y + 1)^4 = \sum_T \binom{n}{k_1,k_2,k_3} x^{k_1} y^{k_2} 1^{k_3} = \sum_T \binom{n}{k_1,k_2,k_3} x^{k_1} y^{k_2}$
is indexed over $T = \{(k_1,k_2,k_3) \ : \ k_1,k_2,k_3 \in \mathbb{N} \text{ and } k_1 + k_2 + k_3 = 4\} =$
$\{(4,0,0), (3,1,0), (0,3,1), (2,2,0), (2,1,1), (2,0,2), (1,3,0), (1,2,1),$
$(1,1,2), (1,0,3), (0,4,0), (0,3,1), (0,2,2), (0,1,3), (0,0,4)\}$.

23. $4x^2+4xy-4xz+y^2-2yz+z^2$.
The sum in $((2x)+y+(-z))^2=\sum_T\binom{2}{k_1,k_2,k_3}(2x)^{k_1}y^{k_2}(-z)^{k_3}$ is indexed over $T=\{(k_1,k_2,k_3)\ :\ k_1,k_2,k_3\in\mathbb{N}$ and $k_1+k_2+k_3=2\}=$
$\{(2,0,0),(1,1,0),(1,0,1),(0,2,0),(0,1,1),(0,0,2)\}$.
So $((2x)+y+(-z))^2=4x^2+4xy-4xz+y^2-2yz+z^2$.

25. $(2w)^2+2(2w)x+2(2w)y+2(2w)(2z)+x^2+2xy+2x(2z)+y^2+2y(2z)+(2z)^2=$ $4w^2+4wx+4wy+8wz+x^2+2xy+4xz+y^2+4yz+4z^2$.

27. $\binom{300}{100,50,40,60,20,30}$.

29. 0, since $10+20+30\neq 80$.

31. $-\binom{100}{25,10,40,25}$. The relevant term is $\binom{100}{25,10,40,25}x^{25}(-y)^{10}z^{40}(-w)^{25}=$ $\binom{100}{25,10,40,25}x^{25}(-1)^{10}y^{10}z^{40}(-1)^{25}w^{25}=\binom{100}{25,10,40,25}x^{25}1y^{10}z^{40}(-1)w^{25}=$ $-\binom{100}{25,10,40,25}x^{25}y^{10}z^{40}w^{25}$.

33. $-2^5\binom{14}{4,2,5,3}=-80720640$. The relevant term is $\binom{14}{4,2,5,3}x^4y^2(-2z)^5w^7=\binom{14}{4,2,5,3}x^4y^2(-2)^5z^5w^7=$ $(-2)^5\binom{14}{4,2,5,3}x^4y^2z^5w^7=-80720640x^4y^2z^5w^7$.

35. $\binom{14+4-1}{14}=\binom{17}{14}=680$.

37. There are $n!$ ways to order the n items. We shall understand that the first k_1 go into category 1, the next k_2 go into category 2, and so forth. Since, within each category, order is not important, we must divide $n!$ by $k_1!k_2!\cdots k_m!$, the number of different orderings leaving items within their categories. We get $\frac{n!}{k_1!k_2!\cdots k_m!}$, which is $\binom{n}{k_1,k_2,\ldots,k_m}$.

39. Consider $3^n=(1+1+1)^n$.
We get

$$\sum_{\substack{0\le k_1,k_2,k_3\le n\\ k_1+k_2+k_3=n}}\binom{n}{k_1,k_2,k_3}\cdot 1^{k_1}\cdot 1^{k_2}\cdot 1^{k_3}.$$

Of course, $1^{k_1}\cdot 1^{k_2}\cdot 1^{k_3}=1$, in each term.

41. Consider $6^n=(3+2+1)^n$.
We get

$$\sum_{\substack{0\le k_1,k_2,k_3\le n\\ k_1+k_2+k_3=n}}\binom{n}{k_1,k_2,k_3}\cdot 3^{k_1}\cdot 2^{k_2}\cdot 1^{k_3}=\sum_{\substack{0\le k_1,k_2\le n\\ k_1+k_2\le n}}\binom{n}{k_1}\binom{n-k_1}{k_2}\cdot 3^{k_1}\cdot 2^{k_2}.$$

Section 7.3

1. $1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{i=0}^{\infty}(i+1)x^i$.
Notice that x^i occurs in the product $(1+x+x^2+x^3+\cdots)(1+x+x^2+x^3+\cdots)$ via products of the form $x^j x^{i-j}$, where $j = 0, 1, \ldots, i$. Since there are $i+1$ such products, the coefficient of x^i is $(i+1)$.

3. $1 + x^2 + x^4 + x^6 + \cdots = \sum_{i=0}^{\infty} x^{2i}$.
Notice that x^k occurs in the product $(1+x+x^2+x^3+\cdots)(1-x+x^2-x^3+\cdots) = (1+x+x^2+x^3+\cdots)(1+(-x)+(-x)^2+(-x)^3+\cdots)$ via products of the form $x^j(-x)^{k-j} = x^j(-1)^{k-j}x^{k-j} = (-1)^{k-j}x^j x^{k-j} = (-1)^{k-j}x^k$, where $j = 0, 1, \ldots, k$. That is, the coefficient of x^k is the alternating sum $\sum_{j=0}^{k}(-1)^{k-j}$. If k is odd, then this alternating sum is 0. If $k = 2i$ is even, then this alternating sum is 1.

5. $1\cdot 1 + 1\cdot x^2 + 1\cdot x^4 + x\cdot 1 + x\cdot x^2 + x\cdot x^4 + x^2\cdot 1 + x^2\cdot x^2 + x^2\cdot x^4 =$
$1 + x^2 + x^4 + x + x^3 + x^5 + x^2 + x^4 + x^6 = 1 + x + 2x^2 + x^3 + 2x^4 + x^5 + x^6$.

7. $(1\cdot x + 1\cdot x^2 + x\cdot x + x\cdot x^2)(1+x^3) =$
$(x+x^2+x^2+x^3)(1+x^3) = x\cdot 1 + x\cdot x^3 + x^2\cdot 1 + x^2\cdot x^3 + x^2\cdot 1 + x^2\cdot x^3 + x^3\cdot 1 + x^3\cdot x^3 =$
$x + x^4 + x^2 + x^5 + x^2 + x^5 + x^3 + x^6 = x + 2x^2 + x^3 + x^4 + 2x^5 + x^6$.

9. $c_0 = 1, c_5 = 1, c_{10} = 2, c_{15} = 2, c_{20} = 2, c_{25} = 1, c_{30} = 1$, and, otherwise, $c_i = 0$.
Note that $(1+x^{10})(1+x^5+x^{10}+x^{15}+x^{20}) = 1+x^5+2x^{10}+2x^{15}+2x^{20}+x^{25}+x^{30}$.

11.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
c_i	1	1	1	1	1	2	1	1	1	1	2	1	1	1	1	2

i	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
c_i	1	1	1	1	2	1	1	1	1	2	1	1	1	1	1

That is, c_i is the coefficient of x^i in the expansion of
$(1 + x^{10} + x^{20})(1 + x^5)(1 + x + x^2 + x^3 + x^4 + x^5)$.

13.

i	0	5	10	15	20	25	30	35	40	45	50	55	60
c_i	1	1	2	2	2	3	2	3	2	2	2	1	1

That is, c_i is the coefficient of x^i in the expansion of
$(1 + x^{25})(1 + x^{10} + x^{20})(1 + x^5 + x^{10} + x^{15})$.

15. 11.
$(1 + x + x^2 + x^3)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2) = \cdots + 11x^5 + \cdots$.

17. 29.
$(1+x+x^2+\cdots+x^6)(1+x+x^2+\cdots+x^6)(1+x+x^2+x^3+x^4) = \cdots+29x^8+\cdots$.

19. (a) 18.
$(x + x^2 + \cdots + x^{10})(x + x^2 + \cdots + x^8)(x + x^2 + x^3 + x^4) = \cdots + 18x^8 + \cdots$.
(b) $\frac{\binom{22}{8}-\binom{12}{8}-\binom{14}{8}-\binom{18}{8}+\binom{10}{8}+\binom{8}{8}}{\binom{22}{8}} = \frac{27256}{31977} \approx .8524$.
(c) $\frac{\binom{10}{8}+\binom{8}{8}}{\binom{22}{8}} = \frac{23}{159885} \approx .00014$.

21. $c_i = \binom{n}{i}a^i$, for $i = 0, 1, \ldots, n$.
By the Binomial Theorem, $(1+ax)^n = \sum_{i=0}^{n} \binom{n}{i} a^i x^i$. If $a \in \mathbb{Z}^+$, then $c_i = \binom{n}{i}a^i$ is the number of n-digit base $(a + 1)$ sequences with exactly i zeros.

23. $\forall\, i \geq 0, c_i = \binom{i+3}{i}$. The generating function is $(1 + x + x^2 + \cdots)^4$. Thus, c_i counts the number of ways to i identical items in 4 distinct categories.

25. $c_{100} = \binom{100+50-1}{100} = \binom{149}{100}$.

27. $\binom{149}{100} + \binom{148}{99}$. It is the coefficient of x^{100} in $\frac{1}{(1-x)^{50}}$ plus the coefficient of x^{100} in $\frac{x}{(1-x)^{50}}$. Note that the latter is the same as the coefficient of x^{99} in $\frac{1}{(1-x)^{50}}$.

29. $\binom{119}{40} + 2\binom{118}{39} + \binom{117}{38}$.

31. 6175. Use $g(x) = (1 + x + x^2 + x^3 + x^4)^{13} = \cdots + 6175x^5 + \cdots$.
What matters here is how many (anywhere from 0 to 4) cards of each denomination are in a hand.

33. 15805. Here, $(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + \cdots)^3 = \frac{1+x+x^2+x^3+x^4}{(1-x)^3}$.
The coefficient of x^{80} is $\binom{82}{80} + \binom{81}{79} + \binom{80}{78} + \binom{79}{77} + \binom{78}{76} = 15805$.
Or,
$\frac{1+x+x^2+x^3+x^4}{(1-x)^3} = \frac{1-x^5}{1-x} \cdot \frac{1}{(1-x)^3} = \frac{1-x^5}{(1-x)^4}$.
The coefficient of x^{80} is $\binom{83}{80} - \binom{78}{75} = 15805$.

35. (a) 351.
$(1 + x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + \cdots)^2 = \frac{1-x^6}{1-x} \cdot \frac{1}{(1-x)^2} = \frac{1-x^6}{(1-x)^3}$.
The coefficient of x^{60} is $\binom{62}{60} - \binom{56}{54} = 351$.
(b) 56, since from 0 to 55 white might be purchased.
(c) 3, since 0, 2, or 4 bottles of champagne would be purchased.

37. 81.
$(\frac{1-x^9}{1-x})^2 \cdot \frac{1}{1-x} = \frac{1-2x^9+x^{18}}{(1-x)^3}$. The coefficient of x^{40} is $\binom{42}{40} - 2\binom{33}{31} + \binom{24}{22} = 81$.

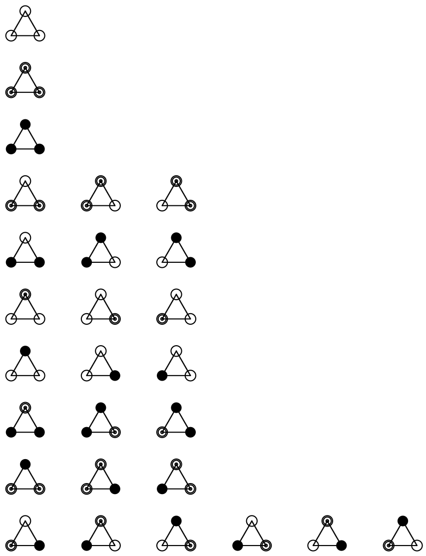
39. 1602.
Use $g(x) = (1 + x + x^2 + \cdots)^3(1 + x)^3 = \frac{1}{(1-x)^3}(1 + x)^3$. The coefficient of x^{20} is $\binom{19}{17}\binom{3}{3} + \binom{20}{18}\binom{3}{2} + \binom{21}{19}\binom{3}{1} + \binom{22}{20}\binom{3}{0} = 1602$.

41. When there are finitely many items, say n, in general, the number of ways of selecting $n-i$ is the same as the number of ways of de-selecting i.
Recall that $\binom{n}{i} = \binom{n}{n-i}$.

43. $\forall\, i \geq 0, c_i = (-1)^i \binom{i+n-1}{i}$, since $\frac{1}{(1+x)^n} = \frac{1}{(1-(-x))^n}$.

Section 7.4

1.



3.

$\circ$	r_0	r_1	r_2	r_3
r_0	r_0	r_1	r_2	r_3
r_1	r_1	r_2	r_3	r_0
r_2	r_2	r_3	r_0	r_1
r_3	r_3	r_0	r_1	r_2

5. (a) f_1.
(b) $r_2 f_4 = f_3$.
(c) No. $r_1 f_2 = f_4 \neq f_3 = f_2 r_1$.

$\circ$	r_0	r_1	r_2	r_3	f_1	f_2	f_3	f_4
r_0	r_0	r_1	r_2	r_3	f_1	f_2	f_3	f_4
r_1	r_1	r_2	r_3	r_0	f_4	f_3	f_1	f_2
r_2	r_2	r_3	r_0	r_1	f_2	f_1	f_4	f_3
r_3	r_3	r_0	r_1	r_2	f_3	f_4	f_2	f_1
f_1	f_1	f_3	f_2	f_4	r_0	r_2	r_1	r_3
f_2	f_2	f_4	f_1	f_3	r_2	r_0	r_3	r_1
f_3	f_3	f_2	f_4	f_1	r_3	r_1	r_0	r_2
f_4	f_4	f_1	f_3	f_2	r_1	r_3	r_2	r_0

7. 6. $G = D_4$ and $N = \frac{1}{8}[2^4 + 2 + 2 + 2^2 + 2^3 + 2^3 + 2^2 + 2^2] = 6$.

9. 10. For each $g \in D_6$, we need to count $|\text{Fix}(g)|$.
Case 1: $g = r_0$ the identity.
Since r_0 moves nothing, each network is fixed. So $|\text{Fix}(r_0)| = 3^3 = 27$.
Case 2: $g = r_i$ for $i = 1$ or 2.
The only networks unchanged by such a rotation are those in which each computer has the same type (from 3 choices). So $|\text{Fix}(r_i)| = 3$.
Case 3: $g = f_i$ for $i = 1$, 2, or 3.
Such a flip fixes one computer and switches the other two. The networks unchanged by this can have any type of computer at the fixed computer, but the two that are switched must have the same type. So $|\text{Fix}(f_i)| = 3^2 = 9$.
Let N be the number of orbits of X. Invoking Theorem 7.8 we get

$$\begin{aligned} N &= \frac{1}{|G|}\sum_{g\in G}|\text{Fix}(g)| \\ &= \frac{1}{6}(|\text{Fix}(r_0)|+|\text{Fix}(r_1)|+|\text{Fix}(r_2)|+|\text{Fix}(f_1)|+|\text{Fix}(f_2)|+|\text{Fix}(f_3)|) \\ &= \frac{1}{6}(27+3+3+9+9+9) = \frac{60}{6} = 10. \end{aligned}$$

Thus, there are 10 different network types.

11. (a) $G = Z_2$ and $N = \frac{1}{2}[6^3 + 6^2] = 126$.
(b) $126 - 6 - 6 \cdot 5 = 90$.
(c) $90 - 6 \cdot 5 = 60$ or $\binom{6}{3} \cdot 3 = 60$.

13. $N = \frac{1}{24}[1(3^6) + 6(3^3) + 3(3^4) + 6(3^3) + 8(3^2)] = 57$.

15. 280. Let M be the number of ways to color the non-base faces and $G = Z_4$. So $M = \frac{1}{4}[4^4 + 4 + 4 + 4^2] = 70$. Since there are then 4 ways to color the base, $N = 4(70) = 280$.

17. $N = \frac{1}{12}[4^4 + 8(4^2) + 3(4^2)] = 36$.

19. 834.
$G = Z_8$ and $N = \frac{1}{8}[3^8 + 3 + 3^2 + 3 + 3^4 + 3 + 3^2 + 3] = 834$.

21. 34.
$G = Z_4$ and $N = \frac{1}{4}[\binom{9}{5} + 2 + \binom{4}{2} + 2] = 34$.

23. 8.
$G = D_8$ and $N = \frac{1}{16}[\binom{8}{4} + 0 + 2 + 0 + \binom{4}{2} + 0 + 2 + 0 + 4\binom{4}{2} + 4\binom{4}{2}] = 8$. They are 11112222, 11121222, 11122122, 11211222, 11221122, 11212122, 11212212, 12121212.

25. (a) $G = D_5$ and $N = \frac{1}{10}[3^5 + 3 + 3 + 3 + 3 + 5(3^3)] = 39$.
(b) $G = Z_2$ and $N = \frac{1}{2}[3^5 + 3^3] = 135$. (c) $3 + 3 \cdot \frac{1}{2}[(2^5 - 2) + (2^3 - 2)] = 57$.

27. 165. $G = Z_4$ and $N = \frac{1}{4}[5^4 + 5 + 5^2 + 5] = 165$.

29. $5 \cdot (\frac{1}{3}[5^3 + 5 + 5])(\frac{1}{2}[5^2 + 5]) = 3375$.

31. 1135. The number of ways to color each of the three outer pairs is $\frac{1}{2}[5^2+5] = 15$. So $N = \frac{1}{3}[15^3 + 15 + 15] = 1135$.

33. *Proof.* Let $g \in G$. ($\subseteq$) Suppose $x \in \mathrm{Fix}(g)$. So $gx = x$. Hence, $g^{-1}gx = g^{-1}x$. So $x = g^{-1}x$. Thus, $x \in \mathrm{Fix}(g^{-1})$. ($\supseteq$) Similar. $\square$

35. $N = \frac{1}{24}[\binom{12}{4,4,4} + 8(0) + 2(3!) + \binom{6}{2,2,2} + 6\binom{6}{2,2,2} + 6\binom{6}{2,2,2}] = 1493$.

37. 2420.
$|\mathrm{Fix}(r_0)| = \binom{12}{6} + 2 \cdot 12\binom{11}{5} + \binom{12}{2}[2\binom{10}{4} + \binom{10}{5}] = 56364$,
$|\mathrm{Fix}(r_1)| = |\mathrm{Fix}(r_5)| = |\mathrm{Fix}(r_7)| = |\mathrm{Fix}(r_{11})| = 0$,
$|\mathrm{Fix}(r_2)| = |\mathrm{Fix}(r_{10})| = 2$,
$|\mathrm{Fix}(r_3)| = |\mathrm{Fix}(r_9)| = 0$,
$|\mathrm{Fix}(r_4)| = |\mathrm{Fix}(r_8)| = \binom{4}{2} = 6$,
$|\mathrm{Fix}(r_6)| = \binom{6}{3} + 6[2\binom{5}{2}] = 140$,
$|\mathrm{Fix}(f_1)| = 2\binom{5}{2} + 5[2\binom{4}{3} + 2\binom{4}{2} + 2 \cdot 2\binom{5}{2} + \binom{6}{3}] = 180$, and
$|\mathrm{Fix}(f_5)| = 6\binom{5}{2} + \binom{6}{3} = 80$.
So $N = \frac{1}{24}[56364 + 4(0) + 2(2) + 2(0) + 2(6) + 140 + 6(180) + 6(80)] = 2420$.

39. $N = \frac{1}{24}[(\binom{6}{4} + \binom{6}{3} + \binom{6}{4}) + 6(2) + 3(2 + 2^3) + 6(2 \cdot 3) + 8(2)] = 6$.

Section 7.5

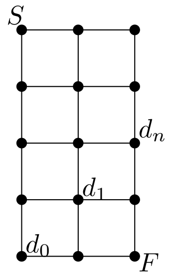
1. Each subset of $\{1, 2, \ldots, n\}$ of size k can be uniquely represented by a binary sequence of length n with k ones, as in Example 7.20(a). For example, $\{1, 2, 6, 8, \ldots\}$ is represented by $11000101\ldots$. For each i, group together those that start with i ones followed by a zero

$$\underbrace{11\cdots1}_{i \text{ times}}0\ldots.$$

The size of this group is $\binom{n-(i+1)}{k-i} = \binom{n-i-1}{k-i}$. The sum of the sizes of the groups $\sum_{i=0}^{k}\binom{n-i-1}{k-i}$ must be the total number $\binom{n}{k}$ of relevant sequences.

3. The $\binom{n}{k}$ subsets of $\{1, 2, \ldots, n\}$ of size k can be broken into three groups.
(i) The $\binom{n-2}{k-2}$ that contain both 1 and 2.
(ii) The $2\binom{n-2}{k-1}$ that contain exactly one of 1 or 2.
(iii) The $\binom{n-2}{k}$ that contain neither 1 nor 2.

5. Of the $\binom{3n}{n}$ paths from $S = (0, 2n)$ to $F = (n, 0)$ in the $(2n + 1)$ by $(n + 1)$ rectangular grid of points $([0, n] \times [0, 2n]) \cap (\mathbb{Z} \times \mathbb{Z})$,



for each $0 \le i \le n$, the number that pass through $d_i = (i, i)$ is $\binom{2n}{i}\binom{n}{n-i}$. So, we have $\binom{3n}{n} = \sum_{i=0}^{n} \binom{2n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{2n}{i}\binom{n}{i}$.

7. A Canadian doubles tournament that starts with 3^n players will ultimately have $3^n - 1$ losers (and one champion). For each $1 \le k \le n$, round k has 3^{n-k+1} competitors and its completion leaves behind $\frac{2}{3}3^{n-k+1} = 2 \cdot 3^{n-k}$ losers. Thus, the total number of losers is $\sum_{k=1}^{n} 2 \cdot 3^{n-k} = 2\sum_{k=0}^{n-1} 3^k$.

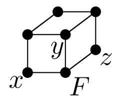
9. Let $\mathcal{U}$ be the set of base-3 sequences of length n, and let A be the subset of those that contain at least one 2. For each $1 \le j \le n$, let A_j be the subset of those that have a 2 in position j. So $A = \cup_{i=1}^{n} A_i$. First, observe that $|A| = |\mathcal{U}| - |A^c| = 3^n - 2^n$. For each $1 \le j_1 < j_2 < \cdots < j_i \le n$, $|A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_i}| = 3^{n-i}$. So, for each $1 \le i \le n$, $S_i = \binom{n}{i}3^{n-i}$. By the Principle of Inclusion-Exclusion, $|A| = \sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}3^{n-i}$.

11. Let $n \ge 1$. For each $1 \le k \le n$, there are $\binom{n}{k}$ choices for a team of size k and then k choices for its captain (one member of the team). In sum, there are $\sum_{k=1}^{n} \binom{n}{k}k$ possible teams with a specified captain. All together, there are n choices for a team captain and then 2^{n-1} choices for the remaining team members. Hence, there are $n2^{n-1}$ possible teams with a specified captain.

13. Assume that you are one of $2n$ people to be split into 2 teams of size n. The number of ways to split $2n$ people into 2 teams of size n is $\frac{\binom{2n}{n}}{2}$. The division by 2 strips away the implied ordering of the teams in the computation $\binom{2n}{n}$. On the other hand, the number of ways for you to pick your $n - 1$ teammates is $\binom{2n-1}{n-1}$.

15. The 3^n base-3 sequences of length n can be partitioned according to triples (k_0, k_1, k_2), where k_0 is the number of zeros, k_1 is the number of ones, and k_2 is the number of twos in the sequence. Of course, there are $\binom{n}{k_0,k_1,k_2}$ sequences that correspond to (k_0, k_1, k_2). Also note that it must be the case that $0 \le k_0, k_1, k_2 \le n$ and $k_0 + k_1 + k_2 = n$.

17. (a) $\binom{a+b+c}{a,b,c}$, since we count the number of $(a+b+c)$-step paths containing a right-, b downward-, and c frontward-steps.
(b) Consider an $(i+1)$ by $(j+1)$ by $(k+1)$ rectangular grid of points. In a path from S to F, the final position before reaching F must be exactly one of the 3 pictured points x, y, or z.



By part (a), the number of paths to x is $\binom{n-1}{i-1,j,k}$, the number to y is $\binom{n-1}{i,j-1,k}$, and the number to z is $\binom{n-1}{i,j,k-1}$. Hence, the sum of these three values must be the total number of paths to F, namely $\binom{n}{i,j,k}$.

19. The identity permutation of $\{1, 2, \ldots, n\}$ moves none of its elements. The size of the set P of nonidentity permutations of $\{1, 2, \ldots, n\}$ is $n! - 1$. For each $1 \le i \le n-1$, let P_i be the set of non-identity permutations for which $i+1$ is the largest position moved. Since, under these conditions, there are i positions to which $i+1$ may be moved, and then $i!$ ways to place $\{1, 2, \ldots, i\}$, we see that $|P_i| = i \cdot i!$. Since $P = P_1 \cup P_2 \cup \cdots \cup P_{n-1}$ is a disjoint union, its cardinality must be $\sum_{i=1}^{n-1}(i \cdot i!)$.

