15

Multiple Integration

15.1 Volume and Average Height

Consider a surface $f(x, y)$; you might temporarily think of this as representing physical topography—a hilly landscape, perhaps. What is the average height of the surface (or average altitude of the landscape) over some region?

As with most such problems, we start by thinking about how we might approximate the answer. Suppose the region is a rectangle, $[a, b] \times [c, d]$. We can divide the rectangle into a grid, m subdivisions in one direction and n in the other, as indicated in figure 15.1.1. We pick x values x_0, x_1,..., x_{m-1} in each subdivision in the x direction, and similarly in the y direction. At each of the points (x_i, y_j) in one of the smaller rectangles in the grid, we compute the height of the surface: $f(x_i, y_j)$. Now the average of these heights should be (depending on the fineness of the grid) close to the average height of the surface:

$$\frac{f(x_0, y_0) + f(x_1, y_0) + \cdots + f(x_0, y_1) + f(x_1, y_1) + \cdots + f(x_{m-1}, y_{n-1})}{mn}.$$

As both m and n go to infinity, we expect this approximation to converge to a fixed value, the actual average height of the surface. For reasonably nice functions this does indeed happen.

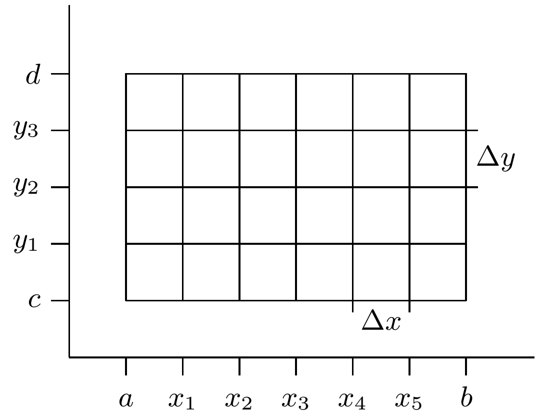

Figure 15.1.1 A rectangular subdivision of $[a, b] \times [c, d]$.

Using sigma notation, we can rewrite the approximation:

$$\begin{aligned}\frac{1}{mn}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j, y_i) &= \frac{1}{(b-a)(d-c)}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j, y_i)\frac{b-a}{m}\frac{d-c}{n}\\ &= \frac{1}{(b-a)(d-c)}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j, y_i)\Delta x\Delta y.\end{aligned}$$

The two parts of this product have useful meaning: $(b-a)(d-c)$ is of course the area of the rectangle, and the double sum adds up mn terms of the form $f(x_j, y_i)\Delta x\Delta y$, which is the height of the surface at a point times the area of one of the small rectangles into which we have divided the large rectangle. In short, each term $f(x_j, y_i)\Delta x\Delta y$ is the volume of a tall, thin, rectangular box, and is approximately the volume under the surface and above one of the small rectangles; see figure 15.1.2. When we add all of these up, we get an approximation to the volume under the surface and above the rectangle $R = [a, b] \times [c, d]$. When we take the limit as m and n go to infinity, the double sum becomes the actual volume under the surface, which we divide by $(b-a)(d-c)$ to get the average height.

Double sums like this come up in many applications, so in a way it is the most important part of this example; dividing by $(b-a)(d-c)$ is a simple extra step that allows the computation of an average. As we did in the single variable case, we introduce a special notation for the limit of such a double sum:

$$\lim_{m,n\to\infty}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j, y_i)\Delta x\Delta y = \iint\limits_R f(x, y)\,dx\,dy = \iint\limits_R f(x, y)\,dA,$$

the **double integral** of f over the region R. The notation dA indicates a small bit of area, without specifying any particular order for the variables x and y; it is shorter and

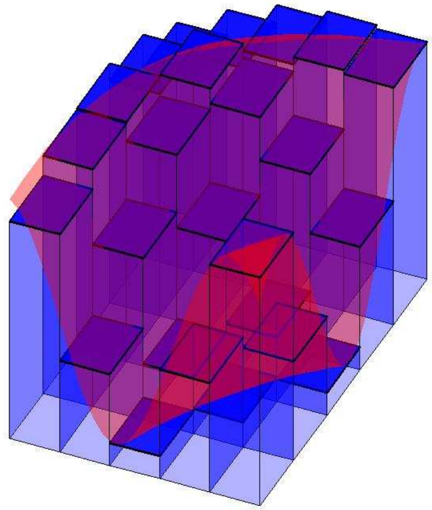

Figure 15.1.2 Approximating the volume under a surface.

more "generic" than writing $dx\,dy$. The average height of the surface in this notation is

$$\frac{1}{(b-a)(d-c)}\iint\limits_R f(x,y)\,dA.$$

The next question, of course, is: How do we compute these double integrals? You might think that we will need some two-dimensional version of the Fundamental Theorem of Calculus, but as it turns out we can get away with just the single variable version, applied twice.