21. If a fair coin is tossed n times, then the probability that at least one head will occur is $1 - \frac{1}{2^n}$. For each $1 \le k \le n$, the probability that the first head occurs on toss k is $(\frac{1}{2})^{k-1}(\frac{1}{2}) = \frac{1}{2^k}$. Hence, the total probability is $\sum_{k=1}^{n} \frac{1}{2^k}$. This must therefore equal $1 - \frac{1}{2^n}$.

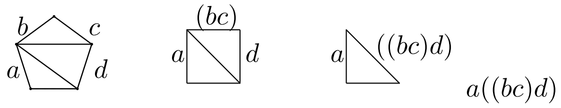
23. Assume that you are one of the 6 people. Of the other 5 people, there must be either 3 whom you have met before or 3 whom you have not met before. (That follows from the Pigeon Hole Principle.) Assume that there are 3 whom you have met before. If two of them have met each other before, then you and those two are a set of 3 who have met each other before. Otherwise, those 3 are a set who have never met each other before. The case in which there are 3 whom you have not met before is handled similarly.

25. 2.

We see that there are 2 ways to triangulate a square.

27. (a) List the $n-1$ terms on the non-base sides of the n-gon. Parentheses go around any pair that sits on a common triangle. Collapse each such triangle to its interior side, write the resulting product from the exterior sides on that

interior side, and regard it as a single term. Now repeat this process on the resulting smaller polygon. For example,



This establishes a one-to-one correspondence between triangulations of the n-gon and parenthesizations of a product of $n-1$ terms.
(b) According to Exercise 45 from Section 4.3, $C_m = \frac{1}{m+1}\binom{2m}{m}$ counts the number of ways of parenthesizing a product consisting of $m+1$ factors. Here, we use $m+1 = n-1$. So $m = n-2$, and $T_n = C_{n-2} = \frac{1}{n-2+1}\binom{2(n-2)}{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$.

29. The coins are on squares of the same color. That leaves 30 squares of that color and 32 of the other color. Since each domino covers one square of each color, they cannot be used to fill the rest of the board. That is, at best, after placing 30 dominos, there will be 2 squares of the other color left. No one domino can then cover those 2 (necessarily nonadjacent) squares of the same color.

31. The first re-deal leaves the selected card within the first 3 rows. This holds, since there are nine cards in the column containing the selected card, and these cards will then fill up three rows. The second re-deal leaves the card in the first row. This holds since, the card must be among the first three cards in its column. Once we know the card is in the first row, the column specifies its location.

Review

1. 92903176.
Let A_i contain those plates without i's.
So $|A_0 \cup A_3 \cup A_6 \cup A_9| = 4 \cdot 9^8 - 6 \cdot 8^8 + 4 \cdot 7^8 - 6^8 = 92903176$.

2. $9100 - (4550+1820+1300+700) + (910+650+350+260+140+100) - (130+70+50+20) + 10 = 2880$.

3. $(\lfloor \frac{4000}{7} \rfloor + \lfloor \frac{4000}{11} \rfloor + \lfloor \frac{4000}{13} \rfloor) - (\lfloor \frac{4000}{77} \rfloor + \lfloor \frac{4000}{91} \rfloor + \lfloor \frac{4000}{143} \rfloor) + \lfloor \frac{4000}{1001} \rfloor =$
$1241 - 121 + 3 = 1123$.

4. $p = \frac{2197}{8330} \approx .2637$.
Let A_{suit} contain those missing a specified suit. $|A_\clubsuit{}^c \cap A_\diamondsuit{}^c \cap A_\heartsuit{}^c \cap A_\spadesuit{}^c| =$
$\binom{52}{5} - 4\binom{39}{5} + 6\binom{26}{5} - 4\binom{13}{5} = 685464$. So $p = \frac{685464}{\binom{52}{5}} = \frac{2197}{8330} \approx .2637$.

5. $p = \frac{887}{2907} \approx .3051$.
Let A_{color} contain those missing a specified color. Containing at most two colors is the same as excluding at least one color. $|A_{\text{red}} \cup A_{\text{white}} \cup A_{\text{blue}}| =$
$[\binom{15}{5} + \binom{14}{5} + \binom{13}{5}] - [\binom{6}{5} + \binom{7}{5} + \binom{8}{5}] = 6209$. So $p = \frac{6209}{\binom{21}{5}} = \frac{887}{2907} \approx .3051$.

6. 31150. Let A_{fruit} contain those missing a specified type of fruit.
$|A_{\text{apple}}{}^c \cap A_{\text{banana}}{}^c \cap A_{\text{orange}}{}^c| =$
$\binom{20}{6} - [\binom{12}{6} + \binom{15}{6} + \binom{13}{6}] + [\binom{8}{6} + \binom{7}{6}] = 38760 - 7645 + 35 = 31150.$

7. $\sum_{i=0}^{4} \frac{(-1)^i}{i!} = \frac{3}{8}$.

8. $\frac{6!}{2!1!3!} = 60$.

9. $\binom{10}{1,2,5,2} = 7560$.

10. $\binom{8}{3,3,2} \cdot 8^2 = 35840$.

11. (a) $\binom{18}{6,5,4,3} = 514594080$. (b) $\frac{\binom{18}{6,6,6}}{3!} = 2858856$. (c) $\frac{\binom{18}{4,4,5,5}}{2 \cdot 2} = 192972780$.

12. $x^2 + 2xy + 2xz + 2xw + y^2 + 2yz + 2yw + z^2 + 2zw + w^2$.
The sum in $(x + y + z + w)^2 = \sum_T \binom{2}{k_1,k_2,k_3,k_4} x^{k_1} y^{k_2} z^{k_3} w^{k_4}$ is indexed over
$T = \{(k_1, k_2, k_3, k_4) \ : \ k_1, k_2, k_3, k_4 \in \mathbb{N} \text{ and } k_1 + k_2 + k_3 + k_4 = 2\} =$
$\{(2, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 2, 0, 0), (0, 1, 1, 0), (0, 1, 0, 1),$
$(0, 0, 2, 0), (0, 0, 1, 1), (0, 0, 0, 2)\}$.
So $(x + y + z + w)^2 = x^2 + 2xy + 2xz + 2xw + y^2 + 2yz + 2yw + z^2 + 2zw + w^2$.

13. $8x^3 - 12x^2y + 12x^2z + 6xy^2 - 12xyz + 6xz^2 - y^3 + 3y^2z - 3yz^2 + z^3$.

14. $3^{20}2^{50}\binom{80}{20,50,10}$.
The relevant term is $\binom{80}{20,50,10}(3x)^{20}(-2y)^{50}z^{10} = \binom{80}{20,50,10}3^{20}(-2)^{50}x^{20}y^{50}z^{10}$.
Note that $(-2)^{50} = 2^{50}$.

15. $\binom{20}{5,4,5,6} = 9777287520$.

16. $1 - x + 2x^2 - 2x^3 + 3x^4 - 3x^5 + \cdots$ has $c_i = (-1)^i \lceil \frac{i+1}{2} \rceil$.
Notice that x^k occurs in the product $(1-x+x^2-x^3+\cdots)(1+x^2+x^4+x^6+\cdots) = (1+(-x)+(-x)^2+(-x)^3+\cdots)(1+x^2+x^4+x^6+\cdots)$ via products of the form $(-x)^{k-2j}x^{2j}$, where $j = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor$. Computing the first several coefficients displays the resulting pattern in the product.

17.

i	0	5	10	15	20	25	30	35	40	45	50
c_i	1	1	2	1	2	2	3	3	3	3	3

i	55	60	65	70	75	80	85	90	95	100
c_i	3	3	3	3	2	2	1	2	1	1

Use the generating function
$(1 + x^{25} + x^{50})(1 + x^{10} + x^{20} + x^{30} + x^{40})(1 + x^5 + x^{10})$. Expand it with a calculator or mathematical software, and read off the coefficients.

18. 17.
$(1+x+x^2+x^3+x^4+x^5)(1+x+x^2+x^3)(1+x+x^2+x^3+x^4) = \cdots + 17x^5 + \cdots$. So 17.

19. 49.
$(1+x+x^2)(x+x^2+x^3+x^4)(x^2+x^3+x^4+x^5+x^6)(x^3+x^4+x^5+x^6+x^7+x^8) = \cdots + 49x^{12} + \cdots$. So 49.

20. 44.
$\frac{1-x^8}{1-x}(\frac{1}{1-x})^2 = \frac{1-x^8}{(1-x)^3}$. Now, $\binom{10}{8} - \binom{2}{0} = 44$.

21. $\binom{20+10-1}{20} = \binom{29}{20} = 10015005$.
See Theorem 7.6(b).

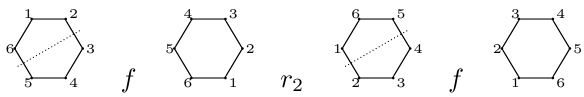
22. $1\binom{7+5-1}{7} + 3\binom{5+5-1}{5} - 1\binom{3+5-1}{3} = \binom{11}{7} + 3\binom{9}{5} - \binom{7}{3} = 673$.

23. 7211.
$(1+x+x^2+x^3+x^4+x^5)(1+x+x^2+\cdots)^3 = (1+x+x^2+x^3+x^4+x^5)(\frac{1}{1-x})^3$.
The coefficient of x^{50} is $\binom{52}{50} + \binom{51}{49} + \binom{50}{48} + \binom{49}{47} + \binom{48}{46} + \binom{47}{45} = 7211$.

24.

$\circ$	r_0	r_1	r_2
r_0	r_0	r_1	r_2
r_1	r_1	r_2	r_0
r_2	r_2	r_0	r_1

25. r_4.



26. 13.
$G = D_6$ and
$N = \frac{1}{12}[2^6+2(2)+2(2^2)+2^3+3(2^4)+3(2^3)] = 13$.

27. 217045.
$G = Z_9$ and $N = \frac{1}{9}[5^9 + 6(5) + 2(5^3)] = 217045$.

28. $\frac{1}{24}[4^6+6(4^3)+3(4^4)+6(4^3)+8(4^2)] = 240$.

29. 4995.
$G = Z_4$ and $N = \frac{1}{4}[3^9 + 3^3 + 3^5 + 3^3] = 4995$.

30. *Proof.* Let $g_1, g_2 \in G$. Suppose $\text{Fix}(g_2) = X$. ($\subseteq$) Suppose $x \in \text{Fix}(g_2g_1)$. So $x = g_2g_1x = g_1x$. Hence $x \in \text{Fix}(g_1)$. ($\supseteq$) Suppose $x \in \text{Fix}(g_1)$. So $g_2g_1x = g_2x = x$. Hence $x \in \text{Fix}(g_2g_1)$. $\square$

31. 4624.
The number of ways to color each of the outer triples is $\frac{1}{3}[4^3+4+4] = 24$. So $N = \frac{1}{3}[24^3+24+24] = 4624$.

32. 94.
$G = D_9$ and $N = \frac{1}{18}[\binom{9}{3,3,3} + 6(0) + 2(3!)] = 94$.

33. Paths from $c_{0,0}$ to $c_{n+1,k}$ must pass through exactly one of $c_{n,k-1}$ or $c_{n,k}$. These are the two points above $c_{n+1,k}$ in the preceding row. The number of paths to $c_{n,k-1}$ is $\binom{n}{k-1}$. The number of paths to $c_{n,k}$ is $\binom{n}{k}$. Hence, $\binom{n+1}{k}$, the number of paths to $c_{n+1,k}$, must be $\binom{n}{k-1} + \binom{n}{k}$.

34. There are $\binom{n}{k}$ subsets of size k from $\{1, 2, \ldots, n\}$. For each $k-1 \le i \le n-1$, there are $\binom{i}{k-1}$ for which $i+1$ is the largest element. The point is that, if $i+1$ is the largest element selected, then the remaining $k-1$ elements must be chosen from the set $\{1, 2, \ldots, i\}$, and there are $\binom{i}{k-1}$ ways to make such a choice.

35. We consider paths through Pascal's triangular grid from $S = c_{0,0}$ to $F = c_{3n,k}$. Note that such a path must go through exactly one point $c_{2n,i}$ in the $2n^{th}$ row. For $i = 0, 1, \ldots, k$, the number of paths from $S = c_{0,0}$ to $F = c_{3n,k}$ through $c_{2n,i}$ is $\binom{2n}{i}\binom{n}{k-i}$. That is, the number of ways from $c_{0,0}$ to $c_{2n,i}$ is $\binom{2n}{i}$, and the number of ways from $c_{2n,i}$ to $c_{3n,k}$ is $\binom{3n-2n}{k-i} = \binom{n}{k-i}$.

36. Here R = "(" and D = ")". Our binary sequences never have more D's than R's at any point. That is, since we want balanced parentheses, as we read from left to right, we can never have more occurrences of ")" than "(". Of course, in the end we must have the same number of each.

37. Note that $|A \triangle \{n\}| = |A| \pm 1$. So $|A|$ is even iff $|A \triangle \{n\}|$ is odd. The same formula $A \mapsto A \triangle \{n\}$ defines both a function and its inverse, since $(A \triangle \{n\}) \triangle \{n\} = A$. This bijection therefore establishes the asserted equality.

38. 4371.
$(\frac{1-x^{100}}{1-x})^3 =$
$(\frac{1}{1-x})^3(1 - 3x^{100} + 3x^{200} - x^{300})$.
The coefficient of x^{205} is
$\binom{207}{205} - 3\binom{107}{105} + 3\binom{7}{5} = 4371$.

39. $\binom{205}{202,3,0} = 1414910$.

40. $\binom{52}{13,13,13,13}$.
The players are ordered North, East, South, West. We are placing 13 items into each of 4 distinct categories. So we employ the multinomial coefficient.

41. $N = \frac{1}{16}[\binom{8}{2,2,4} + 6(0) + \binom{4}{1,1,2} + 4\binom{4}{1,1,2} + 4\binom{4}{1,1,2}] = 33.$

42. (a) $(1+x+x^2+\cdots+x^{10})(1+x+x^2+\cdots+x^8)(1+x+x^2+\cdots+x^{12}) = \cdots + 54x^9 + \cdots$. So $N = 54$.
(b) $\frac{\binom{10}{3}\binom{8}{3}\binom{12}{3}}{\binom{30}{9}} = \frac{896}{8671} \approx .1033.$
(c) The outcomes based on numbers of colors are not equally likely.

43. Let c_k be the coefficient of x^k in $(1+x+x^2)^4 = (1-x^3)^4 \frac{1}{(1-x)^4}$, and suppose $k \geq 12$. From the left-hand side, it is obvious that $c_k = 0$. Since the right-hand side equals $(1-4x^3+6x^6-4x^9+x^{12})\frac{1}{(1-x)^4}$, we see also that $c_k = \binom{k+3}{k} - 4\binom{k}{k-3} + 6\binom{k-3}{k-6} - 4\binom{k-6}{k-9} + \binom{k-9}{k-12}$.

44. (a) $\binom{16}{4,4,4,4} = 63063000.$
(b) $G = Z_4$ and $N = \frac{1}{4}[\binom{16}{4,4,4,4} + 2(4!) + \binom{8}{2,2,2,2}] = 15766392.$

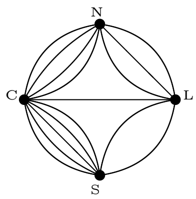
45. 17.
$(x+x^2+x^3+x^4+x^5+x^6)(x+x^2+x^3+x^4)(x+x^2+x^3+x^4+x^5) = \cdots + 17x^8 + \cdots.$

46. 10.
$(x^2+x^4+\cdots+x^{24})(x^3+x^6+\cdots+x^{24})(x^4+x^8+\cdots+x^{24}) = \cdots + 10x^{25} + \cdots.$

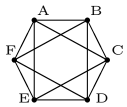
2.8 Chapter 8

Section 8.1

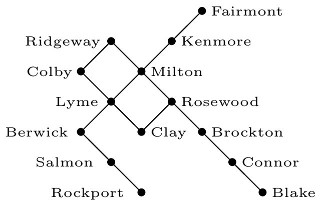
1.



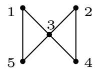
3.



5.

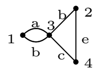


7. The graph is simple.



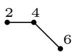
It has no loops and no multiple edges.

9. The graph is not simple, because it has multiple edges.



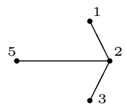
Edges a and b are multiple edges, since they both join 1 with 3.

11. Yes.
The endpoints of $\{2, 4\}$ and $\{4, 6\}$ are in W.

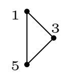


13. No. Edge e needs vertex 4, and $4 \notin W$.

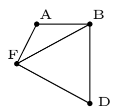
15. $E = \{\{1,2\},\{2,3\},\{2,5\}\}$.



17. $E = \{\{1,3\},\{3,5\},\{1,5\}\}$.



19.



21. No. An endpoint is missing.
Alternatively, the flat line might be a loop that is squashed flat and thus intersects itself illegally.

23. Yes.
The drawing has an allowed crossing of two edges.

25. No. Two edges intersect in infinitely many points.
Alternatively, there may be vertices missing from the two places where three curves meet.

27. Yes.
$1, 3, 4$.

29. Let $e \mapsto \{u, v\}$ be an edge. The walk u, e, v, e, u is not a path.
It repeats the vertex u.

31. Yes.
$1, 3, 5, 1$.

33. 2.
The path $1, 3, 4$ has length 2, and there is no shorter path from 1 to 4. That is, 1 and 4 are not directly joined by an edge.

35. 2.
The path $1, a, 3, e, 4$ has length 2, and there is no shorter path from 1 to 4. That is, 1 and 4 are not directly joined by an edge.

37. 5.
The path

Berwick, Lyme, Clay, Rosewoood, Brockton, Conner

has length 5, and there is no shorter path from Berwick to Conner.

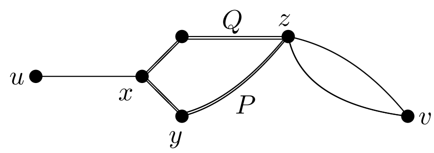
39. Yes.
There are no vertex repetitions, and $\{5,1\}, \{1,2\}, \{2,3\}$ are all edges in the graph.

41. No.
It is a circuit, but vertex 1 is repeated.

43. (a) Colby, Lyme, Clay, Rosewood.
(b) Yes.
It has 7 vertices.

45. *Proof.* Suppose $d = \text{dist}(u, w) \geq 1$. Let P be a path of length d from u to w. Let v be the last vertex on P before w. Let Q be the path from u to v obtained by truncating P. Since Q has length $d-1$, it follows that $\text{dist}(u, v) \leq d-1$. We claim that $\text{dist}(u, v) = d-1$. So suppose to the contrary that $\text{dist}(u, v) < d-1$. Hence, there is a path Q' of length $l < d-1$ from u to v. Form P' from Q' by adding the edge from v to w. So P' is a path of length $l+1 < d$ from u to w. However, there should be no path of length less than d from u to w. From this contradiction it follows that $\text{dist}(u, v) = d-1$. □

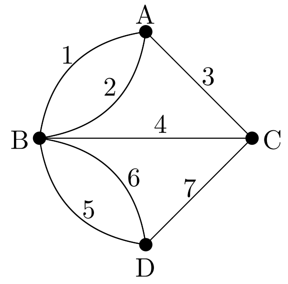
47. *Proof.* Let P and Q be distinct paths from u to v. We can find a portion of P followed by a portion of Q that forms a cycle.



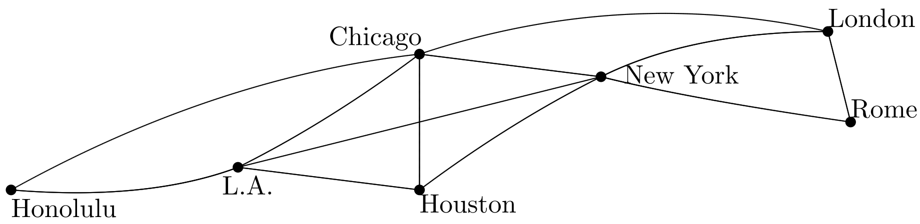
Since P and Q must be different at some point after u, let y be the first vertex in P that is not in Q. So the vertex x in P immediately preceding y must also be in Q. Since P and Q become different after vertex x but both end up at vertex v, let z be the first vertex in P that is after x (and y) and is common to P and Q. Notice that the portion of P strictly between x and z has nothing in common with Q. Therefore, the walk which follows P from x to z and then follows Q backwards from z to x forms a cycle in G. □

49. *Sketch.* By symmetry, we must start along edge 3 or edge 4.
Case 1: If we take edge 3, then without loss of generality, we take edge 1. We must then, without loss of generality, take edge 5. We then finish with $7, 4, 2$ or

$7, 4, 6$ or $6, 2$ or $6, 4, 7$ and get stuck without hitting every edge.
Case 2: If we take edge 4, then without loss of generality, we take edge 1. *Subcase 2a*: Take edge 3 next, then 7, and, without loss of generality, 5. After taking 2 or 6 we get stuck. *Subcase 2b*: Take edge 2 next, and then, without loss of generality, 5. If we take 6, then we get stuck. Otherwise, we take edge 7, then take 3, and get stuck. □

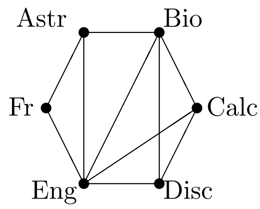


51. (a) $n = 2, 3, 4, 5$, or 6.
(b) 2.



$n = 2$: Honolulu, Chicago, London.
$n = 3$: Honolulu, L.A., New York, London.
$n = 4$: Honolulu, L.A., Houston, New York, London.
$n = 5$: Honolulu, L.A., Houston, New York, Rome, London.
$n = 6$: Honolulu, L.A., Chicago, Houston, New York, Rome, London.
Of course, 2 is the length of the shortest path.

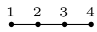
53. Calculus and Discrete Math.



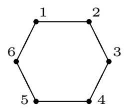
Calculus and Discrete Math are the only classes that are not joined to Astronomy by an edge.

Section 8.2

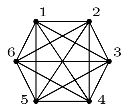
1.



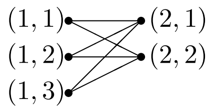
3.



5.



7.



9.

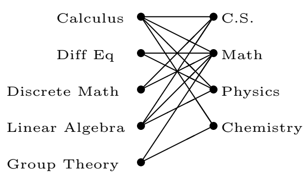
There are two binary sequences of length one.

11. (a) P_7.
Colby, Ridgeway, Milton, Rosewood, Brockton, Connor, Blake.
(b) Yes.
The blue line is P_7, the black line is P_7, and the gray line in P_4.
(c) Colby, Lyme, Milton.
Parts of both the gray line and the blue line are used.

13. (a) No. Nord is not adjacent to Sud.
(b) Just exclude Sud.
That is, consider the subgraph induced by the other three vertices.

15. (a) No.
C_4 is a subgraph of K_4 that is not complete.
(b) Yes.
Suppose u and v are vertices in the induced subgraph. Since there is an edge joining u and v in the complete graph, that edge must be present in the induced subgraph.

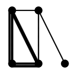
17.



19. Yes. Let V_1 contain the odd-numbered vertices, and V_2 the even.
In P_n, each vertex k can only be adjacent to $k-1$ and $k+1$. Since $k-1$ and $k+1$ do not have the same parity as k, every edge must join an odd-numbered vertex to an even-numbered vertex.

21. *Proof.* Let H be a subgraph of a bipartite graph G. Let V_1, V_2 bipartition G, and let W be the vertex set of H. We claim that $W \cap V_1, W \cap V_2$ bipartition H. Suppose e is an edge of H. Since e is an edge of G, e must have one end v_1 in V_1 and the other end v_2 in V_2. Since v_1 and v_2 must be vertices of H, we have $v_1 \in W \cap V_1$ and $v_2 \in W \cap V_2$. This establishes our claim. □

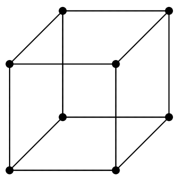
23. It is not bipartite, because it contains a 3-cycle.



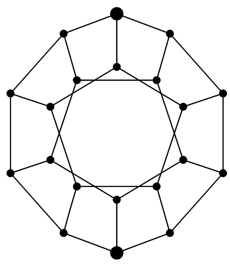
25. $|V| = n$ and $|E| = n - 1$.
Note $V = \{1, 2, \ldots, n\}$ and $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}\}$.

27. $|V| = n$ and $|E| = \binom{n}{2}$.
Note $V = \{1, 2, \ldots, n\}$ and $E = \mathcal{P}_2(V)$.

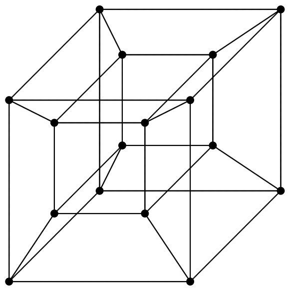
29. $|V| = 8$ and $|E| = 12$.



31. $|V| = 20$ and $|E| = 30$.



33. $|V| = 16$ and $|E| = 32$.



35. No. They differ by more than one digit.
They differ in the fourth and fifth digits.

37. (a) 1101.
The nearest code word is 1101010 and corresponds to the message 1101.
(b) Male, A^+.
See message 1101 in Table 8.2.

39. $n - 1$.
No two vertices are farther apart than 1 and n.

41. 1, for $n \geq 2$.
No two vertices are farther apart than 1 and 2.

43. 3.
No two vertices are farther apart than 000 and 111.

45. 5.
No two vertices are farther apart than the two displayed large in the above answer to Exercise 31.

47. 4.
No two vertices are farther apart than 0000 and 1111.

Section 8.3

1.

$$\begin{array}{c c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

3.
$$\begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}0&0&0&1&0&0\\0&0&0&1&0&0\\0&0&0&1&0&0\\1&1&1&0&1&1\\0&0&0&1&0&1\\0&0&0&1&1&0\end{bmatrix}\end{array} = A$$

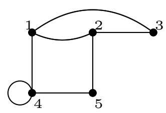
5.
$$\begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}0&1&0&0&0&1\\1&0&1&0&1&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&1&0&1&0&1\\1&0&0&0&1&0\end{bmatrix}\end{array} = A$$

7.
$$\begin{array}{cc} & \begin{array}{cccccc}2&5&1&4&6&3\end{array} \\ \begin{array}{c}2\\5\\1\\4\\6\\3\end{array} & \begin{bmatrix}0&0&0&1&0&0\\0&0&0&1&1&0\\0&0&0&1&0&0\\1&1&1&0&1&1\\0&1&0&1&0&0\\0&0&0&1&0&0\end{bmatrix}\end{array} = B$$

9.
$$\begin{array}{cc} & \begin{array}{ccccc}1&2&3&4&5\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}0&1&1&1&1\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\end{bmatrix}\end{array}$$

11.
$$\begin{array}{cc} & \begin{array}{cccccc}(1,1)&(1,2)&(2,1)&(2,2)&(2,3)&(2,4)\end{array} \\ \begin{array}{c}(1,1)\\(1,2)\\(2,1)\\(2,2)\\(2,3)\\(2,4)\end{array} & \begin{bmatrix}0&0&1&1&1&1\\0&0&1&1&1&1\\1&1&0&0&0&0\\1&1&0&0&0&0\\1&1&0&0&0&0\\1&1&0&0&0&0\end{bmatrix}\end{array}$$

13.



15. $\begin{bmatrix}16&13\\18&9\end{bmatrix}$ $3(4)+2(2)=16,\quad 3(1)+2(5)=13,$
$4(4)+1(2)=18,\quad 4(1)+1(3)=9.$

17. $\begin{bmatrix} 7 & 5 \\ 6 & 8 \end{bmatrix}$

19. $\begin{bmatrix} 0 & 2 & 7 \\ 1 & 0 & 4 \\ 8 & 3 & 0 \end{bmatrix}$

21. $\begin{bmatrix} 0 & 24 & 9 \\ 8 & 4 & 10 \\ 30 & 5 & 35 \end{bmatrix}$

23.

$$(a) \quad P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (b) \quad PA = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$(c) \quad PAP^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(a) Note that the rows of the identity matrix I_6 have been permuted according to the permutation $2, 5, 1, 4, 6, 3$. (b) Note that the rows of A have been permuted according to the permutation $2, 5, 1, 4, 6, 3$. (c) This is the same as the answer from Exercise 7.

25.

$$A^2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

For example, for each $1 \leq i \leq 6$, the entry of A^2 in position (i, i) is the number of edges incident with vertex i.

27.

$$A^2 = \begin{bmatrix} 2 & 0 & 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 & 0 & 2 \\ 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 0 & 3 & 0 \\ 0 & 2 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

For example, the two of length 2 from vertex 1 to vertex 5 are $1, 2, 5$ and $1, 6, 5$.

29. 0, 6, and 0, respectively.
E.g. The number of walks from 1 back to 1 of length 4 is $2 + 2 + 2 = 6$.

31. 2, 0, and 5, respectively.
E.g. The number of walks from 1 back to 1 of length 6 is $1 + 1 + 3 = 5$.

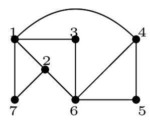
33. *Proof.* Let i, j be vertices. ($\rightarrow$) Suppose there is a path from i to j. So, for some $0 \leq k \leq n-1$, there is a path of length k from i to j. So A^k has a positive value in entry i, j. So $I + A + \cdots + A^k + \cdots + A^{n-1}$ has a positive value in entry i, j. ($\leftarrow$) The previous argument is reversible. $\square$

35.
$1 : 4$
$2 : 4$
$3 : 4$
$4 : 1, 2, 3, 5, 6$
$5 : 4, 6$
$6 : 4, 5.$

37.
$1 : 2, 6$
$2 : 1, 3, 5$
$3 : 2, 4$
$4 : 3, 5$
$5 : 2, 4, 6$
$6 : 1, 5.$

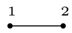
39.
$000 : 001, 010, 100$
$001 : 000, 011, 101$
$010 : 000, 011, 110$
$011 : 001, 010, 111$
$100 : 000, 101, 110$
$101 : 001, 100, 111$
$110 : 010, 100, 111$
$111 : 011, 101, 110.$

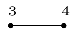
41.



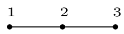
Section 8.4

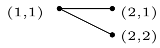
1. Define $f(1) = 3, f(2) = 4$.

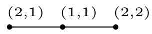




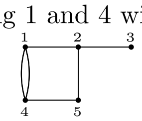
3. Define $f(1) = (2, 1), f(2) = (1, 1), f(3) = (2, 2)$.

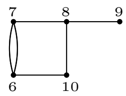




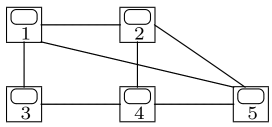


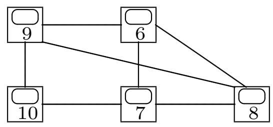
5. Define $f(1) = 7, f(2) = 8, f(3) = 9, f(4) = 6, f(5) = 10$. Also match up the parallel edges connecting 1 and 4 with those connecting 6 and 7.



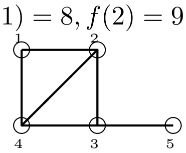


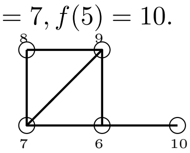
7. Define $f(1) = 9, f(2) = 6, f(3) = 10, f(4) = 7, f(5) = 8$.



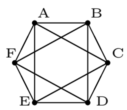


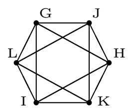
9. Define $f(1) = 8, f(2) = 9, f(3) = 6, f(4) = 7, f(5) = 10$.





11. (a) We *can* define $f(A) = G$, $f(B) = J$, $f(C) = H$, $f(D) = K$, $f(E) = I$, and $f(F) = L$.



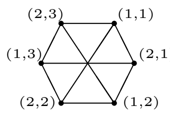


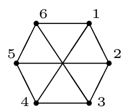
(b)

Time Period	Study Group Meeting
1	German, Kuwait
2	Indochina, Japanese
3	History, Latin

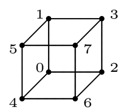
We simply use the isomorphism to make substitutions in the given schedule.

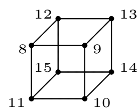
13. Define $f(1, 1) = 1, f(1, 2) = 3, f(1, 3) = 5,\ f(2, 1) = 2, f(2, 2) = 4,$
$f(2, 3) = 6$.



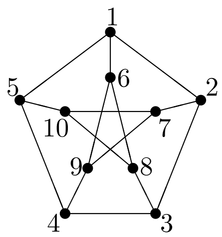


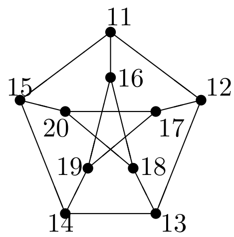
15. Define $f(0) = 15$, $f(1) = 12$, $f(2) = 14$, $f(3) = 13$, $f(4) = 11$, $f(5) = 8$, $f(6) = 10$, $f(7) = 9$.

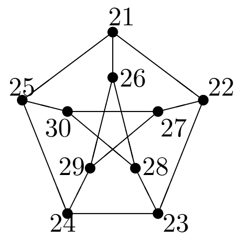




17. From the first graph to the second graph, define $\forall\ i,\ f(i) = i + 10$. The same formula works from the second to the third.







19. $3! \cdot 2 = 12$. The vertices $\{1, 2, 3\}$ can be permuted in any of 3! ways. Vertex 4 must stay put. Vertices $\{5, 6\}$ can be permuted in either of 2 ways.

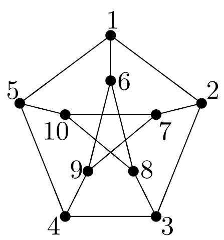
21. $2 \cdot 2 = 4$. There is a vertical line of symmetry and a horizontal line of symmetry. For each line of symmetry, we have the 2 choices: reflect or not.

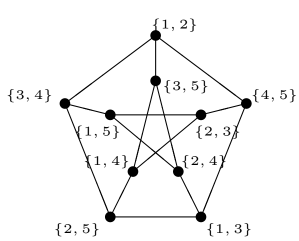
23. $|D_n| = 2n$. We have the n rotations, and the n reflections. See Definition 7.3 for the description of the dihedral group D_n.

25. $2(n!)^2$. Let $V_1 = \{(1, 1), \ldots, (1, n)\}$ and $V_2 = \{(2, 1), \ldots, (2, n)\}$. The set V_1 can be permuted in any of $n!$ ways, the set V_2 can be permuted in any of $n!$ ways, and we can either switch V_1 with V_2 or not (2 choices).

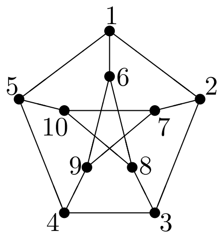
27. *Proof.* Suppose that $v_0, e_1, v_1, \ldots, v_n$ is a path in a graph G and that $f : G \longrightarrow H$ is an isomorphism. Since $v_0, e_1, v_1, \ldots, v_n$ is a walk in G, it follows that $f(v_0), f(e_1), f(v_1), \ldots, f(v_n)$ is a walk in H. Since there are no vertex repetitions in the list $v_0, v_1, \ldots, v_n$ and f_V is a bijection, there cannot be any repetitions in the list $f(v_0), f(v_1), \ldots, f(v_n)$. Hence, $f(v_0), f(e_1), f(v_1), \ldots, f(v_n)$ is a path in H. $\square$

29.





31. *Sketch.* It suffices to show that the cycles $6, 8, 10, 7, 9, 6$ and $1, 2, 7, 10, 5, 1$ and $6, 8, 3, 4, 9, 6$ all work. Rotations handle the rest. $\square$



To send $1, 2, 3, 4, 5, 1$ to $6, 8, 10, 7, 9, 6$, define $f(1) = 6$, $f(2) = 8$, $f(3) = 10$, $f(4) = 7$, $f(5) = 9$, $f(6) = 1$, $f(7) = 3$, $f(8) = 5$, $f(9) = 2$, $f(10) = 4$. To send $1, 2, 3, 4, 5, 1$ to $1, 2, 7, 10, 5, 1$, define $g(1) = 1$, $g(2) = 2$, $g(3) = 7$, $g(4) = 10$, $g(5) = 5$, $g(6) = 6$, $g(7) = 3$, $g(8) = 9$, $g(9) = 8$, $g(10) = 4$. To send $1, 2, 3, 4, 5, 1$ to $6, 8, 3, 4, 9, 6$, define $h(1) = 6$, $h(2) = 8$, $h(3) = 3$, $h(4) = 4$, $h(5) = 9$, $h(6) = 1$, $h(7) = 10$, $h(8) = 2$, $h(9) = 5$, $h(10) = 7$.

33. *Proof.* Note that the vertex set of C_n is $\{1, 2, \ldots, n\}$. Among the automorphisms of C_n are the rotations r_k for $k \in \mathbb{Z}$. Let i and j be arbitrary vertices in C_n. The rotation r_{j-i} moves i to j. That is, $r_{j-i}(i) = i + (j - i) = j$. $\square$

35. *Proof.* Let $G = (V, E)$ and $H = (W, F)$ be graphs. Suppose $G \cong H$. So we have an isomorphism $f : G \longrightarrow H$. Define $f^{-1} : H \longrightarrow G$ by taking $f_V^{-1} : V_H \longrightarrow V_G$ to be the inverse of $f_V : V_G \longrightarrow V_H$ and $f_E^{-1} : E_H \longrightarrow E_G$ to be the inverse of $f_E : E_G \longrightarrow E_H$. The point is that f_V and f_E are bijections. So the inverses f_V^{-1} and f_E^{-1} exist and satisfy $f_V^{-1} \circ f_V = \mathrm{id}_V$ and $f_E^{-1} \circ f_E = \mathrm{id}_E$. (See Theorem 5.10.)

To see that f^{-1} is a graph isomorphism, it suffices to check that f^{-1} is a graph map. So suppose $w_1, w_2 \in W$ and e is an edge joining w_1 and w_2. Since f_V is a bijection, we have $v_1, v_2 \in V$ such that $f(v_1) = w_1$ and $f(v_2) = w_2$. Since $f(v_1)$ and $f(v_2)$ are joined by the edge e, vertices v_1 and v_2 must be joined by an edge d such that $f_E(d) = e$. Thus, $f_E^{-1}(e) = d$ joins $f_V^{-1}(w_1)$ to $f_V^{-1}(w_2)$.

We conclude that $H \cong G$. $\square$

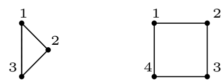
37. *Proof.* ($\rightarrow$) Suppose G is vertex transitive. Let u be a vertex of G. For every vertex v in G, there is an automorphism f of G such that $f(u) = v$ (by the definition of vertex transitive). ($\leftarrow$) Suppose there is a vertex u such that, for all vertices v, there is an automorphism f of G such that $f(u) = v$. Suppose $v_1, v_2 \in V$. So there exist automorphisms f_1 and f_2 such that $f_1(u) = v_1$ and $f_2(u) = v_2$. Let $f = f_2 \circ f_1^{-1}$. Observe that f is an automorphism and $f(v_1) = v_2$. Since v_1 and v_2 are arbitrary, G is vertex transitive. $\square$

39. *Sketch.* ($\rightarrow$) Suppose $f : G \longrightarrow H$ is an isomorphism, and let $D : H \longrightarrow \mathbb{R}^2$ be any drawing of H. Then $D \circ f : G \longrightarrow \mathbb{R}^2$ is a drawing of G with the same

image. $\square$ The point is that a drawing provides a map from the vertex set of a graph to a set of points in the plane $\mathbb{R}^2$. Also, each edge is assigned to a curve in $\mathbb{R}^2$. Using f_V and f_E, we can therefore construct a drawing for G from a drawing for H.

41. Define $f(1) = f(2) = f(3) = f(5) = 5, f(4) = 4, f(6) = 6$.

43. *Sketch.* Suppose there is one. Without loss of generality, say $f(1) = 1$ and $f(2) = 2$. If $f(3) = 3$, then $\{1, 3\}$ needs to be an edge of C_4. If $f(3) = 4$, then $\{2, 4\}$ needs to be an edge of C_4. Thus f cannot exist. $\square$



45. No. Triangle $1, 2, 3$ has no place to go. There is no triangle in graph (c).

47. $\forall\ i$, define $f(1, i) = 1$ and $f(2, i) = 2$.
Recall that $K_{m,n}$ has bipartition V_1, V_2, where $V_1 = \{(1, i)\ :\ 1 \le i \le n\}$ and $V_2 = \{(2, i)\ :\ 1 \le i \le n\}$. Also, P_2 has vertex set $\{1, 2\}$. We map V_1 to 1 and V_2 to 2. Since all edges of $K_{m,n}$ join V_1 to V_2, this is a graph map.

49. $m \le n$. When $m \le n$, we can map K_m into K_n as a subgraph. If $m > n$, then we are forced to map two vertices of K_m to the same vertex of K_n. However, there is no place to put the edge joining those two vertices in K_m. That is, there are no loops.

Section 8.5

1. (a) has 6 vertices, whereas (d) has only 5.
So apply the contrapositive of Theorem 8.8(i).

3. (a) has 6 edges, whereas (c) has 7.
So apply the contrapositive of Theorem 8.8(ii).

5. $5, 2, 2, 1, 1, 1$.
That is, $\deg(4) = 5$, $\deg(5) = 2$, $\deg(6) = 2$, $\deg(1) = 1$, $\deg(2) = 1$, and $\deg(3) = 1$. We happened to use the ordering $4, 5, 6, 1, 2, 3$ for the vertices (which gave us a nonincreasing degree sequence), but any ordering suffices.

7. $3, 3, 2, 2, 2, 2$.

That is, deg(2) = 3, deg(5) = 3, deg(1) = 2, deg(3) = 2, deg(4) = 2, and deg(6) = 2. We happened to use the ordering $2, 5, 1, 3, 4, 6$ for the vertices (which gave us a nonincreasing degree sequence), but any ordering suffices.

9. $\delta((b)) = 2$ and $\delta((g)) = 1$.
So $(b) \ncong (g)$ by Theorem 8.11(iii) (its contrapositive).

11. They do not have a common degree sequence.
So Theorem 8.11(i) tells us that they are not isomorphic.

13. The subgraph induced by the degree 3 vertices is P_2 on the left graph and Φ_2 on the right graph. See Exercise 28 in Section 8.4.
A graph isomorphism would have to map the degree 3 vertices from the left graph to the degree 3 vertices on the right graph. However, the right graph is then missing a needed edge.

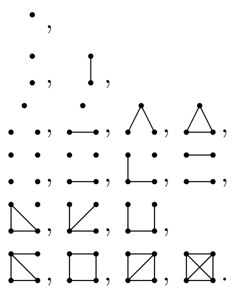
15. The computers of degree 3 are adjacent in the left configurations, but not in the right.
See Exercise 28 in Section 8.4. A graph isomorphism would have to map the degree 3 vertices from the left graph to the degree 3 vertices on the right graph. However, the right graph is then missing a needed edge.

17. The grid on the left has a power station of degree 4, and that on the right does not.
The graphs therefore have different degree sequences. In particular, they have different maximum degrees. So Theorem 8.11 tells us that they are not isomorphic.

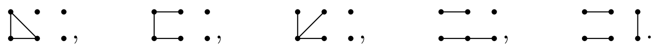
19. All vertices in the first graph have degree 4, while vertex "German" in the second graph has degree 3.
The graphs therefore have different degree sequences. In particular, they have different maximum degrees. So Theorem 8.11 tells us that they are not isomorphic.

21. Group them according to the numbers of vertices n and edges m.

$n = 1$
$n = 2$
$n = 3$
$n = 4, m \leq 2$
$n = 4, m = 3$
$n = 4, m \geq 4$

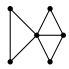


23.



We have grouped them according to how many vertices are isolated. Within that, we see that there are two possibilities with two vertices isolated. Those are distinguished by their maximum degrees.

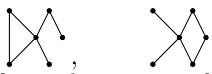
25.



There are 6 vertices. So the degree 5 vertex must be adjacent to each of the other vertices. When the vertex of degree 5 is removed, it leaves a graph with degree sequence $2, 1, 1, 1, 1$. There is only one possibility for this, as shown in Exercise 26. So we have only one possible graph here.

27. None.
An odd number of odd-degree vertices is not possible. See Corollary 8.14.

29.

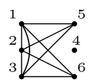


There are 6 vertices. First form the vertex of degree 4. Note that exactly one vertex v is not a neighbor of the degree 4 vertex. Now, there are two edges left to place, and there are only two different ways to place them. Either one or both will be adjacent to v.

31. *Proof.* Suppose not. So there is an odd number of odd-degree vertices. Hence, $\sum_{v \in V} \deg(v)$ is odd. However, $\sum_{v \in V} \deg(v) = 2|E|$, and $2|E|$ is even. This is a contradiction. $\square$

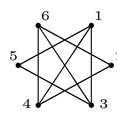
33. $|V| = 2^n$ and $|E| = n2^{n-1}$.
Recall that V is the set of binary sequences of length n. Observe that each vertex has degree n. So $2|E| = \sum_{v \in V} \deg(v) = n2^n$.

35.



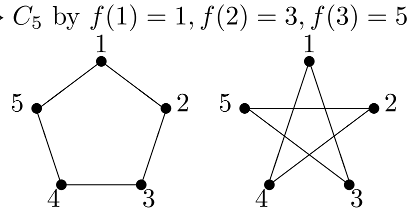
The edges appear here exactly where they do not appear in graph (a).

37.

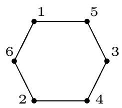


The edges appear here exactly where they do not appear in graph (c).

39. Define $f : {C_5}^c \longrightarrow C_5$ by $f(1) = 1, f(2) = 3, f(3) = 5, f(4) = 2, f(5) = 4$.



41. C_6.



The vertices are in a different order, but the graph is isomorphic to C_6.

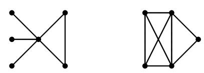
43. The complements C_6 and $C_3 + C_3$, respectively, are not isomorphic. One is connected, and the other is not.
Exercise 27 from Section 8.4 can also be used to see $C_6 \ncong C_3 + C_3$, since the path $1, 2, 3, 4$ in C_6 cannot be mapped to a path in $C_3 + C_3$, under any isomorphism.

45. *Proof.* Suppose $G \cong H$. So we have an isomorphism $f : G \longrightarrow H$. Thus, $f_V : V_G \longrightarrow V_H$ and $f_E : E_G \longrightarrow E_H$ are bijections. Note that the bijection f_V determines a bijection between $\mathcal{P}_2(V_G)$ and $\mathcal{P}_2(V_H)$. Hence, we have a bijection $f_{E^c} : \mathcal{P}_2(V_G) \setminus E_G \longrightarrow \mathcal{P}_2(V_H) \setminus E_H$. The bijections f_V and f_{E^c} thus determine an isomorphism $f : G^c \longrightarrow H^c$. Hence $G^c \cong H^c$. □

47. The idea is in the proof for Exercise 45.
Let $G = (V, E)$ be a simple graph. The point is that a bijection $f : V \longrightarrow V$ determines a bijection $f : E \longrightarrow E$ mapping edges to edges iff it determines a bijection $f : E^c \longrightarrow E^c$ mapping non-edges to non-edges.

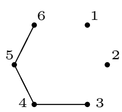
49. True.
Proof. Let G be vertex transitive and $v, w \in V$. So we have an automorphism f with $f(v) = w$. By Lemma 8.10, $\deg(w) = \deg(v)$. Since v, w are arbitrary, G must be regular. □

51.



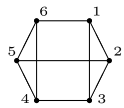
This is one graph with two components.

53.



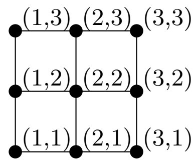
Note that both (a) and (c) have vertex set $\{1,2,3,4,5,6\}$. The edges they have in common are $\{3,4\}$, $\{4,5\}$, and $\{5,6\}$.

55.



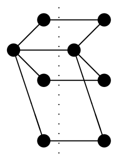
Note that edges $\{1,3\}$ and $\{4,6\}$ are the only edges in (b) that are not already in (c).

57.



We have three copies of P_3 that are connected in a path (like P_3).

59.



Note that the dotted line is not part of the graph. On each side of the dotted line is a copy of $K_{1,3}$. Corresponding vertices in the two copies are then connected by an edge (like the vertices in P_2 are connected by an edge).

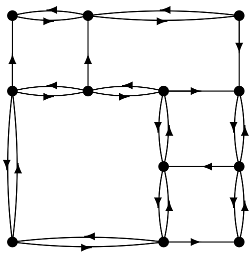
61. (a) The graph $G \cap (H \cup K)$ has vertex set $V_G \cap (V_H \cup V_K)$ and edge set $E_G \cap (E_H \cup E_K)$. The graph $(G \cap H) \cup (G \cap K)$ has vertex set $(V_G \cap V_H) \cup (V_G \cap V_K)$ and edge set $(E_G \cap E_H) \cup (E_G \cap E_K)$. By the distributive laws for sets,
$V_G \cap (V_H \cup V_K) = (V_G \cap V_H) \cup (V_G \cap V_K)$ and
$E_G \cap (E_H \cup E_K) = (E_G \cap E_H) \cup (E_G \cap E_K)$.
So the graphs $G \cap (H \cup K)$ and $(G \cap H) \cup (G \cap K)$ must be the same graph.
(b) Similar.

63. Define $f(1) = (1,1), f(2) = (1,2), f(3) = (1,3), f(4) = (2,1), f(5) = (2,2), f(6) = (2,3)$.

65. *Proof.* Since $V_G \neq \emptyset$ and $V_H \neq \emptyset$, we have vertices $v \in V_G$ and $w \in V_H$. Let G' be the subgraph of G induced by $\{v\}$, and let H' be the subgraph of H induced by $\{w\}$. Observe that $G' \times H \cong H$ and $G \times H' \cong G$, via the maps $(v,h) \mapsto h$ and $(g,w) \mapsto w$, respectively. $\square$

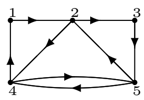
Section 8.6

1.