Going back to the double sum, we can rewrite it to emphasize a particular order in which we want to add the terms:

$$\sum_{i=0}^{n-1}\left(\sum_{j=0}^{m-1} f(x_j,y_i)\Delta x\right)\Delta y.$$

In the sum in parentheses, only the value of x_j is changing; y_i is temporarily constant. As m goes to infinity, this sum has the right form to turn into an integral:

$$\lim_{m\to\infty}\sum_{j=0}^{m-1} f(x_j,y_i)\Delta x = \int_a^b f(x,y_i)\,dx.$$

So after we take the limit as m goes to infinity, the sum is

$$\sum_{i=0}^{n-1}\left(\int_a^b f(x,y_i)\,dx\right)\Delta y.$$

Of course, for different values of y_i this integral has different values; in other words, it is really a function applied to y_i:

$$G(y) = \int_a^b f(x, y)\, dx.$$

If we substitute back into the sum we get

$$\sum_{i=0}^{n-1} G(y_i)\Delta y.$$

This sum has a nice interpretation. The value $G(y_i)$ is the area of a cross section of the region under the surface $f(x, y)$, namely, when $y = y_i$. The quantity $G(y_i)\Delta y$ can be interpreted as the volume of a solid with face area $G(y_i)$ and thickness Δy. Think of the surface $f(x, y)$ as the top of a loaf of sliced bread. Each slice has a cross-sectional area and a thickness; $G(y_i)\Delta y$ corresponds to the volume of a single slice of bread. Adding these up approximates the total volume of the loaf. (This is very similar to the technique we used to compute volumes in section 9.3, except that there we need the cross-sections to be in some way "the same".) Figure 15.1.3 shows this "sliced loaf" approximation using the same surface as shown in figure 15.1.2. Nicely enough, this sum looks just like the sort of sum that turns into an integral, namely,

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=0}^{n-1} G(y_i)\Delta y &= \int_c^d G(y)\, dy \\ &= \int_c^d \int_a^b f(x, y)\, dx\, dy. \end{aligned}$$

Let's be clear about what this means: we first will compute the inner integral, temporarily treating y as a constant. We will do this by finding an anti-derivative with respect to x, then substituting $x = a$ and $x = b$ and subtracting, as usual. The result will be an expression with no x variable but some occurrences of y. Then the outer integral will be an ordinary one-variable problem, with y as the variable.

EXAMPLE 15.1.1 Figure 15.1.2 shows the function $\sin(xy)+6/5$ on $[0.5, 3.5]\times[0.5, 2.5]$. The volume under this surface is

$$\int_{0.5}^{2.5} \int_{0.5}^{3.5} \sin(xy) + \frac{6}{5}\, dx\, dy.$$

The inner integral is

$$\int_{0.5}^{3.5} \sin(xy) + \frac{6}{5}\, dx = \frac{-\cos(xy)}{y} + \frac{6x}{5}\Big|_{0.5}^{3.5} = \frac{-\cos(3.5y)}{y} + \frac{\cos(0.5y)}{y} + \frac{18}{5}.$$

Unfortunately, this gives a function for which we can't find a simple anti-derivative. To complete the problem we could use Sage or similar software to approximate the integral.

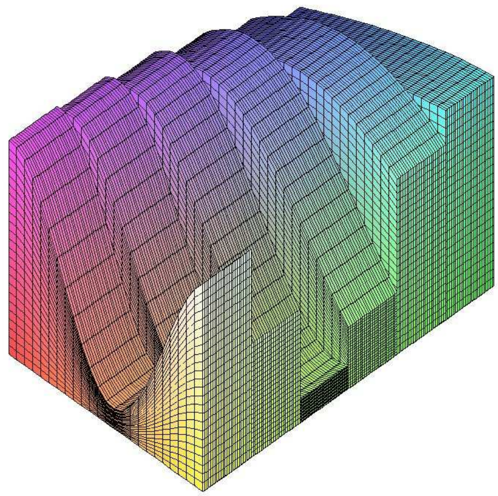

Figure 15.1.3 Approximating the volume under a surface with slices. (AP)

Doing this gives a volume of approximately 8.84, so the average height is approximately $8.84/6 \approx 1.47$. □

Because addition and multiplication are commutative and associative, we can rewrite the original double sum:

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j, y_i)\Delta x\Delta y = \sum_{j=0}^{m-1}\sum_{i=0}^{n-1} f(x_j, y_i)\Delta y\Delta x.$$

Now if we repeat the development above, the inner sum turns into an integral:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_j, y_i)\Delta y = \int_c^d f(x_j, y)\,dy,$$

and then the outer sum turns into an integral:

$$\lim_{m\to\infty}\sum_{j=0}^{m-1}\left(\int_c^d f(x_j, y)\,dy\right)\Delta x = \int_a^b\int_c^d f(x, y)\,dy\,dx.$$

In other words, we can compute the integrals in either order, first with respect to x then y, or vice versa. Thinking of the loaf of bread, this corresponds to slicing the loaf in a direction perpendicular to the first.

We haven't really proved that the value of a double integral is equal to the value of the corresponding two single integrals in either order of integration, but provided the function is reasonably nice, this is true; the result is called **Fubini's Theorem**.

EXAMPLE 15.1.2 We compute $\displaystyle\iint\limits_R 1+(x-1)^2+4y^2\,dA$, where $R=[0,3]\times[0,2]$, in two ways.

First,

$$\begin{aligned}
\int_0^3\int_0^2 1+(x-1)^2+4y^2\,dy\,dx &= \int_0^3 y+(x-1)^2y+\frac{4}{3}y^3\Big|_0^2\,dx\\
&= \int_0^3 2+2(x-1)^2+\frac{32}{3}\,dx\\
&= 2x+\frac{2}{3}(x-1)^3+\frac{32}{3}x\Big|_0^3\\
&= 6+\frac{2}{3}\cdot 8+\frac{32}{3}\cdot 3-(0-1\cdot\frac{2}{3}+0)\\
&= 44.
\end{aligned}$$

In the other order:

$$\begin{aligned}
\int_0^2\int_0^3 1+(x-1)^2+4y^2\,dx\,dy &= \int_0^2 x+\frac{(x-1)^3}{3}+4y^2x\Big|_0^3\,dy\\
&= \int_0^2 3+\frac{8}{3}+12y^2+\frac{1}{3}\,dy\\
&= 3y+\frac{8}{3}y+4y^3+\frac{1}{3}y\Big|_0^2\\
&= 6+\frac{16}{3}+32+\frac{2}{3}\\
&= 44.
\end{aligned}$$

□

In this example there is no particular reason to favor one direction over the other; in some cases, one direction might be much easier than the other, so it's usually worth considering the two different possibilities.

Frequently we will be interested in a region that is not simply a rectangle. Let's compute the volume under the surface $x+2y^2$ above the region described by $0\le x\le 1$ and $0\le y\le x^2$, shown in figure 15.1.4.

In principle there is nothing more difficult about this problem. If we imagine the three-dimensional region under the surface and above the parabolic region as an oddly shaped loaf of bread, we can still slice it up, approximate the volume of each slice, and add these

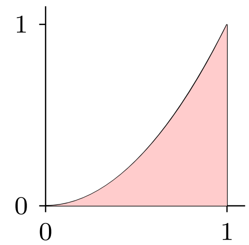

Figure 15.1.4 A parabolic region of integration.

volumes up. For example, if we slice perpendicular to the x axis at x_i, the thickness of a slice will be Δx and the area of the slice will be

$$\int_0^{x_i^2} x_i + 2y^2\,dy.$$

When we add these up and take the limit as Δx goes to 0, we get the double integral

$$\begin{aligned}\int_0^1 \int_0^{x^2} x + 2y^2\,dy\,dx &= \int_0^1 xy + \frac{2}{3}y^3\Big|_0^{x^2}\,dx\\ &= \int_0^1 x^3 + \frac{2}{3}x^6\,dx\\ &= \frac{x^4}{4} + \frac{2}{21}x^7\Big|_0^1\\ &= \frac{1}{4} + \frac{2}{21} = \frac{29}{84}.\end{aligned}$$

We could just as well slice the solid perpendicular to the y axis, in which case we get

$$\begin{aligned}\int_0^1 \int_{\sqrt{y}}^1 x + 2y^2\,dx\,dy &= \int_0^1 \frac{x^2}{2} + 2y^2 x\Big|_{\sqrt{y}}^1\,dy\\ &= \int_0^1 \frac{1}{2} + 2y^2 - \frac{y}{2} - 2y^2\sqrt{y}\,dy\\ &= \frac{y}{2} + \frac{2}{3}y^3 - \frac{y^2}{4} - \frac{4}{7}y^{7/2}\Big|_0^1\\ &= \frac{1}{2} + \frac{2}{3} - \frac{1}{4} - \frac{4}{7} = \frac{29}{84}.\end{aligned}$$

What is the average height of the surface over this region? As before, it is the volume divided by the area of the base, but now we need to use integration to compute the area of the base, since it is not a simple rectangle. The area is

$$\int_0^1 x^2\,dx = \frac{1}{3},$$

so the average height is 29/28.

EXAMPLE 15.1.3 Find the volume under the surface $z = \sqrt{1-x^2}$ and above the triangle formed by $y = x$, $x = 1$, and the x-axis.