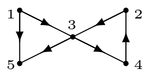
Each two-way street is represented by two edges in opposite directions to each other. Each one-way street is represented by an edge with no corresponding opposite.

3.

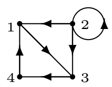


Each computer is represented by a vertex. The two-way communication line is represented by two edges, one in each of the two possible directions of communication.

5. A simple directed graph.



7. Not a simple directed graph,

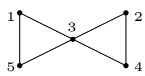


since there is a loop edge incident with vertex 2.

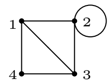
9. Yes.
$2, 1, 3, 4$.

11. $1, 2, 3, 5$ and $1, 2, 4, 5$.
Those are the shortest such paths, and the only ones that do not repeat a vertex.

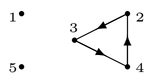
13. We simply remove the directions from each edge in (a).



15. We simply remove the directions from each edge in (c).

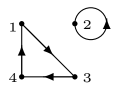


17. Three strong components.



Notice that 1 forms an isolated strong component since there are no edges into 1, and 5 forms an isolated strong component since there are no edges out of 5. Vertices that sit in a circuit, like $2, 3, 4$ here, always sit in the same strong component.

19. Two strong components.



Notice that 2 forms an isolated strong component since there are no edges from other vertices into 2. Vertices that sit in a circuit, like $1, 3, 4$ here, always sit in the same strong component.

21. Yes.
The point is that this graph has just one strong component. That is, the graph is strongly connected.

23. Define $f(1) = 6, f(2) = 5, f(3) = 4$.

25. Define $f(1) = 7, f(2) = 5, f(3) = 8, f(4) = 6$.

27.

vertex	indeg	outdeg
1	0	2
2	1	1
3	2	2
4	1	1
5	2	0

29.

vertex	indeg	outdeg
1	2	1
2	1	3
3	2	1
4	1	1

31.

vertex	indeg	outdeg
1	0	2
2	2	1
3	1	1
4	1	0

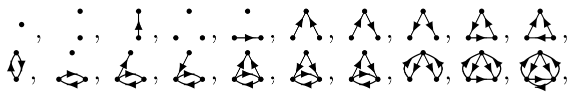
33. In the left graph, vertex 2 has in-degree 3, and no vertex in the right graph has that.
We appeal to the directed version of Lemma 8.10. That is, in-degrees and out-degrees must be preserved by isomorphisms.

35. In the right graph, vertex 5 has in-degree 3, and no vertex in the left graph has that.
We appeal to the directed version of Lemma 8.10. That is, in-degrees and out-degrees must be preserved by isomorphisms.

37. The left graph has one vertex (namely, 1) with in-degree 1, and the right graph has more (namely, $6, 7, 8$). Hence, they are not isomorphic. We appeal to the directed version of Theorem 8.11. That is, in-degree sequences and out-degree sequences must be preserved by isomorphisms.

39. Define $f(1) = 4, f(2) = 3$.

41.



Group them according to the numbers of vertices n and edges m.

$n = 1$

$n = 2$

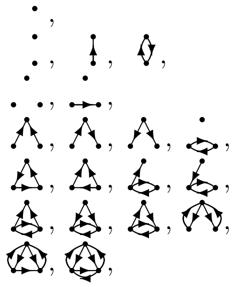
$n = 3, m \leq 1$

$n = 3, m = 2$

$n = 3, m = 3$

$n = 3, m = 4$

$n = 3, m \geq 5$



43. These graphs are distinguished by their in-degree and out-degree sequences.

indeg	outdeg	indeg	outdeg	indeg	outdeg	indeg	outdeg
3	0	3	0	2	1	2	1
1	2	2	1	2	1	2	1
1	2	1	2	2	1	1	2
1	2	0	3	0	3	1	2

45. *Proof.* Suppose G is strongly connected. Let u, v be vertices in $\underline{G}$. We have a path P from u to v in G. The underlying path $\underline{P}$ is a path from u to v in $\underline{G}$. So $\underline{G}$ is connected. $\square$

47. *Proof.* Suppose $G \cong H$. So we have an isomorphism $f : G \longrightarrow H$. In particular, f_V and f_E are bijections. Define $\underline{f} : \underline{G} \longrightarrow \underline{H}$ by $\underline{f}_V(v) = f_V(v)$ and $\underline{f}_E(\underline{e}) = f_E(e)$ for all vertices v and edges e in G. Since G and $\underline{G}$ have the same vertex set, we see that $\underline{f}_V$ is a bijection. Since the map $e \mapsto \underline{e}$ gives a one-to-one correspondence between the edges of G and the edges of $\underline{G}$, we see that $\underline{f}_E$ is a bijection. Since f is a graph map, so is $\underline{f}$. Hence, $\underline{f}$ is an isomorphism. Therefore $\underline{G} \cong \underline{H}$. $\square$
The converse does not hold. In Exercise 43 for example, we see four different directed graphs with underlying graph K_4.

49.
$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} = A, \qquad \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} = A^2.$$

$$\begin{array}{lll} 0(0)+0(1)+1(0)=0, & 0(0)+0(0)+1(1)=1, & 0(1)+0(0)+1(0)=0, \\ 1(0)+0(1)+0(0)=0, & 1(0)+0(0)+0(1)=0, & 1(1)+0(0)+0(0)=1, \\ 0(0)+1(1)+0(0)=1, & 0(0)+1(0)+0(1)=0, & 0(1)+1(0)+0(0)=0. \end{array}$$

51. $A + A^T$.

Let B be the adjacency matrix for $\underline{G}$. Let u and v be any vertices in the common vertex set for G and $\underline{G}$. The number of edges from u to v in $\underline{G}$ plus the number of edges from v to u in $\underline{G}$ equals the number of edges joining u to v in G. That is, the (u, v) entry of A plus the (v, u) entry of A equals the (u, v) entry of B (which is the same as the (v, u) entry of B). Of course, the (v, u) entry of A equals the (u, v) entry of A^T.

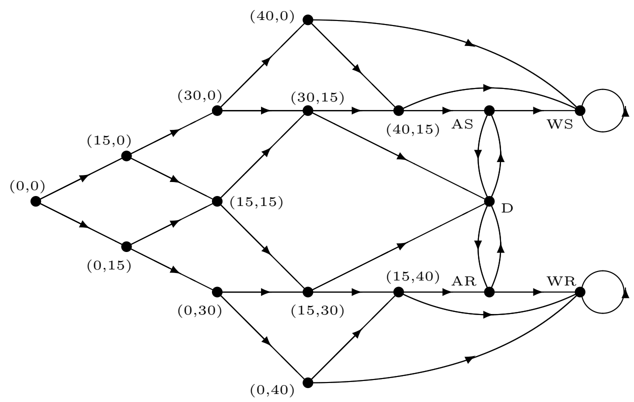
53. Let $A = [a_{i,j}]$ be an adjacency matrix for a loopless directed graph G. Then, for each $1 \leq k \leq n$, we have $\sum_{j=1}^{n} a_{k,j} = \text{outdeg}(v_k)$ and $\sum_{i=1}^{n} a_{i,k} = \text{indeg}(v_k)$. Moreover, $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} = |E|$. The point is that, for each $1 \leq k \leq n$, the sum of the entries in row k is outdeg(v_k) and the sum of the entries in column k is indeg(v_k).

55. No. $p(1, 3) + p(1, 2) = .5 \neq 1$.
That is, the sum of the values assigned to the edges with tail 1 is not 1, as required.

57.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & .4 & .6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & .2 & .8 & 0 \end{bmatrix} \end{array} = M, \qquad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} = M^2.$$

59. (a)



(b) There are 17 states. The three states AS, D, and AR form one class. Each of the other states forms a class by itself. Hence there are 14 single state classes and 1 three state class, for a total of 15.
(c) The 2 states/classes WS and WR are absorbing. The other 13 are transient.

61. For a fixed i, the sum of the weights of the edges of the form (i, j) must be 1, by the definition of a Markov chain graph. The column sums need not be 1. See Exercise 57, for example.
The point is that, for each state i, the sum of the values assigned to the edges

with tail i must be 1. In fact, none of the columns in the matrix in Exercise 57 add up to 1.

63. (a) 0.173 after two at bats, and 0.266 after three at bats.

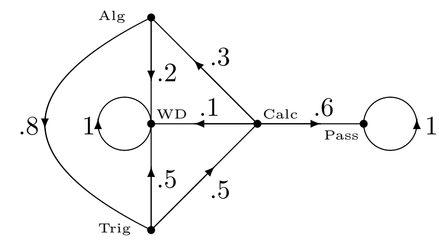
$$\begin{array}{c} \\ 0\\1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & .110 & .058 & .045 & .056 & .731 \\ 0 & .303 & .150 & .163 & .173 & .213 \\ 0 & 0 & .423 & .068 & .343 & .168 \\ 0 & 0 & 0 & .490 & .425 & .085 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \approx M^2$$

$$\begin{array}{c} \\ 0\\1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & .061 & .051 & .048 & .087 & .753 \\ 0 & .166 & .135 & .159 & .266 & .273 \\ 0 & 0 & .275 & .068 & .444 & .213 \\ 0 & 0 & 0 & .343 & .548 & .110 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \approx M^3$$

(b) 0.590. Start matters, since the values in column 4 of M^∞ vary.

$$\begin{array}{c} \\ 0\\1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & .198 & .802 \\ 0 & 0 & 0 & 0 & .590 & .410 \\ 0 & 0 & 0 & 0 & .690 & .310 \\ 0 & 0 & 0 & 0 & .833 & .167 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \approx M^{30}$$

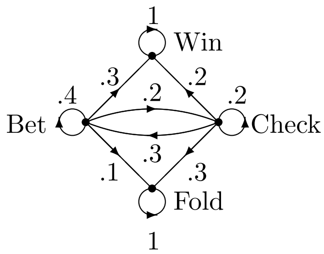
65. (a)



(b) Calc, Trig, and Algebra have period 3. WD and Pass have period 1. Note that WD and Pass are absorbing states, while Alg, Trig, Calc forms a cycle of length 3. (c) 68.2% go from Calc to Pass. 72.7% go from Alg to WD.

$$M^{20} \approx \begin{bmatrix} 0 & 0 & 0 & .318 & .682 \\ 0 & 0 & 0 & .659 & .341 \\ 0 & 0 & 0 & .727 & .273 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

67. (a)



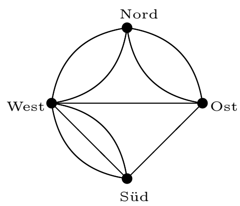
(b)

$$\begin{array}{r c} & \begin{array}{cccc} \text{Bet} & \text{Check} & \text{Fold} & \text{Win} \end{array} \\ \begin{array}{r} \text{Bet} \\ \text{Check} \\ \text{Fold} \\ \text{Win} \end{array} & \left[\begin{array}{cccc} 0 & 0 & .333 & .667 \\ 0 & 0 & .5 & .5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \approx M^{\infty} \end{array}$$

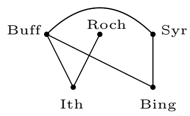
(c) If the computer bets on the first turn, then the computer probably wins. If the computer checks on the first turn, then there is an even chance of winning and losing.
(d) No.
Unless, the computer folds on the first turn, the odds are not in Keith's favor.

Review

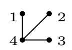
1.



2.

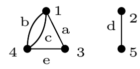


3. (a)



(b) $\emptyset$.
There are no edges among $1, 2, 3$.
(c) 2.
$2, 4, 3$ is a shortest path.
(d) Yes.
$V_1 = \{4\}, V_2 = \{1, 2, 3\}$. That is, $G \cong K_{1,3}$.

4. (a)

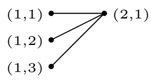


(b) $1, a, 3, e, 4, b, 1$.
The only repetition is that the start equals the end.
(c) No.
There is no path from 1 to 2, for example.
(d) The subgraph induced by $\{1, 3, 4\}$ and the subgraph induced by $\{2, 5\}$.
The two components are shown separated in the picture for part (a).

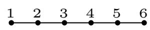
5. (a) Yes.
There are two parallel edges between two vertices.
(b) Yes.
Again, there are two parallel edges between two vertices.

6. (a) Yes.
The are no loops and no multiple edges.
(b) Yes.
The only repetition is that the start equals the end, and there are indeed edges $\{3, 4\}$, $\{4, 5\}$, $\{5, 6\}$, $\{6, 3\}$ that join consecutive vertices in the given list.
(c) Yes.
There are no vertex repetitions, and there are indeed edges $\{4, 5\}$, $\{5, 2\}$, $\{2, 3\}$ that join consecutive vertices in the given list.

7.

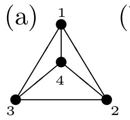


8.



9.

(a)

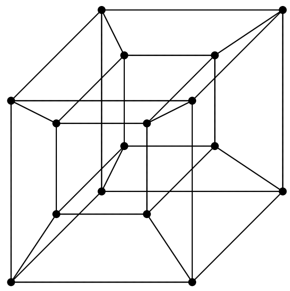


(b) 1,2,3,1,4.

It is a trail, since no edges are repeated. It is not a path, since the vertex 1 is repeated. It is not a cycle, since it does not start and end at the same vertex.

10. (a) See Figure 8.19.
(b) 2,
since they differ in two places.
(c) Yes, 0101, a left-handed male.

(d) 0001 is equidistant from 0000 and 0101.

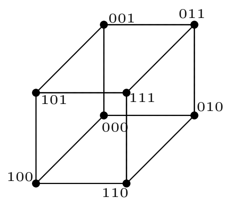


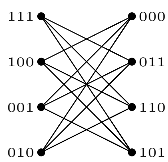
11. *Proof.* Suppose G is complete. Let u, v be vertices in W. Since G is complete, there is an edge $e \mapsto \{u, v\}$ in G. Since u and v are vertices in W, it follows that e is an edge of H. Hence, H is complete. □

12. (a) No.
A loop edge cannot possibly connect vertices from two disjoint sets V_1 and V_2.
(b) Yes.

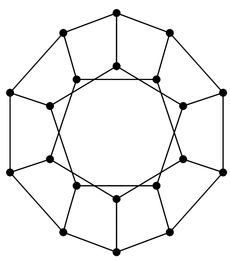
This graph has bipartition $\{1\}, \{2\}$.
(c) Yes.
In fact, C_n is bipartite whenever n is even.

13. Let $V_1 = \{111, 100, 001, 010\}$ and $V_2 = \{000, 011, 110, 101\}$.





14. No. It contains 5-cycles.



Recall that a graph is bipartite iff it contains no cycles of odd length.

15. No. It could contain 3-cycles.
For example, if the class contains a group of three mutual friends.

16. (a)

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(b)

$$\begin{array}{l} 1 : 2, 3 \\ 2 : 1, 4 \\ 3 : 1, 4 \\ 4 : 2, 3, 4 \end{array}$$

17. $\begin{bmatrix} 3 & 7 \\ 6 & 0 \end{bmatrix}$.

18. (a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = P.$

The rows of I_4 are permuted according to the permutation $2, 4, 3, 1$.

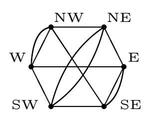
(b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = PA$

E.g., the entry in the second row and third column of PA is
$0(1) + 0(0) + 0(0) + 1(1) = 1$.
(c) It is the adjacency matrix relative to the vertex ordering $2, 4, 3, 1$.

$$\begin{array}{c} \\ 2 \\ 4 \\ 3 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 4 & 3 & 1 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} = PAP^T$$

19. 3. There are 3 walks of length 4 from 1 to 3.

20.



21. Define $f(1) = 5, f(2) = 6, f(3) = 8,\ f(4) = 7$.

22. Define $f(1) = 2, f(a) = c, f(b) = d$.

23. Define $f(1) = 3, f(2) = 6, f(3) = 5,\ f(4) = 4, f(5) = 1, f(6) = 2$.

24. *Proof.* For each vertex $b_1b_2\cdots b_n$ in Q_n, we define an automorphism $f_{b_1b_2\cdots b_n}$ of Q_n such that $f_{b_1b_2\cdots b_n}(00\cdots 0) = b_1b_2\cdots b_n$. That is, for each vertex $a_1a_2\cdots a_n$ in Q_n define $f_{b_1b_2\cdots b_n}(a_1a_2\cdots a_n) = c_1c_2\cdots c_n$, where $c_i = a_i + b_i \bmod 2$, for each $1 \le i \le n$. Now given any two vertices $u_1u_2\cdots u_n$ and $v_1v_2\cdots v_n$ in Q_n, observe that the automorphism $f_{v_1v_2\cdots v_n} \circ f^{-1}_{u_1u_2\cdots u_n}$ sends $u_1u_2\cdots u_n$ to $v_1v_2\cdots v_n$. $\square$

25. No.
They are not regular. Vertex transitive graphs must be regular.

26. No.
The Petersen graph has 10 vertices, not 12.

27. Define $f(1) = f(4) = a, f(2) = b,\ f(3) = c$.

28. 2.
The identity and the automorphism switching b and c are the only automorphisms.

29. $8 \cdot 6 = 48$.
There are only 24 automorphism of a die, since we cannot take the mirror image of the die as an automorphism. Note that $48 = 24 \cdot 2$.

30. No. There is no K_4 in the octahedron.
Note that the Tetrahedron is isomorphic to K_4. That whole graph must map into a subgraph H of the Octahedron with $H \cong K_4$. However, no such H exists.

31. It follows from Corollary 8.14.
We have $2|E| = \sum_{v\in V} \deg(v) = \sum_{v\in V} r = r|V|$. Since 2 divides $r|V|$ and $\gcd(2, r) = 1$, it follows that 2 divides $|V|$. That is, $|V|$ is even.

32. The graph on the left has maximum degree 4, while that on the right has

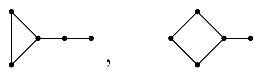
maximum degree 5.
Isomorphic graphs must have the same maximum degree, and these two do not.

33. The degree sequences are different.
A degree sequence for the graph on the left is $5, 3, 3, 3, 3, 3, 2$, and a degree sequence for the graph on the right is $5, 5, 3, 3, 2, 2, 2$. Isomorphic graphs must have a common degree sequence, and these two do not.

34. Let U be the set of vertices in the left graph of degree 2 or 3, and similarly define W for the right graph. The subgraphs induced by U and W are not isomorphic, so the graphs cannot be isomorphic.
The subgraph induced by U is $P_1 + P_3$, and the subgraph induced by W is $P_2 + P_2$.

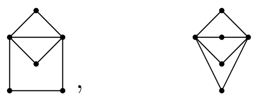
35. No.
The left graph has a vertex of degree 4 and the right graph does not.

36.



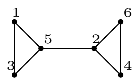
Let v be the vertex of degree 3. The vertex not adjacent to v is adjacent to either 1 or 2 neighbors of v.

37.



The two degree 4 vertices are either adjacent or not.

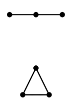
38. This is the complement of the left graph.



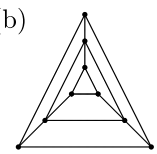
Define $f(1) = 1$, $f(2) = 4$, $f(3) = 2$, $f(4) = 6$, $f(5) = 3$, $f(6) = 5$.

39. *Proof.* Observe that $(G^c)^c$ and G have the same vertex sets and the same edge sets. Namely, $V_{(G^c)^c} = V_{G^c} = V_G$ and $E_{(G^c)^c} = (E_{G^c})^c = ({E_G}^c)^c = E_G$. Since the edge sets are subsets of $\mathcal{P}_2(V)$, the graphs $(G^c)^c$ and G are the same. $\square$

40. (a)

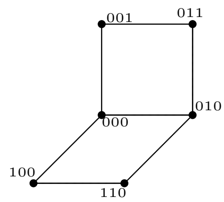


(b)



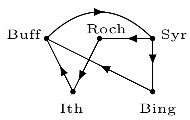
Part (a) is a disjoint union of the path P_3 and the cycle C_3. Part (b) is the product of the path P_3 with the cycle C_3. That is, we have 3 copies of C_3, with corresponding vertices connected in a path (namely, P_3).

41. (a) The subgraph induced by $\{000, 010\}$.
(b) The subgraph induced by $\{000, 001, 010, 011, 100, 110\}$.
That is, $G \cap H$ is isomorphic to P_2, and $G \cup H$ is the following graph.

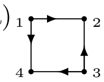


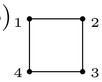
42. *Sketch.* For all $u \in V_G$, $v \in V_H$, $d \in E_G$, and $e \in E_H$, define $f(u, v) = (v, u)$, $f(d, v) = (v, d)$, and $f(u, e) = (e, u)$. This gives an isomorphism
$f : G \times H \longrightarrow H \times G$. $\square$
Recall that $G \times H$ has vertex set $V_G \times V_H$ and edge set $(E_G \times V_H) \cup (V_G \times E_H)$. We have defined f on each of the sets $V_G \times V_H$, $E_G \times V_H$, and $V_G \times E_H$. It is easy to see that f_V and f_E are bijections. Also, f is a graph map.

43.

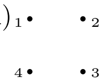


44.

(a) 

(b) 

(c) All of G is one.

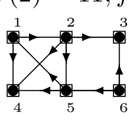
(d) 

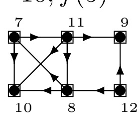
(e)

vert.	in.	out.
1	0	2
2	2	0
3	0	2
4	2	0

45. Yes.
The two edges that are parallel in the underlying graph have distinct directions in the directed graph. So there are no loops and no multiple edges.

46. Define $f(1) = 7, f(2) = 11, f(3) = 9, f(4) = 10, f(5) = 8, f(6) = 12$.





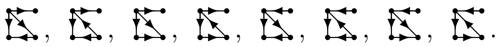
47. The left graph has a vertex with out-degree 3, and the right graph does not. The out-degree of a vertex must be preserved by an isomorphism. So no isomorphism is possible.

48. They are not. The left graph has a vertex with out-degree 0, and the right graph does not.
The out-degree of a vertex must be preserved by an isomorphism. So no isomorphism is possible.

49.

They are distinguished by the number of edges.

50.

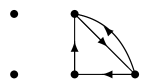


The edge incident with the pendant vertex in $\underline{G}$ can be assigned two possible directions. For each such choice, there are four ways that the remaining triangle in $\underline{G}$ can be directed.

51.

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

52. No, its strong components are two isolated vertices and the subgraph induced by the other three.

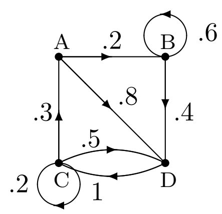


53. No. The only edge out of vertex 3 has value .2 and not 1 as required.

54.

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & .2 & .8 & 0 \\ .3 & 0 & .7 & 0 \end{bmatrix} \end{array}$$

55. (a)



(b) 0.08 is the relevant value in M^2.
(c) Irreducible, since M^3 has all nonzero entries. Regular, since all states are aperiodic.
(d) $\frac{4}{9} \approx 0.4444$.

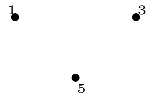
$$\begin{bmatrix} 0 & .12 & .8 & .08 \\ 0 & .36 & .4 & .24 \\ .06 & .06 & .54 & .34 \\ .3 & 0 & .2 & .5 \end{bmatrix} = M^2 \qquad \begin{bmatrix} .24 & .072 & .24 & .448 \\ .12 & .216 & .32 & .344 \\ .162 & .048 & .448 & .342 \\ .06 & .06 & .54 & .34 \end{bmatrix} = M^3$$

$$\begin{bmatrix} .1333 & .0666 & .4444 & .3555 \\ .1333 & .0666 & .4444 & .3555 \\ .1333 & .0666 & .4444 & .3555 \\ .1333 & .0666 & .4444 & .3555 \end{bmatrix} = M^{30} \qquad \text{to 4 decimal places.}$$

2.9 Chapter 9

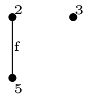
Section 9.1

1.



Since each edge is incident with vertex 2 or vertex 4, all edges get removed.

3.



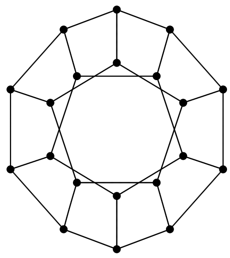
Note that edge c is removed since it is in F and not due to any of the vertex removals from W.

5. $\kappa = 1$.
Proof. Since (a) is connected, $\kappa \geq 1$. Since $\{4\}$ is a disconnecting set of size 1, $\kappa \leq 1$. Therefore, $\kappa = 1$. □

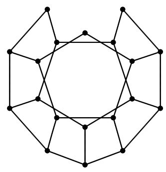
7. $\kappa = 2$.
Proof. Observe that (c) is connected and that the removal of any one vertex leaves a connected graph (check all 6 cases). Hence, $\kappa \geq 2$. Since $\{1, 5\}$ is a disconnecting set of size 2, $\kappa \leq 2$. Therefore, $\kappa = 2$. □

9. $\kappa = 1$.
Proof. Since (e) is connected, $\kappa \geq 1$. Since $\{4\}$ is a disconnecting set of size 1, $\kappa \leq 1$. Therefore, $\kappa = 1$. □

11. $\kappa = 3$.



Proof. The graph resulting from the removal of the topmost pictured vertex cannot be disconnected by the removal of only one more vertex.



Since the dodecahedron is vertex transitive, this shows that no two vertices can form a disconnecting set. Hence, $\kappa \geq 3$. Since $\kappa \leq \delta = 3$, it follows that $\kappa = 3$. $\square$

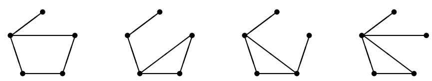
13. (a) $K_{3,3}$. See Exercise 13 from Section 8.4.
(b) $\kappa = 3$, by Theorem 9.4. That is, $\kappa(K_{3,3}) = \min\{3, 3\} = 3$.
(c) No. That makes $\kappa = \delta = 2 < 3$. No matter what cable is removed, the resulting graph will have minimum degree 2.

15. *Proof.* Suppose G is connected and $\delta(G) = 1$. So $1 \leq \kappa(G) \leq \delta(G) = 1$. Hence, $\kappa(G) = 1$. $\square$
Since G is connected, $1 \leq \kappa(G)$. Whenever we have an inequality $x \leq y \leq z \leq x$ with the same number on both ends, all of the inequalities are forced to be equalities.

17. (a) *Proof.* Suppose G has n vertices and n edges. Then,

$$n\delta(G) \leq \sum_{v \in V} \deg(v) = 2|E| = 2n.$$

Thus, $\kappa(G) \leq \delta(G) \leq 2$. Since $\kappa(C_n) = 2$, the cycle C_n has the highest possible connectivity. $\square$
(b) See Theorem 9.3.
(c)



(d) Probably the last, since the greatest number of components can be left by removing a single vertex.

19. *Proof.* Suppose a graph $G = (V, E)$ has $|V| = n$ and $|E| < \lceil \frac{3n}{2} \rceil$. Hence, $n\delta(G) \leq \sum_{v \in V} \deg(v) = 2|E| < 2\lceil \frac{3n}{2} \rceil$. That is, $\delta(G) < \frac{2}{n} \lceil \frac{3n}{2} \rceil$. In both the case that n is even and the case that n is odd, we see that $\delta(G) < 3$. Since $\delta(G)$ is an integer, it follows that $\kappa(G) \leq \delta(G) \leq 2$. $\square$

21. *Proof.* Suppose G is a 2-regular graph. By Theorem 9.3, each component of G must be a cycle. Of course, G is the disjoint union of its components. So G is a disjoint union of cycles. $\square$

23. *Sketch.* Let D be a κ-set for G, and let $v \in D$. Suppose v is not adjacent to some component of $G \setminus D$. Then, $D' = D \setminus \{v\}$ disconnects G. Thus, $\kappa(G) \leq |D'| < |D|$, a contradiction. $\square$
Let C_1 be a component of $G \setminus D$ to which v is not adjacent, and let C_2 be a different component. The point is that C_1 and $C_2 \cup \{v\}$ cannot be joined by a path in $G \setminus D'$. Hence, $G \setminus D'$ will have at least two components.

25. Let H and J be two disjoint copies of K_{d+1}. Let u be a vertex from H,

and let v be a vertex from J. Make a new graph G from the disjoint union of $H \setminus \{u\}$ and $J \setminus \{v\}$ by adding a new single vertex w. Connect w to all of the neighbors of u from H and all of the neighbors of v from J, so w will have degree $2d$. Note that $\delta(G) = d$, and $\{w\}$ is a κ-set. Hence, $\kappa(G) = 1$.

27. Note that $G \times P_2$ contains subgraphs $G \times \{1\}$ and $G \times \{2\}$ that are isomorphic copies of G. Let D be a κ-set for $G \times P_2$. At least one of the copies of G must be disconnected in $(G \times P_2) \setminus D$. Moreover, D must contain at least one vertex in each copy of G. So at least one of the vertices of D was not needed to disconnect the copy of G that got disconnected. Hence, $\kappa(G) \leq \kappa(G \times P_2) - 1$.
Say $G \times \{1\}$ is a copy of G that gets disconnected by the removal of D. There must be at least one vertex v in both $G \times \{2\}$ and D. So in fact, $G \times \{1\}$ is disconnected by $D \setminus \{v\}$, a set of size $\kappa(G \times P_2) - 1$.

29. $\lambda = 1$.
Proof. Observe that (a) is connected. Hence, $\lambda \geq 1$. Since $\{\{1, 4\}\}$ is a disconnecting set of edges of size 1, $\lambda \leq 1$. Therefore, $\lambda = 1$. □

31. $\lambda = 2$.
Proof. By the result in Exercise 7, $2 = \kappa \leq \lambda$. Since $\{\{6, 1\}, \{1, 2\}\}$ is a disconnecting set of edges of size 2, $\lambda \leq 2$. Therefore, $\lambda = 2$. □

33. $\lambda = 2$.
Proof. Observe that (e) is connected and the removal of a single edge will not disconnect the graph. Hence, $\lambda \geq 2$. Since $\{\{4, 5\}, \{5, 6\}\}$ is a disconnecting set of edges of size 2, $\lambda \leq 2$. Therefore, $\lambda = 2$. □

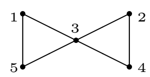
35. $\lambda = 3$.
The graph is 3-regular. It follows from Theorem 9.6 and Exercise 11 that $3 = \kappa = \lambda$.

37. $\lambda = 3$. There are multiple λ-sets.
The graph is 3-regular. It follows from Theorem 9.6 and Exercise 13(b) that $3 = \kappa = \lambda$. For each vertex, the set of edges incident with it forms a λ-set.

39. *Proof.* Suppose $\lambda(G) \leq 1$. If G is connected, then $1 \leq \kappa \leq \lambda \leq 1$. So $\kappa = \lambda = 1$. If G is not connected, then $\kappa = \lambda = 0$. □

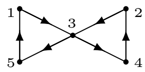
41. $n - 1 = \kappa(K_n) \leq \lambda(K_n) \leq \delta(K_n) = n - 1$.
See Remark 9.1 and Theorem 9.5. Note that the inequalities force $\lambda(K_n) = n-1$.

43.



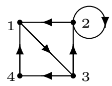
$\kappa = 1$ and $\lambda = \delta = 2$ for the pictured graph.

45. (a) $\kappa = 1$. (b) $\lambda = 1$.
The graph is strongly connected, $\{3\}$ is a κ-set, and $\{(1,3)\}$ is a λ-set.



Note that the resulting graph (after removal of $\{3\}$ or $\{(1,3)\}$) is not strongly connected.

47. (a) $\kappa = 1$. (b) $\lambda = 1$.
The graph is strongly connected, $\{1\}$ is a κ-set, and $\{(2,1)\}$ is a λ-set.



Note that the resulting graph (after removal of $\{1\}$ or $\{(2,1)\}$) is not strongly connected.

49. $\kappa = \lambda = 1$. G is strongly connected, so something must be removed to disconnect it. The unique east-west one-way street is the key. Removing it or one of its ends causes the graph to no longer be strongly connected.

51. If S disconnects $\underline{G}$, then S disconnects G.
This works for both a disconnecting set S and a disconnecting set of edges S.

Section 9.2

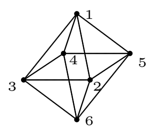
1. Neither. There are four vertices of odd degree. Namely, $1, 2, 4, 5$. We must have none for an Euler circuit, and two for an Euler trail.

3. An Euler trail. $1, c, 3, f, 4, d, 2, a, 1, b, 2, e, 5, h, 5, g, 4$.
Since vertices 1 and 4 have odd degree, there is no Euler circuit.

5. An Euler trail. $5, 6, 1, 2, 3, 4, 5, 2$.
Since vertices 2 and 5 have odd degree, there is no Euler circuit.

7. An Euler circuit. $6, n, 8, l, 7, j, 6, k, 7, o, 10, q, 9, p, 9, m, 6$.
Note that each vertex has even degree.

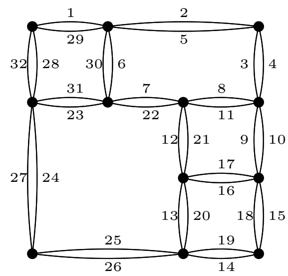
9. An Euler circuit. The graph is 4-regular.



$1, 2, 3, 6, 5, 4, 1$.

11. A (romantic) Euler trail.
The graph is connected and has exactly 2 vertices of odd degree (Nord and Sud). Starting at Nord the following sequence of bridge crossings ends at Sud: $9, 10, 11, 8, 7, 12, 13, 6, 5, 15, 14, 1, 4, 3, 2$.

13.



An Euler circuit is specified that starts and ends at the upper-left vertex in this graph. It corresponds to a route that covers each side of each street exactly once.

15. Yes. Connect the two odd-degree vertices with an edge.
The degrees of these two vertices now become even, and the remaining vertices retain their even degrees. Thus, Euler's Theorem applies to this new graph.

17. No. c.
Since a component is a graph, it cannot have an odd number of odd degree vertices. Pick one vertex in each component and form a cycle with those vertices.

19. The layout of the hallways determines a graph in which vertices represent intersections (cross ways) or corners and edges represent hallways. Since each hallway must be both mopped and waxed, each edge is doubled. Thus, every vertex has even degree and Euler's Theorem applies.
Note that, since we are in a single building, we assume that this graph is connected.

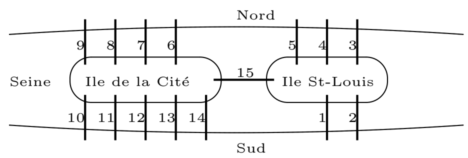
21. Vertex repetitions in an Euler circuit determine where to begin and end the cycles. Each cycle will use two edges at each of its vertices.
See the proof of Euler's Theorem and its preceding example. An Euler circuit is built by pasting together cycles.

23. Even n.
In Q_n, the degree of each vertex is n, and we need these degrees to be even.

25. Both m and n even.
In $K_{m,n}$, some vertices have degree n and some have degree m. We need all degrees to be even.

27. 2.
There is an Euler trail between the two vertices of degree 3. Then two more edges are required to return to the start.

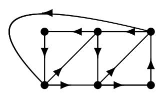
29. (a) 2. (b) 2.



In (a), you can double bridges 9 and 10. In (b), you can double bridges 9 and 1.

31. Neither.
outdeg$(1) = 2 +$ indeg(1). See Theorem 9.8.

33. An Euler circuit.
Observe that the graph is strongly connected and each vertex v has outdeg$(v) =$ indeg(v).



35. Neither.
outdeg$(a) = 3 +$ indeg(a). See Theorem 9.8.

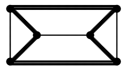
37. For each edge e, the addition of the edge e' forces a balance between in- and out-degrees in G'. Let u and v be two vertices in G. Since G is weakly connected, there is a path from u to v, say. Using the the new edges in G', we can 'reverse' that path to form a path from v to u. Hence G' is strongly connected.

39. Mimic the proof of Theorem 9.7(a).
We must travel along edges in their correct direction, but that does not restrict us in our argument.

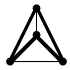
41. Use a directed graph.
That is, make each side of the street a directed edge that points in the legal direction of travel. Theorem 9.8 now does the work for us and guarantees an Euler circuit.

Section 9.3

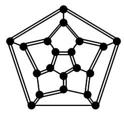
1.



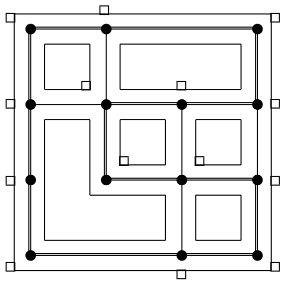
3.



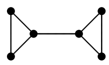
5.



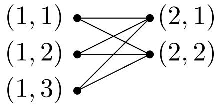
7.



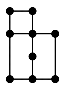
9. The edges incident with degree-2 vertices must be included. However, premature 3-cycles are then formed.



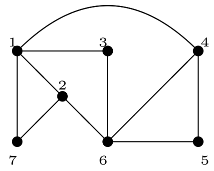
11. Since, the edges incident with degree-2 vertices must be included, all edges get included. However, a Hamiltonian cycle is not formed.



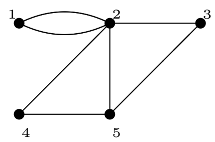
13. After all of the edges incident with degree-2 vertices get included, the bottom middle vertex is then incident with 3 edges of the cycle, which is impossible.



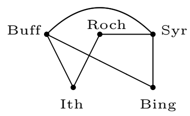
15. It is not Hamiltonian. After all of the edges incident with degree-2 vertices get included, the result is a Hamiltonian path that cannot be completed to a Hamiltonian cycle.



17. It is not Hamiltonian. Since vertex 1 has degree 2, the two parallel edges must be included, and form a premature 2-cycle.

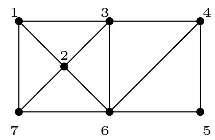


19. Yes. Syracuse, Rochester, Ithaca, Buffalo, Binghamton, Syracuse.



21. $K_{m,n}$ is Hamiltonian iff $m = n$.
Notice that a Hamiltonian cycle must alternate between the sets V_1 and V_2 of a bipartition. If $m \neq n$, so $|V_1| \neq |V_2|$, then it will be impossible to end in the same set from which you started, while covering every vertex.

23. There are 3. Namely, $1, 2, 3, 4, 5, 6, 7, 1$ and $1, 2, 7, 6, 5, 4, 3, 1$ and $1, 3, 4, 5, 6, 2, 7, 1$.



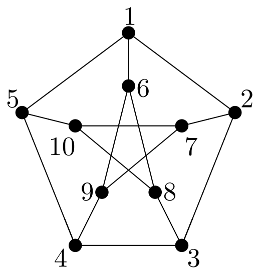
Notice that the different Hamiltonian cycles are distinguished by the choice of edges that pass through vertex 2.

25. $\forall\, n \geq 3$, $\frac{(n-1)!}{2}$. Fix a vertex to be considered the starting vertex for the Hamiltonian cycles. There are $n-1$ choices for the second vertex, $n-2$ choices for the third vertex, and so on. Multiplication gives $(n-1)!$. However, we must divide this count by 2, since each sequence counted above gives the same Hamiltonian cycle as its reverse.

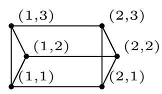
27. Note that $\kappa(K_{k+1,k}) = k$ but $K_{k+1,k}$ is not Hamiltonian, by Exercise 21.

29. Suppose $\frac{n}{2} \leq \kappa(G)$. By Theorem 9.2, $\frac{n}{2} \leq \kappa(G) \leq \delta(G)$. Hence, Theorem 9.11 says that G must be Hamiltonian.

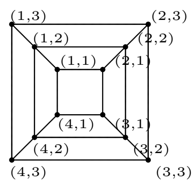
31. See Example 8.23, and take advantage of the symmetries. We may assume that $1, 2, 3$ is part of a Hamiltonian cycle. If the cycle further contains $1, 2, 3, 4$, then it suffices to assume that it contains $1, 2, 3, 4, 5$, and we see that this cannot be extended to a Hamiltonian cycle. If the cycle instead contains $1, 2, 3, 8$, then it suffices to assume that it contains $6, 1, 2, 3, 8$, and we see that this cannot be extended to a Hamiltonian cycle.



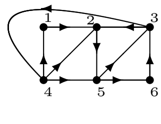
33. $(1,1), (1,2), \ldots, (1,n), (2,n), (2,n-1), \ldots, (2,1), (1,1)$ is a Hamiltonian cycle. For a concrete example, consider the case in which $n = 3$.



35. $(1,1), (1,2), \ldots, (1,n), (2,n), (2,n-1), \ldots, (2,2), (3,2), (3,3), \ldots, (3,n),$ $(3,1), (2,1), (1,1)$ is a Hamiltonian cycle. For a concrete example, consider the case in which $n = 4$.

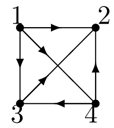


37. $1, 2, 5, 6, 3, 4, 1$.



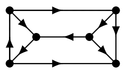
Notice that each edge is traversed in its correct direction.

39. $1, 4, 3, 2$.



Notice that each edge is traversed in its correct direction.

41. The lower-right vertex has out-degree zero.



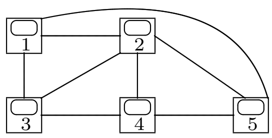
That vertex can be entered but not exited.

43. (a) No. Ann defeated everyone else.
(b) Yes. Ed, Ann, Bob, Cari, Dan.
(c) Yes. There is a Hamiltonian cycle, whose start can be freely chosen.

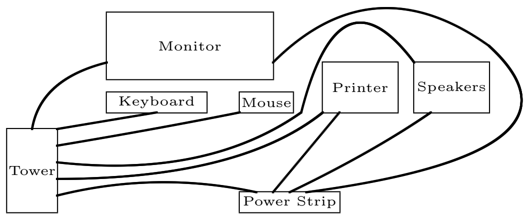
45. *Sketch.* Let v be a vertex with the maximum possible out-degree. Let A_{out} be the set of heads of edges with tail v. Let A_{in} be the set of tails of edges with head v. Let u be any vertex in A_{in}. If there is no edge with tail in A_{out} and head u, then u has higher out-degree than v, a contradiction. Hence, u is distance 2 from v. $\square$

Section 9.4

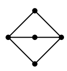
1.



3.

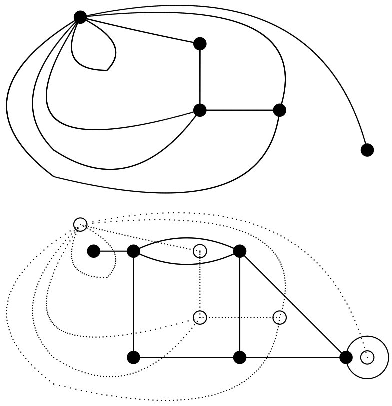


5.

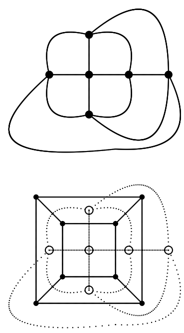


7.

9.

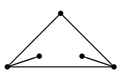
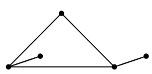


11.



13. $|R| = 8$.
$|V| - |E| + |R| = 2$ gives $10 - 16 + |R| = 2$.

15.

 versus 

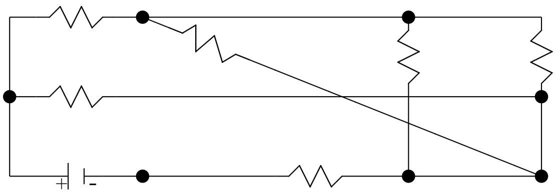
These two graphs are clearly isomorphic. The dual of the embedding on the left has a vertex with two loops, while the dual of the embedding on the right does not. So the duals are not isomorphic.

17. $|V| - |E| + |R| = c + 1$.
For $1 \leq i \leq c$, the i^{th} component by itself satisfies $|R_i| = |E_i| - |V_i| + 2$. Since they all share the same outer region, adding these equations gives $|R| + (c-1) = |E| - |V| + 2c$.

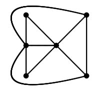
19. *Proof.* Suppose G contains a subdivision H of K_5 or $K_{3,3}$. Suppose to the contrary that G is planar. Then H is planar, and hence K_5 or $K_{3,3}$ is planar. This is a contradiction. □

21. $\forall\, n \geq 5$, K_n contains K_5 as a subgraph.
By Kuratowski's Theorem, K_n is therefore not planar.

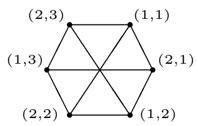
23. It is not planar, since it contains a $K_{3,3}$ subdivision.



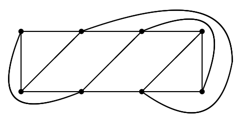
25. It is planar, as shown.



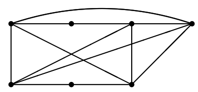
27. This is $K_{3,3}$, which is not planar.



29. It is planar, as shown.



31. It is not planar. Deleting the bottom right vertex leaves a K_5-subdivision.

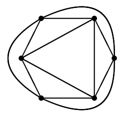


33. K_5 is the only one. It is the only one that contains a subdivision of K_5, and a subdivision of $K_{3,3}$ is not possible on 5 vertices.

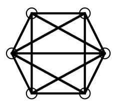
35. *Proof.* Let G be a planar graph. Suppose to the contrary that $\delta(G) \geq 6$. Then $3|V| = \frac{1}{2}(6|V|) \leq \frac{1}{2}\sum_{v\in V}\deg(v) = |E| \leq 3|V| - 6$, a contradiction. $\square$
Note that $\forall\, v \in V, \deg(v) \geq 6$. Hence, $\sum_{v\in V}\deg(v) \geq 6|V|$.

37. Let $V = \{0, 1, \ldots, n-1\}$, $E_0 = \{\{0,1\}, \{0,2\}, \ldots, \{0, n-2\}\}$, $E_1 = \{\{1,2\}, \{2,3\}, \ldots, \{n-3, n-2\}, \{n-2, 1\}\}$, $E_2 = \{\{n-1, 1\}, \{n-1, 2\}, \ldots, \{n-1, n-2\}\}$, $E = E_0 \cup E_1 \cup E_2$, and $G = (V, E)$. Draw the cycle induced by $\{1, 2, \ldots, n-2\}$ in the unit circle, put 0 at the origin, and put $n+1$ outside of the unit circle. So G can be seen to be planar. Note that $|V| = n$ and $|E| = 3(n-2) = 3n-6$.
Note that, when $n = 6$, the graph G is the Octahedron.

39. A planar embedding is pictured.



A single edge added must be a diameter across the picture. The resulting graph then contains $K_{3,3}$ as a subgraph. So it is not planar.

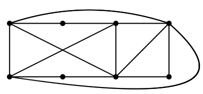


41. $\nu = 0$ by Exercise 25.
The graph is planar.

43. $\nu = 1$ by Exercise 27 and Example 9.19.
The graph is not planar, and there exists a drawing with one crossing.

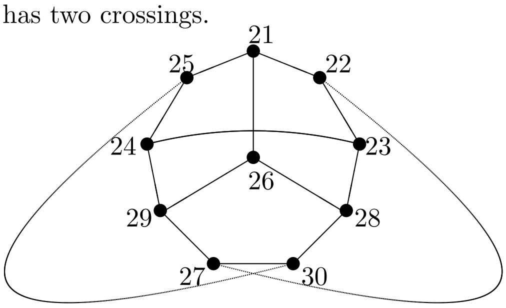
45. $\nu = 0$ by Exercise 29.
The graph is planar.

47. $\nu = 1$,



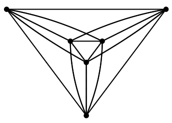
by Exercise 31 and the pictured drawing.

49. This drawing has two crossings.

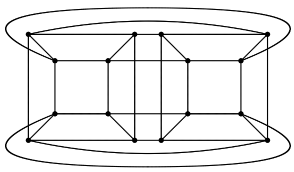


See Example 8.23.

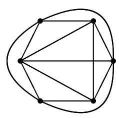
51.



53.



55. One crossing is possible as shown.



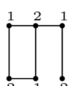
Now apply the result in Exercise 39. An edge has been added to the graph from Exercise 39. So, at least one crossing is necessary.

57. True. The properties of being connected and having only even degrees are preserved in subdivisions. Subdividing edges simply introduces new vertices of degree 2.

Section 9.5

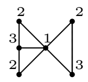
1. No. Two adjacent vertices are both colored with color 2.

3. $\chi = 2$.



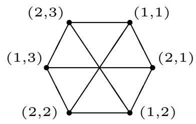
We see that a 2-coloring exists. The presence of an edge makes 2 colors necessary.

5. Since $\omega = 3$ and a 3-coloring is pictured, $\chi = 3$.



That is, $3 = \omega \leq \chi \leq 3$. So, equalities must hold throughout.

7. $\chi = 2$. This graph is $K_{3,3}$.



See Theorem 9.19.

9. Let (V_1, V_2) be a bipartition. Use color 1 on the vertices in V_1 and color 2 on V_2. Since edges only join V_1 to V_2, no edge will join vertices of the same color.

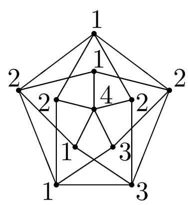
11. 2.