Let's consider the two possible ways to set this up:

$$\int_0^1 \int_0^x \sqrt{1-x^2}\,dy\,dx \qquad \text{or} \qquad \int_0^1 \int_y^1 \sqrt{1-x^2}\,dx\,dy.$$

Which appears easier? In the first, the inner integral is easy, because we need an antiderivative with respect to y, and the entire integrand $\sqrt{1-x^2}$ is constant with respect to y. Of course, the outer integral may be more difficult. In the second, the inner integral is mildly unpleasant—a trig substitution. So let's try the first one, since the first step is easy, and see where that leaves us.

$$\int_0^1 \int_0^x \sqrt{1-x^2}\,dy\,dx = \int_0^1 y\sqrt{1-x^2}\Big|_0^x\,dx = \int_0^1 x\sqrt{1-x^2}\,dx.$$

This is quite easy, since the substitution $u = 1-x^2$ works:

$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{3}u^{3/2} = -\frac{1}{3}(1-x^2)^{3/2}.$$

Then

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{3}(1-x^2)^{3/2}\Big|_0^1 = \frac{1}{3}.$$

This is a good example of how the order of integration can affect the complexity of the problem. In this case it is possible to do the other order, but it is a bit messier. In some cases one order may lead to a very difficult or impossible integral; it's usually worth considering both possibilities before going very far. □

Exercises 15.1.

1. Compute $\displaystyle\int_0^2 \int_0^4 1 + x\,dy\,dx$. ⇒

2. Compute $\displaystyle\int_{-1}^1 \int_0^2 x + y\,dy\,dx$. ⇒

3. Compute $\displaystyle\int_1^2 \int_0^y xy\,dx\,dy$. ⇒

4. Compute $\displaystyle\int_0^1 \int_{y^2/2}^{\sqrt{y}} dx\,dy$. ⇒

5. Compute $\displaystyle\int_1^2 \int_1^x \frac{x^2}{y^2}\,dy\,dx$. ⇒

6. Compute $\displaystyle\int_0^1 \int_0^{x^2} \frac{y}{e^x}\,dy\,dx$. ⇒

7. Compute $\displaystyle\int_0^{\sqrt{\pi/2}} \int_0^{x^2} x\cos y\,dy\,dx$. ⇒

8. Compute $\displaystyle\int_0^{\pi/2} \int_0^{\cos\theta} r^2(\cos\theta - r)\,dr\,d\theta$. ⇒

9. Compute: $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy$. ⇒

10. Compute: $\displaystyle\int_0^1 \int_{y^2}^1 y\sin(x^2)\,dx\,dy$. ⇒

11. Compute: $\displaystyle\int_0^1 \int_{x^2}^1 x\sqrt{1+y^2}\,dy\,dx$ ⇒

12. Compute: $\displaystyle\int_0^1 \int_0^y \frac{2}{\sqrt{1-x^2}}\,dx\,dy$ ⇒

13. Compute: $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$ ⇒

14. Compute $\displaystyle\int_{-1}^1 \int_0^{1-x^2} x^2 - \sqrt{y}\,dy\,dx$. ⇒

15. Compute $\displaystyle\int_0^{\sqrt{2}/2} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} x\,dy\,dx$. ⇒

16. Evaluate $\displaystyle\iint x^2\,dA$ over the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$, and $x = 8$. ⇒

17. Find the volume below $z = 1 - y$ above the region $-1 \le x \le 1$, $0 \le y \le 1 - x^2$. ⇒

18. Find the volume bounded by $z = x^2 + y^2$ and $z = 4$. ⇒

19. Find the volume in the first octant bounded by $y^2 = 4 - x$ and $y = 2z$. ⇒

20. Find the volume in the first octant bounded by $y^2 = 4x$, $2x + y = 4$, $z = y$, and $y = 0$. ⇒

21. Find the volume in the first octant bounded by $x + y + z = 9$, $2x + 3y = 18$, and $x + 3y = 9$. $\Rightarrow$

22. Find the volume in the first octant bounded by $x^2 + y^2 = a^2$ and $z = x + y$. $\Rightarrow$

23. Find the volume bounded by $4x^2 + y^2 = 4z$ and $z = 2$. $\Rightarrow$

24. Find the volume bounded by $z = x^2 + y^2$ and $z = y$. $\Rightarrow$

25. Find the volume under the surface $z = xy$ above the triangle with vertices $(1, 1, 0)$, $(4, 1, 0)$, $(1, 2, 0)$. $\Rightarrow$

26. Find the volume enclosed by $y = x^2$, $y = 4$, $z = x^2$, $z = 0$. $\Rightarrow$

27. A swimming pool is circular with a 40 meter diameter. The depth is constant along east-west lines and increases linearly from 2 meters at the south end to 7 meters at the north end. Find the volume of the pool. $\Rightarrow$

28. Find the average value of $f(x, y) = e^y \sqrt{x + e^y}$ on the rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 1)$ and $(0, 1)$. $\Rightarrow$

29. Figure 15.1.5 shows a temperature map of Colorado. Use the data to estimate the average temperature in the state using 4, 16 and 25 subdivisions. Give both an upper and lower estimate. Why do we like Colorado for this problem? What other state(s) might we like?