13. 6

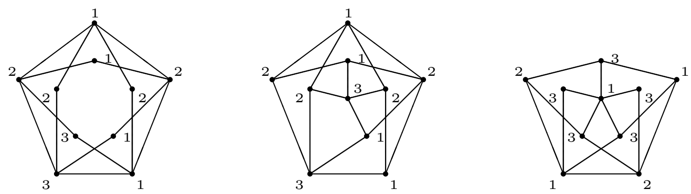
15. 2

17. 1

19. *Sketch.* The outer 5-cycle is "uniquely" 3-colorable. This then forces all three colors to be used on the 5 neighbors of the center vertex. Now, the center vertex requires a fourth color. □



21. By symmetry, it suffices to consider three cases.

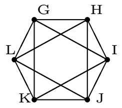


None is bipartite, and each has a 3-coloring.

23.

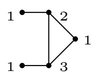
Time Period	Committee Meeting
1	German, Japanese
2	History, Kuwait
3	Indochina, Latin

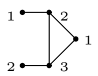
Note that German, History, and Indochina form a clique requiring three colors.



25. Since C_5 is not bipartite, 3 colors are needed. One color class must be of size 1. The other color classes are then forced.
Let G and H be graphs isomorphic to C_5 that have been colored. Let v in G and w in H be the vertices comprising the color classes of size one. Note that the neighbors of v must receive different colors (think about this). Similarly, the neighbors of w receive different colors. Map v to w and map the neighbors of v to the neighbors of w (in either of two possible ways). This determines the rest of an isomorphism from G to H as well.

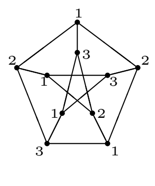
27. The sizes of the color classes do not match up in the pictured colorings.



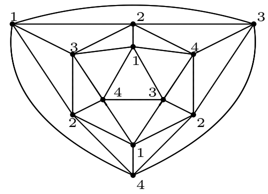


The left coloring has a color class of size 3, and the right coloring does not. Hence, no isomorphism from the left graph to the right can send the color class of size 3 to another color class of size 3.

29. $\chi = 3$. The 5-cycles need 3 colors, and a 3-coloring is easily found.

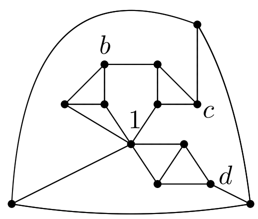


31. $\chi = 4$. A 4-coloring is easily found. A 3-coloring is seen to be impossible, by trying to construct one.



Color the vertices of a triangle with three colors. This forces the colors on adjacent triangles. Further extending this process leads to a conflict.

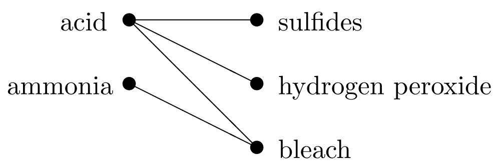
33. $\chi = 4$. Try starting a 3-coloring from the middle vertex, and observe that it fails.



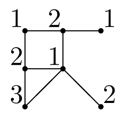
Observe that vertices b, c, and d must also receive color 1. It is now impossible to color the outer triangle.

35. For any subset W of V, the set W is independent in G^c iff W induces a clique in G.
Hence, an independent set W in G^c of largest possible size will also be a clique in G of largest possible size. That is, $\alpha(G^c) = \omega(G)$. Since $G^{cc} \cong G$, we also have $\alpha(G) = \alpha(G^{cc}) = \omega(G^c)$.

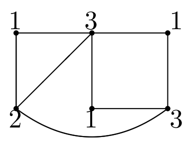
37. The graph with vertices {acid, bleach, sulfides, ammonia, hydrogen peroxide}, whose edges reflect potential dangerous chemical reactions, has chromatic number 2. Putting acids and ammonia in cabinet 1 and bleach, sulfides and hydrogen peroxide in cabinet 2 is safe.



39.

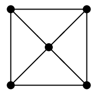


41.



43. *Proof.* ($\rightarrow$) Suppose G and H are bipartite. So $\chi(G), \chi(H) \leq 2$, by Theorem 9.19. Thus, $\chi(G \times H) \leq 2$, by Theorem 9.28. Hence, $G \times H$ is bipartite, by Theorem 9.19. ($\leftarrow$) Suppose $G \times H$ is bipartite. So $\chi(G), \chi(H) \leq \max\{\chi(G), \chi(H)\} = \chi(G \times H) \leq 2$. Hence, G and H are bipartite. $\square$

45. (a)



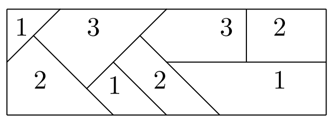
(b) Use the result from Exercise 22, together with the observation that vertex n must receive a color different from those of $1, 2, \ldots, n-1$.
That is, $\chi(W_n) = 1 + \chi(C_{n-1})$.

47. Use the Greedy Coloring Algorithm, and color the highest-degree vertex first. At most $d_2 + 1$ colors will be used.
That is, when coloring a particular vertex, no more than d_2 colors can ever be adjacent to it. So some color in $\{1, 2, \ldots, d_2 + 1\}$ must suffice.

49. (a) Say v_1 and v_2 are combined to form v. A $\chi(G')$-coloring of G' gives a $\chi(G')$-coloring of G with v_1 and v_2 the same color.
(b) Let G be C_6 and identify two opposite vertices (such as 1 and 4).

51. *Sketch.* Let v be a vertex such that $G \setminus \{v\}$ is disconnected, and let $H_1, \ldots, H_c$ be the components of $G \setminus \{v\}$. Argue that, for each $1 \leq i \leq c$, the subgraph induced by $H_i \cup \{v\}$ can be colored with at most $\Delta(G)$ colors. Further, all of these colorings can be arranged to give v the same color. $\square$

53. 3.



55. Refer to an atlas.
(a) By the Four Color Theorem, the map can be colored using only four colors.
(b) 4.
(c) The states PA, MD, VA, KY, and OH form a 5-cycle that requires 3 colors by Exercise 22. Since WV is adjacent to each of these, a fourth color is required. See Exercise 45 as well.

Review

1. (a) G is connected, and the removal of the central vertex disconnects it.
(b) No single edge disconnects G, but two do. Note that $\delta = 2$ and $\lambda \leq \delta$.

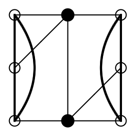
2. $5 = \min\{5, 7\}$. See Theorem 9.4.

3. 4, since $4 = \kappa \leq \lambda \leq \delta = 4$. See Remark 9.1 and Theorem 9.5. Note that the inequalities are forced to be equalities.

4. $\kappa = 1$ for all paths on 2 or more vertices.
See Theorem 9.1.

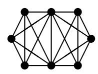
5. 2, since $2 = \kappa \leq \lambda \leq \delta = 2$.
See Theorems 9.1 and 9.5. Note that the inequalities are forced to be equalities.

6. $\kappa(G) = 2$, since the top-middle and bottom-middle vertices form a disconnecting set, and no single vertex does.



7. False. $\kappa(G) \leq \lambda(G)$.
See Theorem 9.5.

8. True.



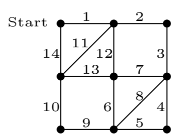
The pictured graph G has $\kappa(G) = 2$ and $\lambda(G) = 4$. The top-middle and bottom-middle vertices form a κ-set. The edges incident with the left-most vertex form a λ-set.

9. *Proof.* Let u and v be the unique pair of vertices not joined by an edge. Then $V \setminus \{u, v\}$ is the *only* disconnecting set for G, and its size is $n - 2$. Thus $\kappa(G) = n - 2$. $\square$

10. Since $\kappa(G) \leq \lambda(G) \leq \delta(G) = 3$, it suffices to observe that no two vertices disconnect G.
Observe that the graph is vertex transitive. The graph obtained by removing any one vertex cannot be disconnected by the removal of just one more vertex. Hence, disconnecting sets for the original graph have size at least 3.

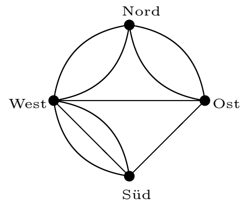
11. $\lambda = \min\{m, n\}$, since $\min\{m, n\} = \kappa \leq \lambda \leq \delta = \min\{m, n\}$.
The inequalities are forced to be equalities. See Theorems 9.4 and 9.5.

12.

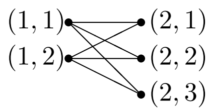


13. There are four vertices of odd degree.
There need to be exactly two vertices of odd degree for an Euler trail to exist.

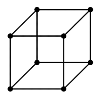
14. An Euler circuit exists, since each vertex has even degree and the graph is connected.



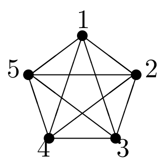
15. There is no Euler circuit, since two vertices have odd degree. There is an Euler trail $(1,1),(2,1),(1,2),(2,2),(1,1),(2,3),(1,2)$.



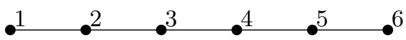
16. Neither. There are eight vertices of degree 3.



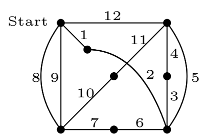
17. $1,2,3,4,1,3,5,2,4,1$ is an Euler circuit.



18. There is no Euler circuit, since two vertices have odd degree. There is an Euler trail $1,2,3,4,5,6$.

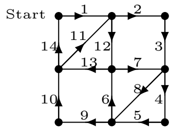


19. An Euler circuit is shown.



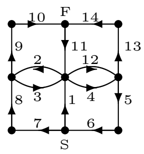
Since there are no vertices of odd degree, there is no Euler trail.

20. An Euler circuit is shown.



Since each vertex has outdeg = indeg, there is no Euler trail.

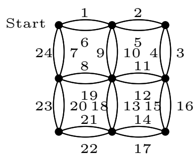
21. An Euler trail is shown.



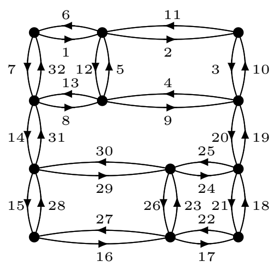
Since vertex S has outdeg = indeg + 1, there is no Euler circuit.

22. Odd $n \geq 5$,
since ${C_n}^c$ is $(n-3)$-regular and $n-3$ is even iff n is odd.

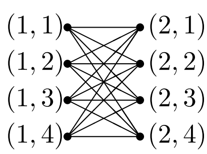
23.



24.



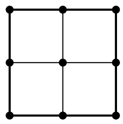
25. (1,1), (2,1), (1,2), (2,2), (1,3), (2,3), (1,4), (2,4), (1,1).



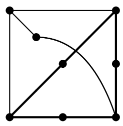
26. $\frac{8 \cdot 3 \cdot 2 \cdot 2}{8 \cdot 2} = 6$.
There are 8 choices for the first vertex, 3 for the second, 2 for the third, 2 for the

fourth, and then a Hamiltonian cycle is uniquely completed. However, we must then divide out for the choice of the first vertex and the choice of the direction in which to traverse the cycle.

27. The edges incident with the degree-2 vertices form a cycle prematurely.



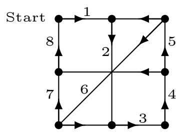
28. No. Edges incident with the degree 2 vertices form a cycle prematurely.



29. *Proof.* Suppose G is Hamiltonian with Hamiltonian cycle C. Since C is a subgraph of G, we have $2 = \delta(C) \leq \delta(G)$. $\square$

30. No. The corresponding graph does not have a Hamiltonian cycle.
In fact, the graph is that in Exercise 27.

31. A Hamiltonian cycle is shown.

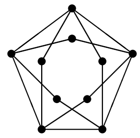


32. (a) Zed.
(b) Zed, Xia, Quo, Jack.
(c) Yes, the Hamiltonian path is unique.

33. From any one vertex, you can follow the Hamiltonian cycle to any other.
Let G be a directed graph with a Hamiltonian cycle C. Let u and v be any two vertices in G. Starting at u, follow C until v is reached. This gives a path from u to v.

34. $|V| = 15$.
$|V| - |E| + |R| = 2$ gives $|V| - 23 + 10 = 2$.

35. The subgraph obtained by deleting the center vertex is a subdivision of K_5.



36. *Proof.* Suppose $G = (V, E)$ is a planar graph all of whose cycles have length at least 5. It suffices to assume that G is connected, because a disconnected graph has fewer edges than the connected graph obtained by joining its components with additional edges. Since the dual graph $\mathrm{D}(G)$ has vertex set R and edge set E, Theorem 8.12 tells us that $\sum_{r \in R} \deg(r) = 2|E|$. Since each region of G must have at least 5 edges, for each $r \in R$, $\deg(r) \geq 5$. Hence, $5|R| \leq 2|E|$. Theorem 9.13 then gives that

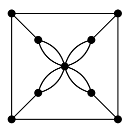
$$2 = |V| - |E| + |R| \leq |V| - |E| + \frac{2}{5}|E| = |V| - \frac{3}{5}|E|.$$

Thus, $\frac{3}{5}|E| \leq |V| - 2$, and the result follows. □

37. The values $|V| = 10$ and $|E| = 15$ do not satisfy the inequality in Exercise 36. That is, all cycles in the Petersen graph have length at least 5. However, $15 \not\leq \frac{5}{3}(10 - 2)$.

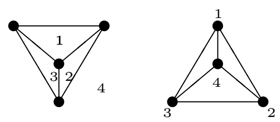
38. K_4 is planar, and all others are subgraphs of K_4.
That is, subgraphs of a planar graph are planar. Each graph on 4 or fewer vertices is a subgraph of the planar graph K_4.

39.



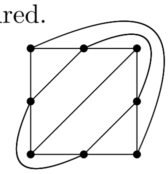
The middle vertex shown here corresponds to the outer region in the embedding from Exercise 13.

40. The face labels on the left correspond to the vertex labels on the right.

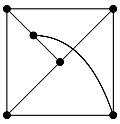


41. No. Subdivide an edge of K_5. The result contains neither K_5 nor $K_{3,3}$.
The correct statement (Kuratowski's Theorem) is that every nonplanar graph contains a *subdivision* of K_5 or $K_{3,3}$.

42. A planar embedding is pictured.

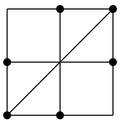


43. The following graph is isomorphic to $K_{3,3}$,



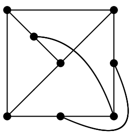
and the circuit board contains a subgraph that is a subdivision of it.

44. It is not planar. It is a subdivision of $K_{3,3}$.
The pictured graph is isomorphic to $K_{3,3}$,

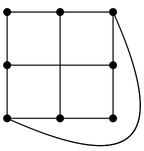


and the power grid is a subdivision of it.

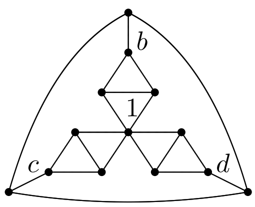
45. $\nu = 1$. By Exercise 43, it is not planar. A drawing with 1 crossing is pictured.



46. $\nu = 1$. By Exercise 44, it is not planar. A drawing with 1 crossing is pictured.

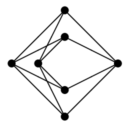


47. $\chi = 4$. Try starting a 3-coloring from the middle vertex, and observe that it fails.

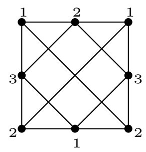


Observe that vertices b, c, and d must also receive color 1. It is now impossible to color the outer triangle.

48.

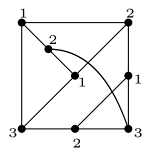


49. $\omega = 3$, $\alpha = 3$, $\chi = 3$.



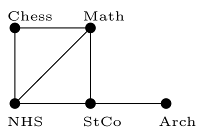
The upper-left triangle colored $1, 2, 3$ forms a clique of maximum size. The three vertices colored 1 form an independent set of maximum size. The displayed 3-coloring uses the fewest colors possible.

50. $\omega = 3$, $\alpha = 3$, $\chi = 3$.

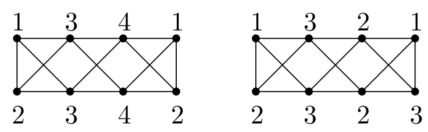


The lower-right triangle colored $1, 2, 3$ forms a clique of maximum size. The three vertices colored 1 form an independent set of maximum size. The displayed 3-coloring uses the fewest colors possible.

51. 3 sessions are needed since Chess, Math, and NHS form a 3-clique. It is possible by the schedule:
(1) Archery, NHS
(2) Chess, Student Council
(3) Math.



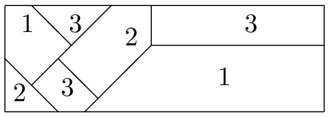
52. The Greedy Coloring Algorithm gives the 4-coloring on the left, whereas the 3-coloring on the right is optimal.



Note that $\omega = 3$.

53. $\Delta(G) + \delta(G^c) = n - 1$.
For every vertex v, $\deg_G(v) + \deg_{G^c}(v) = n - 1$. A vertex in G incident to the largest number of edges will be a vertex in G^c incident to the smallest number of edges.

54.



55. Refer to an atlas.
(a) It is easy to do in three colors.
(b) Three.
(c) The regions Alberta, Northwest Territories, and Saskatchewan form a 3-clique that requires three colors.

2.10 Chapter 10

Section 10.1

1. No.
It contains a cycle of length 3.

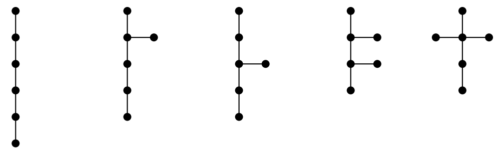
3. $\min\{m, n\} = 1$.
If $m, n \geq 2$, then $K_{m,n}$ contains a cycle of length 4.

5. No.
It has multiple edges.

7. No.
A loop forms a cycle of length 1.

9. Yes.
It is a forest but not a tree (unless $n = 1$). It consists of n disjoint copies of T.

11. 5.



These are the possible carbon trees.

13. (a) Yes. There are other walks but only one path.
(b) No. Hawk, Center, Park and Hawk, Center, Anselm, Main, Park are two different paths.
(c) No. Center, Park, Main and Center, Anselm, Main are distinct shortest paths from Center to Main.

15. Converse: Let G be a graph. If there is a unique path between any pair of vertices in G, then G is a tree.
Proof. Suppose that between any pair of vertices in G there is a unique path. The existence of paths shows that G is connected. So suppose G contains a cycle C. Let u and v be distinct vertices of C. Then C provides two distinct paths from u to v. This contradiction shows that G contains no cycles. Hence, G must be a tree. $\square$

17. *Proof.* ($\rightarrow$) Suppose G has a unique spanning tree T. So $V_T = V_G$. Suppose that outside of T there is some edge $e \mapsto \{u, v\}$. Hence, in T there is a path P from u to v. If we let d be the first edge on P, then $(T \setminus \{d\}) \cup \{e\}$ is a spanning tree, different from T. So it must be that $E_T = E_G$, and hence $T = G$. ($\leftarrow$) If G is a tree, then only G itself can be its own spanning tree. $\square$

19. *Proof.* Suppose G is connected and $|E| = |V| - 1$. Let T be a spanning tree for G. So $V_T = V$ and $E_T \subseteq E$. By Theorem 10.3, $|E_T| = |V_T| - 1$. Since $|E_T| = |V_T| - 1 = |V| - 1 = |E|$, it follows that $E_T = E$ and thus $G = T$ is a tree. $\square$

21. *Proof.* $(\rightarrow)$ Suppose T is a tree with exactly 2 leaves, and let P be a longest path in T. Note that the two ends of P must be leaves. Suppose that there are vertices outside of P, and let v be one of greatest possible distance from P. Then, v must be a third leaf. Since there can be no vertices outside of P, it follows that $T = P$. $(\leftarrow)$ Obvious. $\square$

23. $n - c$.
For each $1 \leq i \leq c$, let n_i be the number of vertices in component i. The number of edges in component i is thus $n_i - 1$. Now,

$$|E| = \sum_{i=1}^{c}(n_i - 1) = (\sum_{i=1}^{c} n_i) - c = n - c.$$

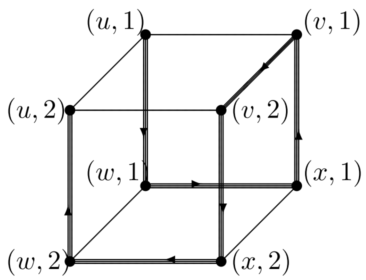
25. Just $K_{1,n}$. All $n - 1$ leaves must be joined to the only other vertex.

27. Theorem: If $G = (V, E)$ is a tree, then $|E| = |V| - 1$.
We prove by induction that: If $G = (V, E)$ is a tree, then G is planar.
Proof. Base case: $|V| = 1$. So $|E| = 0$ and G is planar. *Inductive step*: Suppose $k \geq 1$ and any tree on k vertices is planar. Let T be a tree on $k + 1$ vertices, and let v be a leaf of T. Since $T \setminus \{v\}$ is a tree on k vertices, it must be planar. The leaf v can now be added to give a planar embedding of T. $\square$
By Euler's Formula, $|V| - |E| + 1 = 2$. So the theorem follows.

29. $\binom{n}{2} + n$.
There are n subgraphs isomorphic to P_1. Every other path subgraph corresponds to the pair of vertices it joins. There are $\binom{n}{2}$ pairs of vertices.

31. $E_T = \{\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}\}$. That is, P_n is a spanning tree for C_n.

33. Remove an edge from the Hamiltonian cycle guaranteed in Example 9.13.



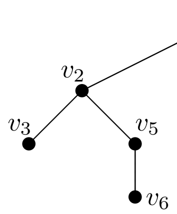
35. Converse: If a graph G has a spanning tree, then G is connected.
Proof. Suppose G has a spanning tree T. Let u and v be vertices in G. The path from u to v in T is also a path in G. So G is connected. $\square$

37. (a) $3, 5$. (b) 2. (c) 2. (d) 3. (e) Yes. (f) Yes.

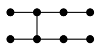
39. (a) 4. (b) 2. (c) 2. (d) 3. (e) No. Note that 2 has three children.
(f) No. Note that 2 and 4 have different numbers of children.

41. (a) No, grandchild. (b) Parent. (c) Leaves. (d) 3. (e) Yes.

43. No. The tree in Figure 10.9 is balanced with v_2 as the root, but not with v_1 as the root.



45. False. The pictured graph is a counterexample.



No matter which vertex is chosen to be the root, the resulting rooted tree will not be balanced.

47. Theorem: If T is a full m-ary tree with n vertices, l leaves, and i internal vertices, then $n = i + l = mi + 1$.
Proof. Suppose T is a full m-ary tree with n vertices, l leaves, and i internal vertices. Since each vertex is either internal or a leaf, $n = i + l$. Make T a directed graph by directing each edge from parent to child. There are $n - 1$ vertices other than the root (and there are $n - 1$ edges). Each non-root vertex is the head (child) of a unique edge with internal tail (parent). Each internal vertex is the tail of m edges, and there are i internal vertices. So $mi = n - 1$. $\square$

49. $n = \frac{ml-1}{m-1}$ and $i = \frac{l-1}{m-1}$.
Note that $l - 1 = n - i - 1 = mi + 1 - i - 1 = (m-1)i$. Note that $n(m-1) = nm - n = nm - mi - 1 = m(n-i) - 1 = ml - 1$.

51. 31.
A full 3-ary tree with $i = 10$ internal vertices has $n = 3(10) + 1 = 31$ vertices.

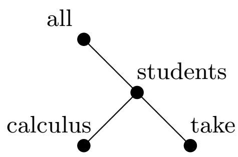
53. *Proof.* Our proof is by induction on h. If $h = 0$, then $l = 1$ and we have

equalities in our desired results, for any choice of m. So suppose that $h \geq 1$, that T is a full m-ary tree of height h with l leaves all at level h, and suppose that our desired results hold for all trees with smaller height. Let v_0 be the root of T, and let $v_1, \ldots, v_m$ be its children. If we remove v_0, then there remain m rooted trees. For each $1 \leq j \leq m$, let T_j be the tree with root v_j, height h_j, and l_j leaves. So $h_j = h - 1$, and the inductive hypothesis applies to T_j. Hence,

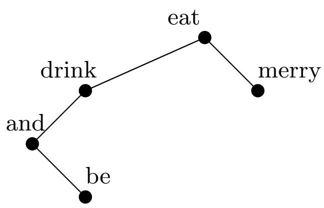
$$l = l_1 + \cdots + l_m = m^{h-1} + \cdots + m^{h-1} = m \cdot m^{h-1} = m^h.$$

□

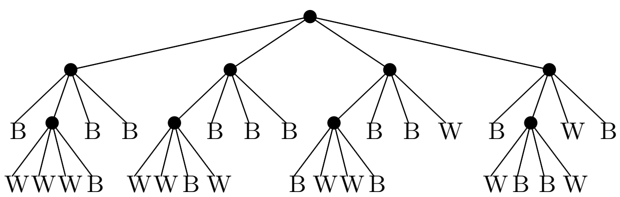
55.



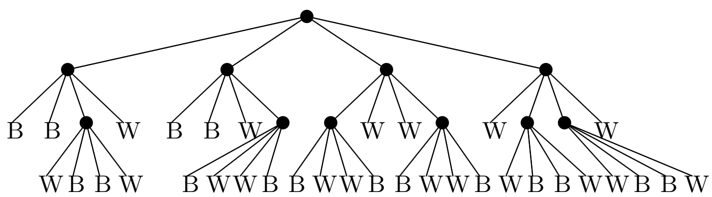
57.