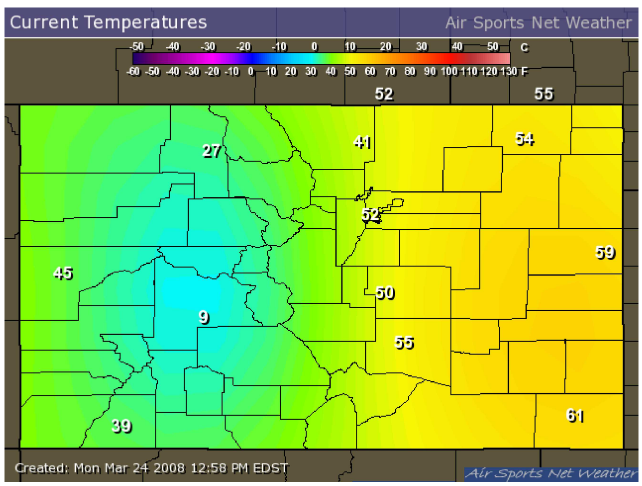

Figure 15.1.5 Colorado temperatures.

30. Three cylinders of radius 1 intersect at right angles at the origin, as shown in figure 15.1.6. Find the volume contained inside all three cylinders. $\Rightarrow$

31. Prove that if $f(x, y)$ is integrable and if $g(x, y) = \displaystyle\int_a^x \int_b^y f(s, t)\, dt\, ds$ then $g_{xy} = g_{yx} = f(x, y)$.

32. Reverse the order of integration on each of the following integrals

a. $\displaystyle\int_0^9 \int_0^{\sqrt{9-y}} f(x, y)\, dx\, dy$

b. $\displaystyle\int_1^2 \int_0^{\ln x} f(x, y)\, dy\, dx$

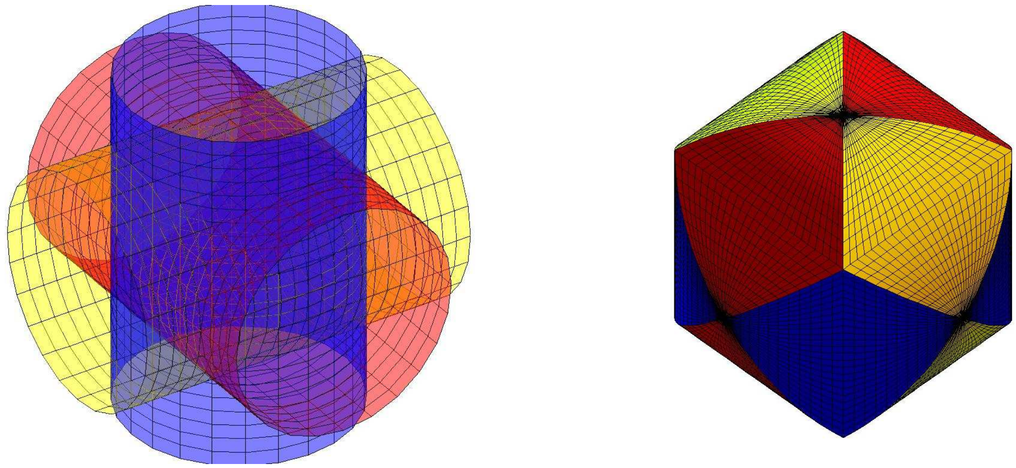

Figure 15.1.6 Intersection of three cylinders. (AP)

c. $\displaystyle\int_0^1 \int_{\arcsin y}^{\pi/2} f(x,y)\,dx\,dy$

d. $\displaystyle\int_0^1 \int_{4x}^{4} f(x,y)\,dy\,dx$

e. $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} f(x,y)\,dx\,dy$

$\Rightarrow$

33. What are the parallels between Fubini's Theorem and Clairaut's Theorem (14.6.2)?

15.2 Double Integrals in Cylindrical Coordinates

Suppose we have a surface given in cylindrical coordinates as $z = f(r,\theta)$ and we wish to find the integral over some region. We could attempt to translate into rectangular coordinates and do the integration there, but it is often easier to stay in cylindrical coordinates.

How might we approximate the volume under such a surface in a way that uses cylindrical coordinates directly? The basic idea is the same as before: we divide the region into many small regions, multiply the area of each small region by the height of the surface somewhere in that little region, and add them up. What changes is the shape of the small regions; in order to have a nice representation in terms of r and θ, we use small pieces of ring-shaped areas, as shown in figure 15.2.1. Each small region is roughly rectangular, except that two sides are segments of a circle and the other two sides are not quite parallel. Near a point (r,θ), the length of either circular arc is about $r\Delta\theta$ and the length of each straight side is simply Δr. When Δr and $\Delta\theta$ are very small, the region is nearly a rectangle with area $r\Delta r\Delta\theta$, and the volume under the surface is approximately

$$\sum\sum f(r_i,\theta_j) r_i \Delta r \Delta\theta.$$

In the limit, this turns into a double integral

$$\int_{\theta_0}^{\theta_1}\int_{r_0}^{r_1} f(r,\theta) r\,dr\,d\theta.$$

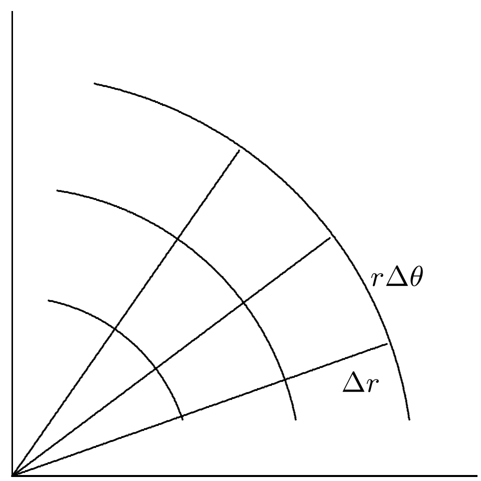

Figure 15.2.1 A cylindrical coordinates "grid".