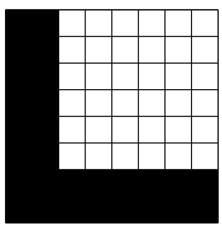
59.



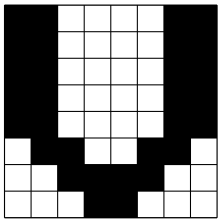
61.



63. L.

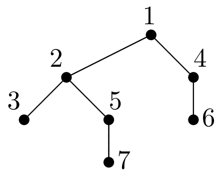


65. V.

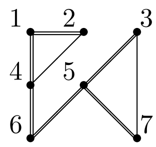


Section 10.2

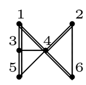
1. [1, 2, 4, 3, 5, 6, 7].
The following picture better reflects the order in which the vertices are encountered.



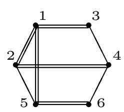
3. [1, 2, 4, 6, 5, 3, 7].



5. [1, 3, 4, 5, 2, 6].

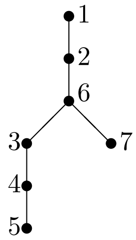


7. [1, 2, 3, 5, 4, 6].

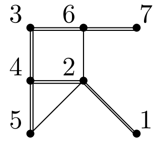


9. [5, 4, 3, 7, 6, 2, 1].
The following picture better reflects the order in which the vertices are encoun-

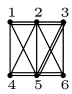
tered.



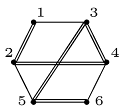
11. $[7, 6, 3, 5, 4, 2, 1]$.



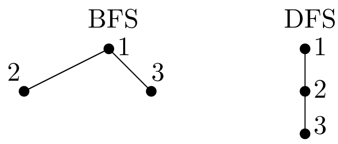
13. $[4, 6, 5, 3, 2, 1]$.



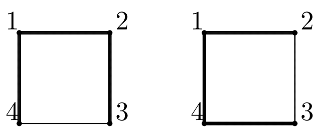
15. $[6, 5, 3, 4, 2, 1]$.



17. Yes. K_3 with labels $1, 2, 3$ is the smallest example.

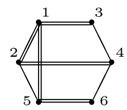


19. Yes. For C_4, input orderings $1, 2, 3, 4$ and $1, 4, 2, 3$ give different trees (from root 1).



21. No. $K_{1,3}$ with the degree-3 vertex labeled 1 and the others $2, 3, 4$ yields the same list L as does P_4 with consecutive labels $2, 3, 1, 4$.
They both yield the list $[2, 3, 4, 1]$.

23. $1, 2, 4$. No; $1, 3, 4$ is another.



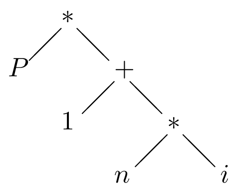
25. True. From each vertex, the path in the breadth-first search tree to the root is a shortest such path.
That is, its level in the tree is its distance in the graph from the root. For the vertices at the highest level, there can be no shorter path to the root as would need to exist in a tree of smaller height.

27. Yes. (V, F) is connected at each stage, and has no cycles at the end.

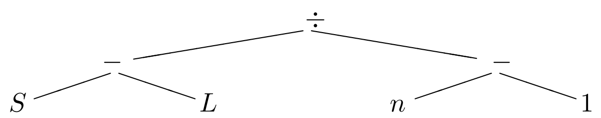
29. Let v be a vertex of $G = (V, E)$.
Perform Depth-First Search for G starting at v to obtain edge set F.
If $|F| = |V| - 1$, then G is connected.
Otherwise G is not.

31. Yes. The first m vertices in L are the children of the $(m+1)^{st}$, the next m are the children of the $2(m+1)^{st}$, and so on. The truth of this fact in the case $m = 2$ makes postfix notation work.

33.



35. (a) $S \quad L \quad - \quad n \quad 1 \quad - \quad \div$.
(b) $\div \quad - \quad S \quad L \quad - \quad n \quad 1$.
(c) $S \quad - \quad L \quad \div \quad n \quad - \quad 1$.



37. 4.

$$\begin{array}{llllllll} 6 & 5 & 3 & - & \div & 1 & + \\ 6 & 2 & \div & 1 & + \\ 3 & 1 & + \\ 4 \end{array}$$

39. 7.

41. 3.

$$\begin{array}{llllll} - & 5 & \div & + & 8 & 6 \quad 7 \\ - & 5 & \div & 14 & 7 & \\ - & 5 & 2 & & & \\ 3 & & & & & \end{array}$$

43. -6.

45. Boston, New York, Toronto, Baltimore, Tampa Bay.

47. Hamiltonian cycle *adcbe* is found.
a, *ad*, *adb*, *adbc*, *adbe*, *adc*, *adcb*, *adcbe*.

49. Depth-First Search completes finding no Hamiltonian cycle.
a, *ab*, *abc*, *abcd*, *abcde*, *abced*, *acb*, *acd*, *acde*, *aced*, failure.

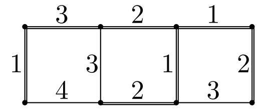
51. Colorings $\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & ? & \\ \hline\end{array}$ and $\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 1 & ? & \\ \hline\end{array}$ are attempted, before realizing none can be found.

53. Coloring $\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & ? & & \\ \hline\end{array}$ is attempted, before realizing none can be found.

55. Consider a graph H in which the vertices are paths of length at most k in K_n. Also add a trivial vertex to H that connects to each path of length 0. The graph H will be a tree with leaves corresponding to permutations of size k. So Depth-First Search will find each of these leaves.

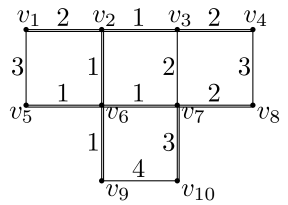
Section 10.3

1.(a)



(b) \$120,000.

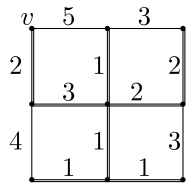
9.(a)



(b) \$140,000.

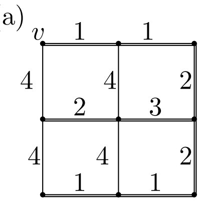
Order:
1,1,1,1,1,2,2,2,3

3.(a)



(b) \$130,000.

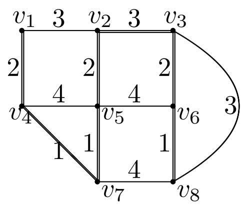
11.(a)



(b) \$130,000.

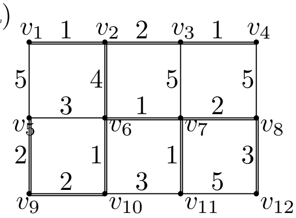
Order:
1,1,2,2,1,1,3,2
Assuming the top left edge of weight 1 is added first.

5.(a)



(b) \$120,000.

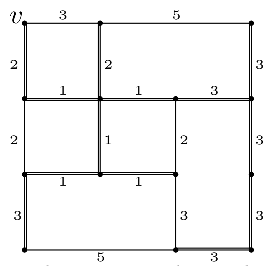
13.(a)



(b) \$200,000.

Order:
1,2,1,4,1,1,1,2,2,2,3
Assuming the top left edge of weight 1 is added first.

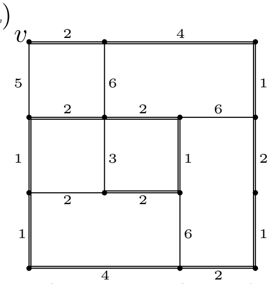
7.(a)



The answer depends on an edge ordering.

(b) \$270,000.

15.(a)



The answer depends on an edge ordering.

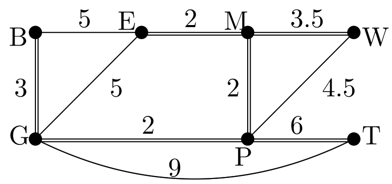
(b) \$250,000.

Order:
1,1,2,2,1,2,4,2,1,2,1,4,2
Assuming the bottom left edge of weight 1 is added first.

17. Certainly not if it is a loop, but yes otherwise.
If the minimum weight edge is not in the tree, then that edge forms a cycle with the tree. So use that edge in place of the one of higher weight in the tree.

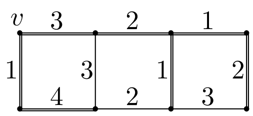
19. True.
Sketch. Let T be the tree produced by Kruskal's algorithm, and suppose T is not the unique minimum spanning tree. Let T' be a minimum spanning tree with the maximum possible number of edges in common with T. Now follow the proof of Theorem 10.8 to obtain a contradiction. $\square$

21. \$18,500.



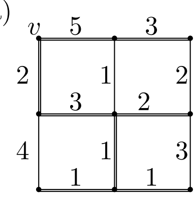
23. *Sketch.* With the ordering of the edges, we may as well regard the weights as distinct. Hence, this result follows from Exercise 19. $\square$

25.(a)



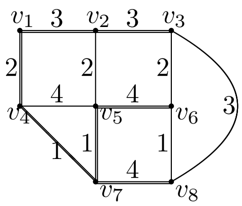
(b) 8 PSI.

27.(a)



(b) 8 PSI.

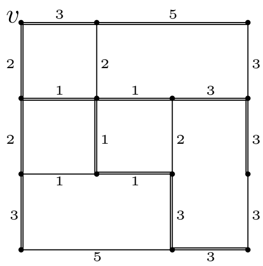
29.(a)



The answer is not unique since any of the three edges incident with the right middle vertex may be chosen.

(b) 8 PSI.

31.(a)



(b) 11 PSI.

33. False.
Let $G = C_4$ with each edge of weight 1, and v any vertex.

35. Take C_4 with three edges of weight 1 and one of weight 2, and let v be an

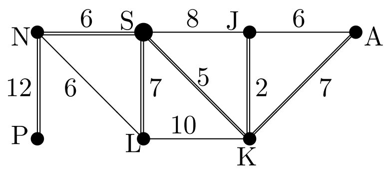
endpoint of the weight-2 edge. The shortest path tree is on the left,

and the minimum spanning tree is on the right.

37. No. Label the edges of C_3 with 1,2,3, and put v incident to 2 and 3.

39. True. The distance function has merely been multiplied by a constant.

41. Adams 12, Johnson 7, Kennedy 5, Lincoln 7, Nixon 6, Polk 18.



43. Let $G = C_4$ with three edges of weight 1 and one of weight 2. Suppose v_1 is an endpoint of the weight-2 edge, and v_2 is not.

The shortest path tree from v_1 is on the left, and the shortest path tree from v_2 is on the right.

45. Yes. It can be proven by induction on the number of cycles.

47. A shortest path tree, since the commander wants the shortest message path to each unit.

Section 10.4

1. 3.

3. location $= 0$.
low $= 1$, high $= 6$, mid $= 3$.
low $= 4$, high $= 6$, mid $= 5$.
low $= 4$, high $= 5$, mid $= 4$.
low $= 4$, high $= 4$.
location $= 4$.

5. location $= 0$.
low $= 1$, high $= 7$, mid $= 4$.
low $= 1$, high $= 4$, mid $= 2$.
low $= 3$, high $= 4$, mid $= 3$.
low $= 4$, high $= 4$.
location $= 0$. Value not found.

7. The first. Sequential Search moves through the array in order. As soon as the desired value is found, that location is returned.

9. (a) For an array of length $n = 10^6$, at most $1 + \lceil \log_2 n \rceil = 21$ comparisons are done. That would take $\frac{21}{2\times 10^9} = 0.0000000105$ seconds.
(b) For an array of length $n = 10^6$, at most $n^2 = 10^{12}$ comparisons are done. That would take $\frac{10^{12}}{2\times 10^9} = 500$ seconds.

11. n.

13. *Proof.* Certainly, $\lceil \log_2(k+2) \rceil \geq \lceil \log_2(k+1) \rceil$. So suppose, to the contrary, that $\lceil \log_2(k+1) \rceil = n < \lceil \log_2(k+2) \rceil$. Hence, $\log_2(k+1) \leq n < \log_2(k+2)$. That is, $k+1 \leq 2^n < k+2$. Since $1 \leq 2^n - k < 2$, it follows that $2^n - k = 1$. However, $k = 2^n - 1$ is now odd, a contradiction. □

15. **Maximum.**
    Let max $= 1$.
    For $i = 2$ to n,
        If $A[i] > A[\text{max}]$, then
            Let max $= i$.
    Return max.

17. $n - 1$.
A comparison is done for each value of i from 2 to n. There are $n - 1$ such values.

19. $\forall\, x > 0,\ |g(x)| \leq |g(x)|$.
That is, $C = 1$ and $d = 0$ works in Definition 10.8. Here $f = g$.

21. $\forall\, x > 0,\ |cg(x)| \leq |c||g(x)|$ and $|g(x)| \leq |\frac{1}{c}||cg(x)|$.
Here we show that $cg(x) \in O(g(x))$ and $g(x) \in O(cg(x))$ to conclude that $O(cg(x)) = O(g(x))$. In this case, both required inequalities are actually equalities given by properties of absolute value.

23. False.
By Exercise 21, $O(\frac{3}{2}x) = O(x)$.
By Lemma 10.15, since $1 < \frac{3}{2}$, we have $x^{\frac{3}{2}} \notin O(x)$.

25. True.
Apply Theorem 10.12 with $m = 4$ together with Lemma 10.10.

27. *Sketch.* Say, for $i = 1, 2$, that $\forall\, x > d_i,\ |f_i(x)| \geq C_i|g(x)|$. Let $C = C_1 + C_2$ and $d = \max\{d_1, d_2\}$. So $\forall\ x > d,\ |f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)| \leq C_1|g(x)| + C_2|g(x)| = C|g(x)|$. □
The Triangle Inequality gives $|f_1(x)+f_2(x)| \leq |f_1(x)|+|f_2(x)|$. When $x > d$, we have both $x > d_1$ and $x > d_2$. Thus, $|f_1(x)| \leq C_1|g(x)|$ and $|f_2(x)| \leq C_2|g(x)|$. Of course, $C_1|g(x)| + C_2|g(x)| = (C_1 + C_2)|g(x)| = C|g(x)|$.

29. *Sketch.* Suppose $f(x) \in O(g(x))$. By Exercise 19, $g(x) \in O(g(x))$. So Exercise 27 finishes the job. □
That is, we have $f(x), g(x) \in O(g(x))$. So $f(x) + g(x) \in O(g(x))$.

31. *Sketch.* $c = c \cdot 1 \in O(g(x))$ by Exercise 21. So Exercise 27 finishes the job. □
That is, since $c, g(x) \in O(g(x))$, we have $c + g(x) \in O(g(x))$.

33. (a) $A_{\text{simple}} = P(1 + .04(60)) = 3.4P$. $A_{\text{compound}} = P(1.02)^{60} \approx 3.28P$. So simple interest is better.
(b) $A_{\text{simple}} = P(1 + .04(72)) = 3.88P$. $A_{\text{compound}} = P(1.02)^{72} \approx 4.16P$. So compound interest is better.
(c) $A_{\text{compound}} = A_{\text{simple}}$ iff $P(1.02)^n = P(1 + .04n)$ iff $n = 0$ or $n \approx 64.2787$. Use $n \geq 65$.

35. *Sketch.* $(\subseteq)\ \forall\ n > 0,\ \log_2 n \leq 1 + \lceil \log_2 n \rceil$. $(\supseteq)\ \forall\ n > 0,\ 1 + \lceil \log_2 n \rceil \leq 2\lceil \log_2 n \rceil \leq 4 \log_2 n$. □
Note that $\forall\, n \geq 2,\ 1 \leq \log_2 n \leq \lceil \log_2 n \rceil \leq \log_2 n + 1 \leq 2 \log_2 n$.
Since $\log_2 n \in O(1 + \lceil \log_2 n \rceil)$, we have $O(\log_2 n) \subseteq O(1 + \lceil \log_2 n \rceil)$.
Since $1 + \lceil \log_2 n \rceil \in O(\log_2 n)$, we have $O(1 + \lceil \log_2 n \rceil) \subseteq O(\log_2 n)$.
Hence, $O(\log_2 n) = O(1 + \lceil \log_2 n \rceil)$.

37. *Sketch.* $(\subseteq)\ \forall\ n > 0,\ \frac{n}{2} \log_2 \frac{n}{2} \leq n \log_2 n$. $(\supseteq)\ \forall\ n > 3,\ n \log_2 n \leq n \log_2 n + n(\log_2 n - 2) = 2n(\log_2 n - 1) = 4(\frac{n}{2} \log_2 \frac{n}{2})$. □
The first inequality holds since $\frac{n}{2} \leq n$, and the second inequality holds since $\forall\, n \geq 4,\ \log_2 n - 2 \geq 0$.

39. *Proof.* $(\rightarrow)$ Suppose $f(x) \in O(g(x))$ and $g(x) \in O(f(x))$. So we have $C_1', d_1, C_2', d_2 > 0$ such that $\forall\, x > d_1,\ |f(x)| \leq C_1'|g(x)|$ and $\forall\, x > d_2,\ |g(x)| \leq C_2'|f(x)|2$. Let $d = \max\{d_1, d_2\}$, $C_1 = \frac{1}{C_2'}$, and $C_2 = C_1'$. So $\forall\, x > d,\ C_1|g(x)| = \frac{1}{C_2'}|g(x)| \leq |f(x)| \leq C_1'|g(x)| = C_2|g(x)|$ $(\leftarrow)$ Suppose there exist positive constants C_1, C_2, and d for which $\forall\ x > d,\ C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$. Let $d_1 = d_2 = d$, $C_1' = C_2$, and $C_2' = \frac{1}{C_1}$. Observe that $\forall\, x > d_1,\ |f(x)| \leq C_1'|g(x)|$ and $\forall\, x > d_2,\ |g(x)| \leq C_2'|f(x)|$. So $f(x) \in O(g(x))$ and $g(x) \in O(f(x))$. □

41. Apply Exercise 39.
Lemma: $f(x) \in \Theta(g(x))$ iff $\Theta(f(x)) = \Theta(g(x))$ iff $O(f(x)) = O(g(x))$.
The symmetry in Definition 10.9 gives that $f(x) \in \Theta(g(x))$ iff $g(x) \in \Theta(f(x))$.
Observe that $C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$ iff $\frac{1}{C_2}|f(x)| \leq |g(x)| \leq \frac{1}{C_1}|g(x)|$.

43. False.
By Theorem 10.16, $O(n) \subset O(n \log_b n)$.

45. True.
$\log_2(n^2) = 2\log_2 n$. Now apply Exercise 21.

47. (a) The second.
(b) The first.
(c) The first.
Let $f(x) = 64\lfloor \log_2 x \rfloor + 108x + 18$ and $g(x) = 2x^2 + 4x + 8$. Observe that $f(55) = 6278 = g(55)$. Graph $f(x)$ and $g(x)$ on the same set of axes (for $x \geq 1$). Observe that $f(x) > g(x)$ for $x < 55$ and $f(x) < g(x)$ for $x > 55$. Note that $\Theta(f(n)) = \Theta(n)$ and $\Theta(g(n)) = \Theta(n^2)$. Also, n is a smaller order of growth than n^2.

49. Apply Exercise 21.
The equation $\log_b n = \frac{1}{\log_2 b}\log_2 n$ shows that $\log_b n$ is a constant multiple of $\log_2 n$. Here, $c = \frac{1}{\log_2 b} \neq 0$.

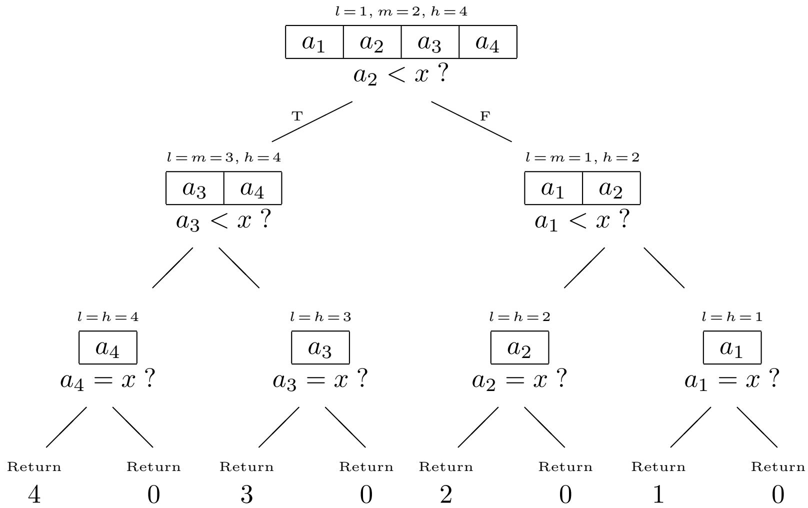
51. *Sketch.* (i) For $2 \leq n$, we have $1 = \log_2 2 \leq \log_2 n$. (ii) Let C be given. For any choice of $n > 2^C$, we have $\log_2 n > C$. $\Box$
By Lemma 10.14, it suffices to show that $1 \in O(\log_2 n)$ and $\log_2 n \notin O(1)$. In (i), we see that $1 \in O(\log_2 n)$ by using $C = 1$ and $d = 1$. In (ii), we see that there is no value of C such that $\log_2 n \leq C(1)$ eventually. Hence, $\log_2 n \notin O(1)$.

53. *Sketch.* This follows from Exercise 51 by multiplying by n. $\Box$
(i) For $2 \leq n$, we have $n = n\log_2 2 \leq n\log_2 n$. (ii) Let C be given. For any choice of $n > 2^C$, we have $n\log_2 n > Cn$.

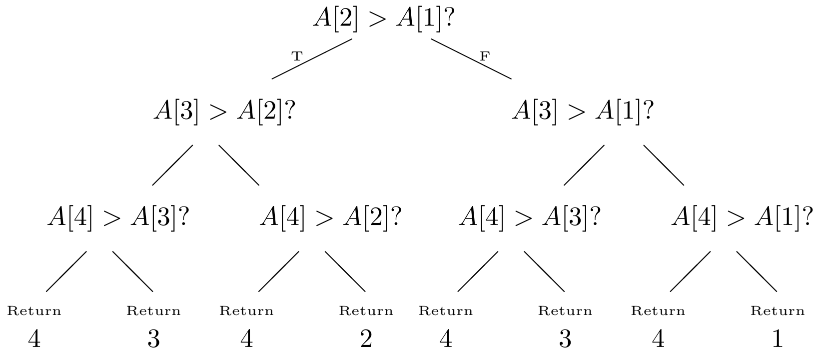
55. *Sketch.* (i) By induction, we see that $\forall\ n \geq 4,\ n^2 \leq 2^n$. (ii) Let C be given. It suffices to consider $C \in \mathbb{Z}$ with $C \geq 10$. By induction, we see that $\forall\ C \geq 10,\ 2^C > C^3$. That is, for $n = C$, $2^n > Cn^2$. $\Box$
By Lemma 10.14, it suffices to show that $n^2 \in O(2^n)$ and $2^n \notin O(n^2)$. In (i), we see that $n^2 \in O(2^n)$ by using $C = 1$ and $d = 3$. In (ii), we see that there is no value of C such that eventually $2^n < Cn^2$. Hence, $2^n \notin O(n^2)$.

Section 10.5

1.



3.