EXAMPLE 15.2.1 Find the volume under $z = \sqrt{4-r^2}$ above the quarter circle bounded by the two axes and the circle $x^2+y^2=4$ in the first quadrant.

In terms of r and θ, this region is described by the restrictions $0 \le r \le 2$ and $0 \le \theta \le \pi/2$, so we have

$$\begin{aligned}\int_0^{\pi/2}\int_0^2 \sqrt{4-r^2}\, r\,dr\,d\theta &= \int_0^{\pi/2} -\frac{1}{3}(4-r^2)^{3/2}\Big|_0^2\,d\theta\\ &= \int_0^{\pi/2}\frac{8}{3}\,d\theta\\ &= \frac{4\pi}{3}.\end{aligned}$$

The surface is a portion of the sphere of radius 2 centered at the origin, in fact exactly one-eighth of the sphere. We know the formula for volume of a sphere is $(4/3)\pi r^3$, so the volume we have computed is $(1/8)(4/3)\pi 2^3 = (4/3)\pi$, in agreement with our answer. (From another point of view, what we've done is prove that the volume of a sphere of radius 2 is (32/3). If you replace 2 by a and do the integral again, it is not any more difficult, and you will prove that the volume of a sphere of radius a is $(4/3)\pi a^3$.) $\square$

This example is much like a simple one in rectangular coordinates: the region of interest may be described exactly by a constant range for each of the variables. As with rectangular coordinates, we can adapt the method to deal with more complicated regions.

EXAMPLE 15.2.2 Find the volume under $z = \sqrt{4 - r^2}$ above the region enclosed by the curve $r = 2\cos\theta$, $-\pi/2 \le \theta \le \pi/2$; see figure 15.2.2. The region is described in polar coordinates by the inequalities $-\pi/2 \le \theta \le \pi/2$ and $0 \le r \le 2\cos\theta$, so the double integral is

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \sqrt{4 - r^2}\, r\, dr\, d\theta = 2\int_0^{\pi/2} \int_0^{2\cos\theta} \sqrt{4 - r^2}\, r\, dr\, d\theta.$$

We can rewrite the integral as shown because of the symmetry of the volume; this avoids a complication during the evaluation. Proceeding:

$$\begin{aligned}
2\int_0^{\pi/2} \int_0^{2\cos\theta} \sqrt{4 - r^2}\, r\, dr\, d\theta &= 2\int_0^{\pi/2} -\frac{1}{3}\,(4 - r^2)^{3/2}\Big|_0^{2\cos\theta}\, d\theta \\
&= 2\int_0^{\pi/2} -\frac{8}{3}\sin^3\theta + \frac{8}{3}\, d\theta \\
&= 2\left(-\frac{8}{3}\frac{\cos^3\theta}{3} - \cos\theta + \frac{8}{3}\theta\right)\Big|_0^{\pi/2} \\
&= \frac{8}{3}\pi - \frac{32}{9}.
\end{aligned}$$

□

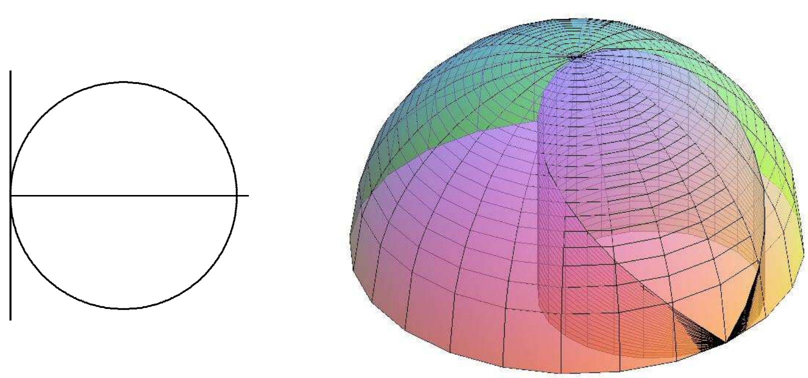

Figure 15.2.2 Volume over a region with non-constant limits.

You might have learned a formula for computing areas in polar coordinates. It is possible to compute areas as volumes, so that you need only remember one technique.

Consider the surface $z = 1$, a horizontal plane. The volume under this surface and above a region in the x-y plane is simply $1 \cdot$ (area of the region), so computing the volume really just computes the area of the region.

EXAMPLE 15.2.3 Find the area outside the circle $r = 2$ and inside $r = 4\sin\theta$; see figure 15.2.3. The region is described by $\pi/6 \le \theta \le 5\pi/6$ and $2 \le r \le 4\sin\theta$, so the integral is

$$\begin{aligned}\int_{\pi/6}^{5\pi/6}\int_{2}^{4\sin\theta} 1\, r\,dr\,d\theta &= \int_{\pi/6}^{5\pi/6} \left.\frac{1}{2}r^2\right|_{2}^{4\sin\theta}\,d\theta\\ &= \int_{\pi/6}^{5\pi/6} 8\sin^2\theta - 2\,d\theta\\ &= \frac{4}{3}\pi + 2\sqrt{3}.\end{aligned}$$

□

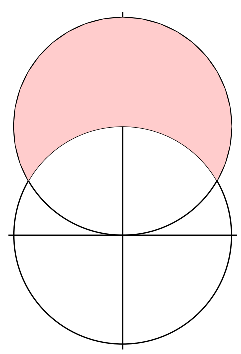

Figure 15.2.3 Finding area by computing volume.

Exercises 15.2.

1. Find the volume above the x-y plane, under the surface $r^2 = 2z$, and inside $r = 2$. ⇒
2. Find the volume inside both $r = 1$ and $r^2 + z^2 = 4$. ⇒
3. Find the volume below $z = \sqrt{1 - r^2}$ and above the top half of the cone $z = r$. ⇒
4. Find the volume below $z = r$, above the x-y plane, and inside $r = \cos\theta$. ⇒
5. Find the volume below $z = r$, above the x-y plane, and inside $r = 1 + \cos\theta$. ⇒
6. Find the volume between $x^2 + y^2 = z^2$ and $x^2 + y^2 = z$. ⇒
7. Find the area inside $r = 1 + \sin\theta$ and outside $r = 2\sin\theta$. ⇒
8. Find the area inside both $r = 2\sin\theta$ and $r = 2\cos\theta$. ⇒
9. Find the area inside the four-leaf rose $r = \cos(2\theta)$ and outside $r = 1/2$. ⇒
10. Find the area inside the cardioid $r = 2(1 + \cos\theta)$ and outside $r = 2$. ⇒
11. Find the area of one loop of the three-leaf rose $r = \cos(3\theta)$. ⇒

12. Compute $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$ by converting to cylindrical coordinates. ⇒

13. Compute $\int_{0}^{a}\int_{-\sqrt{a^2-x^2}}^{0} x^2 y\,dy\,dx$ by converting to cylindrical coordinates. ⇒

14. Find the volume under $z = y^2 + x + 2$ above the region $x^2 + y^2 \le 4$ ⇒

15. Find the volume between $z = x^2y^3$ and $z = 1$ above the region $x^2 + y^2 \le 1$ ⇒

16. Find the volume inside $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$. ⇒

17. Find the volume under $z = r$ above $r = 3 + \cos\theta$. ⇒

18. Figure 15.2.4 shows the plot of $r = 1 + 4\sin(5\theta)$.

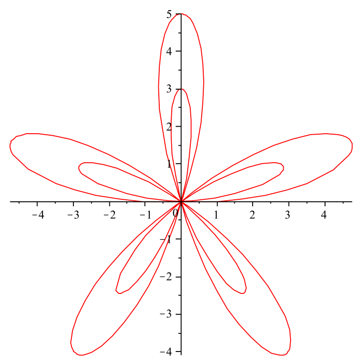

Figure 15.2.4 $r = 1 + 4\sin(5\theta)$

a. Describe the behavior of the graph in terms of the given equation. Specifically, explain maximum and minimum values, number of leaves, and the 'leaves within leaves'.

b. Give an integral or integrals to determine the area outside a smaller leaf but inside a larger leaf.

c. How would changing the value of a in the equation $r = 1 + a\cos(5\theta)$ change the relative sizes of the inner and outer leaves? Focus on values $a \ge 1$. (Hint: How would we change the maximum and minimum values?)

19. Consider the integral $\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dA$, where D is the unit disk centered at the origin. (This is the same shape described in a different way in exercise 13 in section 9.7.) (See the graph here.)

a. Why might this integral be considered improper?

b. Calculate the value of the integral of the same function $1/\sqrt{x^2+y^2}$ over the annulus with outer radius 1 and inner radius δ.

c. Obtain a value for the integral on the whole disk by letting δ approach 0. ⇒

d. For which values λ can we replace the denominator with $(x^2+y^2)^\lambda$ in the original integral and still get a finite value for the improper integral?

15.3 Moment and Center of Mass

Using a single integral we were able to compute the center of mass for a one-dimensional object with variable density, and a two dimensional object with constant density. With a double integral we can handle two dimensions and variable density.