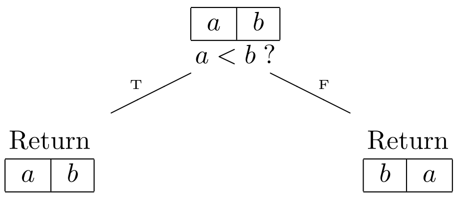
5.

i=2: 7 5 3 1
i=3: 5 7 3 1
i=4: 3 5 7 1
end: 1 3 5 7

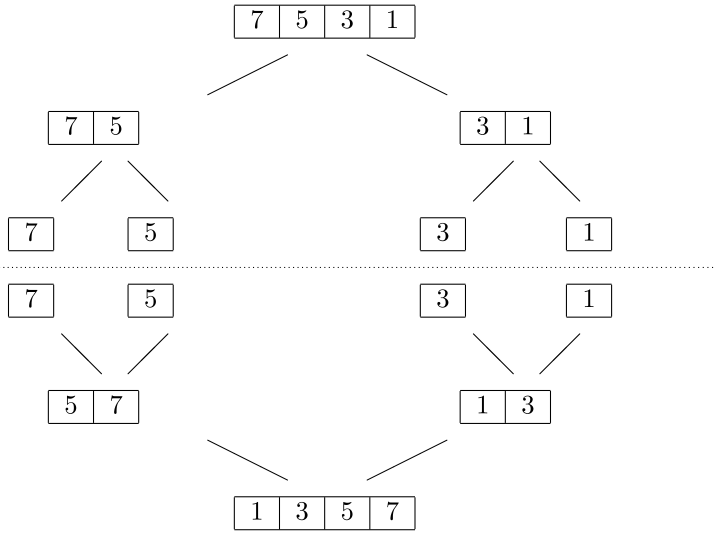
7.

i=2: 4 2 6 1 5 3
i=3: 2 4 6 1 5 3
i=4: 2 4 6 1 5 3
i=5: 1 2 4 6 5 3
i=6: 1 2 4 5 6 3
end: 1 2 3 4 5 6

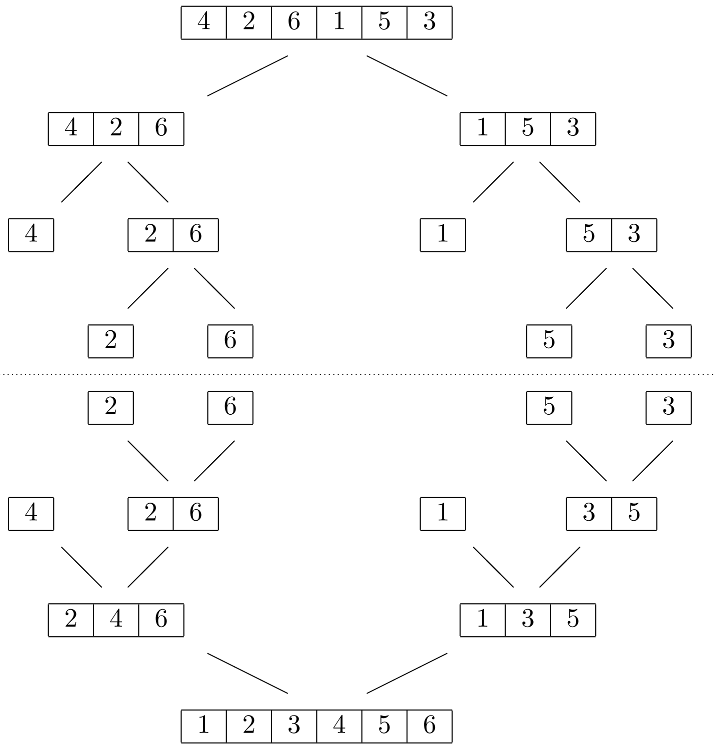
9.



11.


7 5 3 1
7 5
3 1
7
5
3
1
7
5
3
1
5 7
1 3
1 3 5 7

13.


4 2 6 1 5 3
4 2 6
1 5 3
4
2 6
1
5 3
2
6
5
3
2
6
5
3
4
2 6
1
3 5
2 4 6
1 3 5
1 2 3 4 5 6

15.

i=6,j=1:	4 2 6 1 5 3
j=2:	2 4 6 1 5 3
j=3:	2 4 6 1 5 3
j=4:	2 4 1 6 5 3
j=5:	2 4 1 5 6 3
i=5,j=1:	2 4 1 5 3 6
j=2:	2 4 1 5 3 6
j=3:	2 1 4 5 3 6
j=4:	2 1 4 5 3 6
i=4,j=1:	2 1 4 3 5 6
j=2:	1 2 4 3 5 6
j=3:	1 2 4 3 5 6
i=3,j=1:	1 2 3 4 5 6
j=2:	1 2 3 4 5 6
i=2,j=1:	1 2 3 4 5 6
end:	1 2 3 4 5 6

17.

i=5,j=1:	7 5 4 2 1
j=2:	5 7 4 2 1
j=3:	5 4 7 2 1
j=4:	5 4 2 7 1
i=4,j=1:	5 4 2 1 7
j=2:	4 5 2 1 7
j=3:	4 2 5 1 7
i=3,j=1:	4 2 1 5 7
j=2:	2 4 1 5 7
i=2,j=1:	2 1 4 5 7
end:	1 2 4 5 7

19.

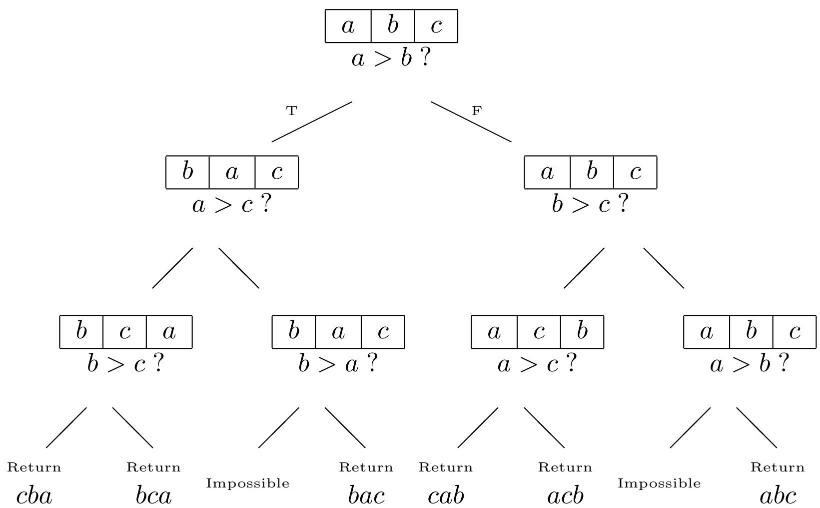
i=1:	3 8 6 1 4
i=2:	1 8 6 3 4
i=3:	1 3 6 8 4
i=4:	1 3 4 8 6
end:	1 3 4 6 8

21.

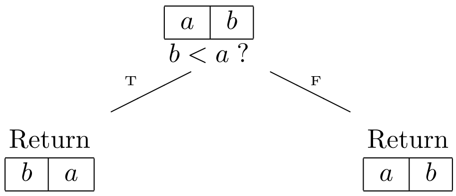
i=1:	6 5 4 3 2 1
i=2:	1 5 4 3 2 6
i=3:	1 2 4 3 5 6
i=4:	1 2 3 4 5 6
i=5:	1 2 3 4 5 6
end:	1 2 3 4 5 6

23. $\frac{n(n-1)}{2}$. For each $n \geq i \geq 2$, there are $i-1$ comparisons. The total number of comparisons is $\sum_{i=2}^{n}\sum_{j=1}^{i-1} 1 = \sum_{i=2}^{n}(i-1) = \sum_{k=1}^{n-1} k = \frac{(n-1)n}{2}$.

25.



27.



29. No.
By Exercise 23, Bubble Sort is $\Theta(\frac{n(n-1)}{2}) = \Theta(n^2)$. However, $\Theta(n \log_2 n)$ is maximally efficient.

31. Bubble Sort.
Trace Algorithm 10.11.

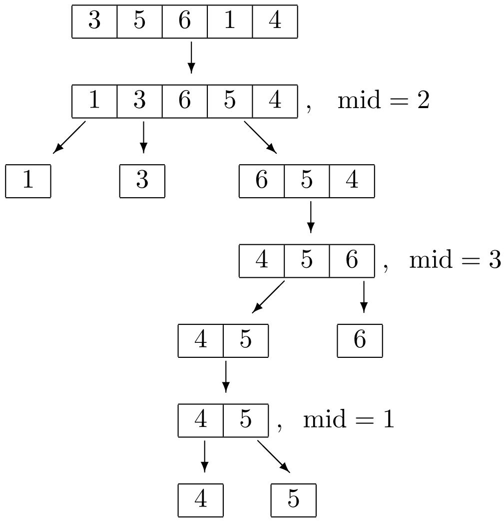
33. Order can be changed.
Consider Algorithm 10.8 in the case that $n = 2$ and $A[1] = A[2]$. The entries get switched.

35. Same order.
See Algorithm 10.11. A switch only occurs if $A[j] > A[j+1]$, never when $A[j] = A[j+1]$.

37. *Proof. Base case*: $(n = 1)$. Note that $c_1 = 0$ and $2^0 \leq 1^2$. *Inductive step*: Suppose $k \geq 1$ and that, for each $0 \leq i \leq k$, $2^{c_i} \leq i^{2i}$. (Goal: $2^{c_{k+1}} \leq (k+1)^{2(k+1)}$.) Observe that $2^{c_{k+1}} = 2^{c_{\lfloor\frac{k+1}{2}\rfloor}+c_{\lceil\frac{k+1}{2}\rceil}+k} = 2^{c_{\lfloor\frac{k+1}{2}\rfloor}}2^{c_{\lceil\frac{k+1}{2}\rceil}}2^k$
$\leq 2^k \lfloor\frac{k+1}{2}\rfloor^{2\lfloor\frac{k+1}{2}\rfloor}\lceil\frac{k+1}{2}\rceil^{2\lceil\frac{k+1}{2}\rceil} = \frac{1}{2^{k+2}} \cdot \begin{cases} (k+1)^{2(k+1)} & \text{if } k \text{ is odd} \\ k^k(k+2)^{k+2} & \text{if } k \text{ is even} \end{cases}$
$\leq (k+1)^{2(k+1)}$. $\square$
Since $2^{c_n} \leq 2^{2n\log_2 n} = n^{2n}$, we have $c_n = \log_2 2^{c_n} \leq \log_2 n^{2n} = 2n\log_2 n$.

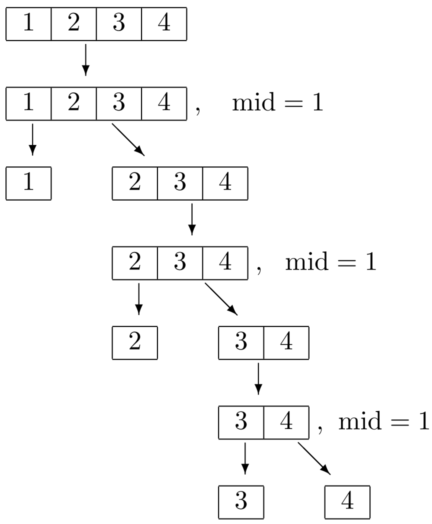
39. The worst-case complexity is $n-1$. Moreover, every input of size n uses $n-1$ comparisons.
Notice that, no matter what the input, a comparison is done for each $2 \leq i \leq n$. Also note that $\Theta(n-1) = \Theta(n)$.

41.



A left-to-right reading of the leaves shows the correct order for A.
| 1 | 3 | 4 | 5 | 6 |.

43.



A left-to-right reading of the leaves shows the correct order for A.
| 1 | 2 | 3 | 4 |.

45. $\sum_{j=1}^{n-1}(n-j) = \frac{n(n-1)}{2}$.

Review

1. We prove the contrapositive.
Proof. Suppose $G = (V, E)$ is connected. So G has a spanning tree $T = (V, F)$.

Thus, $|E| \geq |F| = |V| - 1$. $\square$

2. $4, 2, \ldots, 2, 1, 1, 1, 1$ or $3, 3, 2, \ldots, 2, 1, 1, 1, 1$.
The sum of the degrees must be $2n - 2$, while exactly 4 vertices have degree 1. In particular, this forces $\Delta \geq 3$. Since the number of leaves must be at least as big as the maximum degree, $\Delta \leq 4$. Finally, observe that there is only one possibility with $\Delta = 4$ and one with $\Delta = 3$.

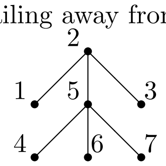
3. 18.
We count the number of trees on 8 vertices with $\Delta \leq 4$. There is 1 with degree sequence $2, 2, 2, 2, 2, 2, 1, 1$. There are 4 with degree sequence $3, 2, 2, 2, 2, 1, 1, 1$. There are 5 with degree sequence $3, 3, 2, 2, 1, 1, 1, 1$. There is 1 with degree sequence $3, 3, 3, 1, 1, 1, 1, 1$. There is 1 with degree sequence $4, 4, 1, 1, 1, 1, 1, 1$. There are 3 with degree sequence $4, 3, 2, 1, 1, 1, 1$. There are 3 with degree sequence $4, 2, 2, 2, 1, 1, 1, 1$. Thus, a total of 18.

4. 30. In general, a forest on n vertices with c components has $n - c$ edges. Hence $n - c = 30 - 4 = 26$.

5. $3n + 1$. The bonds correspond to the edges in the tree whose vertices correspond to the carbon and the hydrogen atoms. There are $n + (2n + 2) = 3n + 2$ vertices and hence $3n + 1$ edges.

6. *Proof.* Let P be a path from u to v. Let w be any vertex in P besides u and v. Since w has degree 2 in P, its degree must be at least 2 in the graph. Hence, w cannot be a leaf. $\square$

7. (a) $4, 6, 7$. (b) 2. (c) 1. (d) 2. (e) No. (f) Yes. (g) Yes. The following picture reflects the levels of the vertices trailing away from root 2.



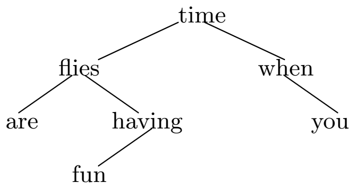
8. No. Two opposite edges on the square Q_2 form a subgraph that is a forest on the vertex set. However, it is not a spanning forest, which should be a tree in this case.
To characterize a spanning forest, we need to require a spanning tree on each component.

9. $i = n - l$ and $m = \frac{n-1}{n-l}$.
We have $n = i + l$ and $n = mi + 1$. So $i = n - l$ and $m = \frac{n-1}{i} = \frac{n-1}{n-l}$.

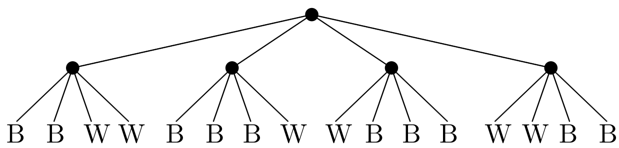
10. 31.
Since $n = 4(10) + 1$, there are $l = 41 - 10 = 31$ files.

11. True.
Proof. Let v be the root. Let P be a path whose length is the diameter, and let u_1 and u_2 be its endpoints. There is a path Q_1 from u_1 to v and a path Q_2 from v to u_2. Note that the lengths of Q_1 and Q_2 cannot exceed the height of the tree. In fact, they are the levels of u_1 and u_2, respectively. Since Q_1 followed by Q_2 is a walk from u_1 to u_2, its length is at least that of P (a shortest possible walk from u_1 to u_2). Thus, twice the height is at least the sum of the levels of u_1 and u_2, which is at least the length of P (which equals the diameter). □
Recall that the diameter of a connected graph is the maximum possible distance between two vertices. Hence, it is the length of a longest possible path.

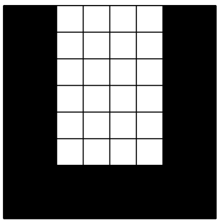
12.



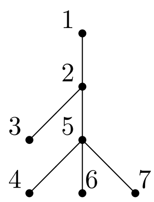
13.



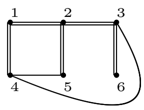
14. U.



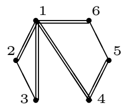
15. $1, 2, 3, 5, 4, 6, 7$.
The following picture better reflects the order in which the vertices are encountered.



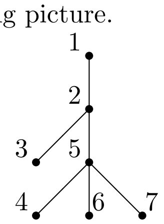
16. $1, 2, 4, 3, 5, 6$.



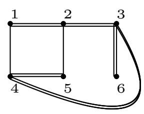
17. $1, 2, 3, 4, 6, 5$.



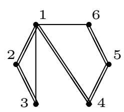
18. $3, 4, 6, 7, 5, 2, 1$. See the following picture.



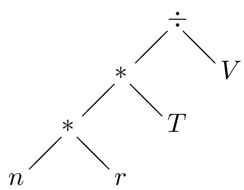
19. $5, 4, 6, 3, 2, 1$.



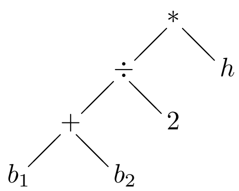
20. $3, 2, 6, 5, 4, 1$.



21.



22. (a) $b_1 \quad b_2 \quad + \quad 2 \quad \div \quad h \quad *$.
(b) $* \quad \div \quad + \quad b_1 \quad b_2 \quad 2 \quad h$.
(c) $b_1 \quad + \quad b_2 \quad \div \quad 2 \quad * \quad h$.



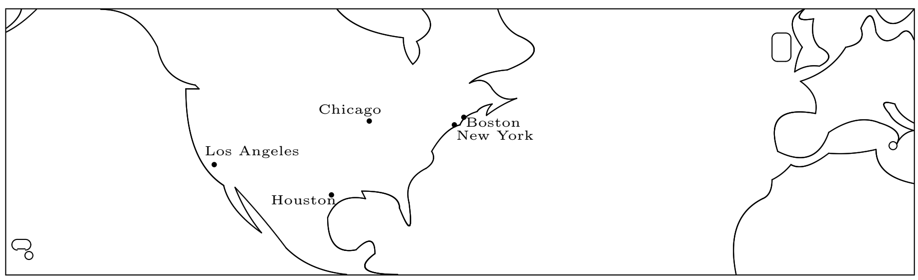
23. 12.

4	5	1	$+$	2	$\div$	$*$
4	6	2	$\div$	$*$		
4	3	$*$				
12						

24. 7.

$+$	1	$*$	$-$	8	5	2
$+$	1	$*$	3	2		
$+$	1	6				
7						

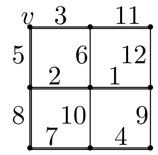
25. Los Angeles, Houston, Chicago, New York, Boston.



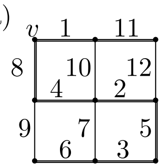
Pay attention to the algorithm and not the fact that the answer agrees with the west-to-east ordering on this map.

26. vw, vwy.
The entire list is v, vw, vwx, vw, vwy, vw, v, failure.

27.

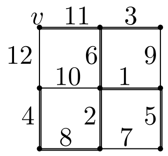


28.(a)



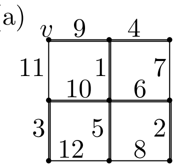
(b) $400,000.

29.



$1, 2, 5, 6, 3, 8, 4, 11$

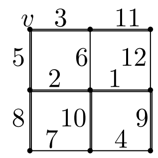
30.(a)



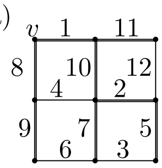
$1, 4, 5, 6, 2, 9, 10, 3$

(b) $400,000.

31.



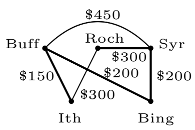
32.(a)



(b) 18 minutes.

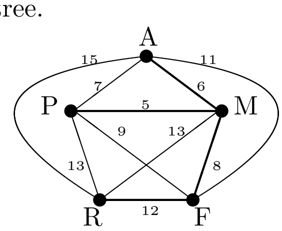
33. (a) True. In Section 10.3, see Exercises 19 and 23.
(b) True. The relative sizes of the distinct weights is all that matters. An edge ordering is irrelevant in this case.

34. (a) 200+200+150 = $550.
(b) A shortest path tree.



A shortest path tree from Syracuse is shown in bold. The path from Syracuse to Ithaca is the longest.

35. (a) $31,000.
(b) A minimum spanning tree.



The weight of the minimum spanning tree is $5 + 6 + 8 + 12 = 31$.

36.

$l=1$		$m=3$		$h=5$
1	2	4	6	8
			$l=m=4$	$h=5$
			6	8
			$l=h=4$	
			6	

Return location = 0.

37. 50 seconds. The worst case complexity is n. The time required is $\frac{n}{1\times 10^9} = \frac{5\times 10^{10} \text{ instructions}}{1\times 10^9 \text{ instructions per second}} = 50$ seconds.

38. $n-1$. At worst, we have one comparison for each $2 \le i \le n$. There are $n-1$ such i.

39. True. Both $3x$ and $x-1$ are polynomials of degree 1.

40. False. x^2+1 is a higher degree polynomial than $25x$.

41. False. $x^3 - x^2 + 7$ is a lower degree polynomial than x^4.

42. True. Both x^2 and $4x^2+5x$ are polynomials of degree 2.

43. False. In fact, $O(n \log_2 n) \subset O(n^2)$, in Theorem 10.16.

44. (a) The first, since $f(n) = 30n^3 + 500n$ has is a higher degree polynomial than $g(n) = 600n^2 + 10000$.
(b) The second, since $g(15) > f(15)$.
(c) 21 years, since $g(n) \ge f(n)$ iff $n \le 20$.
In particular, solve $f(n) - g(n) = 0$ and get $n = 20$.

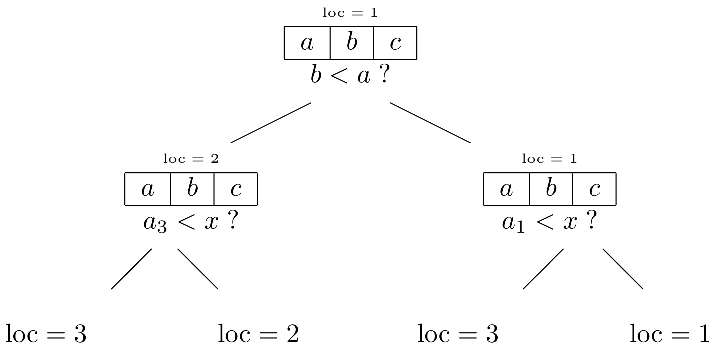
45. $\exists\, C \in \mathbb{R}^+$ such that $\forall\, x > 0,\ |f(x)| \le C \cdot 1$.
This is the definition of bounded with $M = C$. It also characterizes the assertion that $f(x) \in O(1)$.

46. $O(\log_2 n) \subset O(n)$ by Theorem 10.16. $O(n) \subseteq O(n^{\frac{3}{2}})$ by Lemma 10.11.
Hence, $O(\log_2 n) \subseteq O(n^{\frac{3}{2}})$ by the transitivity of $\subseteq$.

47. Yes, $O(n-1) \subseteq O(n^2)$.
The worst-case complexity of Minimum is $n-1$ and $O(n-1) = O(n) \subseteq O(n^2)$.

48. Yes. Minimum always uses $n-1$ comparisons.
Note that the while loop is never exited until $i = n$. Also $\Theta(n-1) = \Theta(n)$.

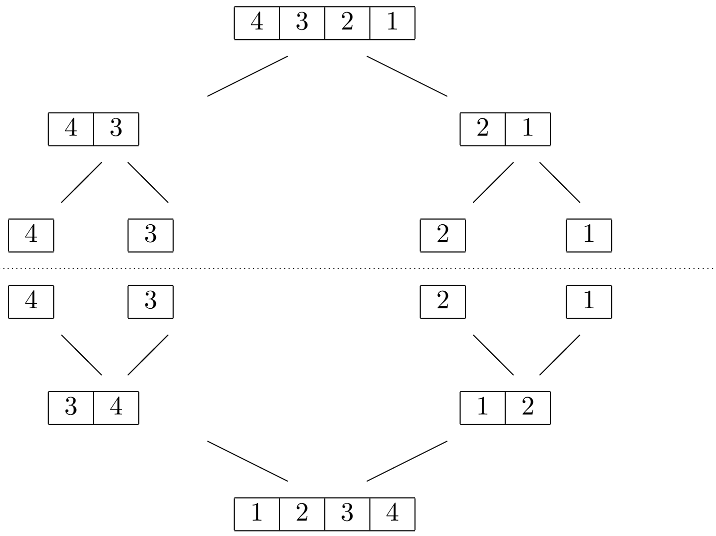
49.



50.

$i=2:$	6	3	8	2	5
$i=3:$	3	6	8	2	5
$i=4:$	3	6	8	2	5
$i=5:$	2	3	6	8	5
end :	2	3	5	6	8

51.



52.

$i=4, j=1:$	4	2	1	3
$j=2:$	2	4	1	3
$j=3:$	2	1	4	3
$i=3, j=1:$	2	1	3	4
$j=2:$	1	2	3	4
end :	1	2	3	4

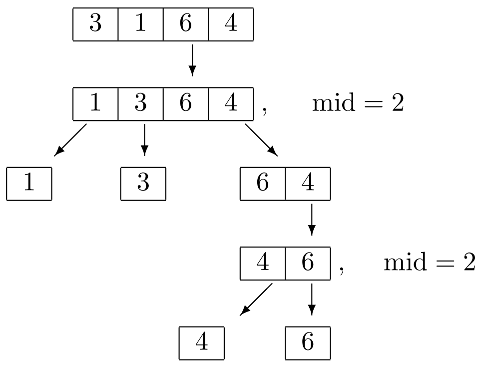
53.

$$\begin{array}{rcccc} i=1: & 2 & 4 & 3 & 1 \\ i=2: & 1 & 4 & 3 & 2 \\ i=3: & 1 & 2 & 3 & 4 \\ \text{end}: & 1 & 2 & 3 & 4 \end{array}$$

54. In order.
See Algorithm 10.8. When A is ordered, the comparison $A[j] < A[i]$ holds true for every $j < i$. Also, see the proof of Example 10.23.

55. No.
See Algorithm 10.11. The comparison $A[j] > A[j+1]$ is done for each $n \geq i \geq 2$ and $1 \leq j \leq i-1$, regardless of the order of A.

56.



A left-to-right reading of the leaves shows the correct order for A.

57. No.
It is $\Theta(n^2)$, since we are measuring worst-case complexity.