Just as before, the coordinates of the center of mass are

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M},$$

where M is the total mass, M_y is the moment around the y-axis, and M_x is the moment around the x-axis. (You may want to review the concepts in section 9.6.)

The key to the computation, just as before, is the approximation of mass. In the two-dimensional case, we treat density σ as mass per square area, so when density is constant, mass is (density)(area). If we have a two-dimensional region with varying density given by $\sigma(x,y)$, and we divide the region into small subregions with area ΔA, then the mass of one subregion is approximately $\sigma(x_i,y_j)\Delta A$, the total mass is approximately the sum of many of these, and as usual the sum turns into an integral in the limit:

$$M = \int_{x_0}^{x_1}\int_{y_0}^{y_1} \sigma(x,y)\,dy\,dx,$$

and similarly for computations in cylindrical coordinates. Then as before

$$\begin{aligned} M_x &= \int_{x_0}^{x_1}\int_{y_0}^{y_1} y\sigma(x,y)\,dy\,dx \\ M_y &= \int_{x_0}^{x_1}\int_{y_0}^{y_1} x\sigma(x,y)\,dy\,dx. \end{aligned}$$

EXAMPLE 15.3.1 Find the center of mass of a thin, uniform plate whose shape is the region between $y = \cos x$ and the x-axis between $x = -\pi/2$ and $x = \pi/2$. Since the density is constant, we may take $\sigma(x,y) = 1$.

It is clear that $\bar{x} = 0$, but for practice let's compute it anyway. First we compute the mass:

$$M = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} 1\,dy\,dx = \int_{-\pi/2}^{\pi/2} \cos x\,dx = \sin x\big|_{-\pi/2}^{\pi/2} = 2.$$

Next,

$$M_x = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} y\,dy\,dx = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\cos^2 x\,dx = \frac{\pi}{4}.$$

Finally,

$$M_y = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} x\,dy\,dx = \int_{-\pi/2}^{\pi/2} x\cos x\,dx = 0.$$

So $\bar{x} = 0$ as expected, and $\bar{y} = \pi/4/2 = \pi/8$. This is the same problem as in example 9.6.4; it may be helpful to compare the two solutions. □

EXAMPLE 15.3.2 Find the center of mass of a two-dimensional plate that occupies the quarter circle $x^2 + y^2 \le 1$ in the first quadrant and has density $k(x^2 + y^2)$. It seems clear that because of the symmetry of both the region and the density function (both are important!), $\bar{x} = \bar{y}$. We'll do both to check our work.

Jumping right in:

$$M = \int_0^1 \int_0^{\sqrt{1-x^2}} k(x^2 + y^2)\,dy\,dx = k\int_0^1 x^2\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3}\,dx.$$

This integral is something we can do, but it's a bit unpleasant. Since everything in sight is related to a circle, let's back up and try polar coordinates. Then $x^2 + y^2 = r^2$ and

$$M = \int_0^{\pi/2} \int_0^1 k(r^2)\,r\,dr\,d\theta = k\int_0^{\pi/2} \left.\frac{r^4}{4}\right|_0^1 d\theta = k\int_0^{\pi/2} \frac{1}{4}\,d\theta = k\frac{\pi}{8}.$$

Much better. Next, since $y = r\sin\theta$,

$$M_x = k\int_0^{\pi/2} \int_0^1 r^4 \sin\theta\,dr\,d\theta = k\int_0^{\pi/2} \frac{1}{5}\sin\theta\,d\theta = k\left.(-\frac{1}{5}\cos\theta)\right|_0^{\pi/2} = \frac{k}{5}.$$

Similarly,

$$M_y = k\int_0^{\pi/2} \int_0^1 r^4 \cos\theta\,dr\,d\theta = k\int_0^{\pi/2} \frac{1}{5}\cos\theta\,d\theta = k\left.\frac{1}{5}\sin\theta\right|_0^{\pi/2} = \frac{k}{5}.$$

Finally, $\bar{x} = \bar{y} = \dfrac{8}{5\pi}$. □

Exercises 15.3.

1. Find the center of mass of a two-dimensional plate that occupies the square $[0,1] \times [0,1]$ and has density function xy. ⇒
2. Find the center of mass of a two-dimensional plate that occupies the triangle $0 \le x \le 1$, $0 \le y \le x$, and has density function xy. ⇒
3. Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at $(0,0)$ and has density function y. ⇒

4. Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at $(0,0)$ and has density function x^2. ⇒

5. Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x = 2$, $y = x$, and $y = 2x$ and has density function $2x$. ⇒

6. Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x = 0$, $y = x$, and $2x + y = 6$ and has density function x^2. ⇒

7. Find the center of mass of a two-dimensional plate that occupies the region enclosed by the parabolas $x = y^2$, $y = x^2$ and has density function $\sqrt{x}$. ⇒

8. Find the centroid of the area in the first quadrant bounded by $x^2 - 8y + 4 = 0$, $x^2 = 4y$, and $x = 0$. (Recall that the centroid is the center of mass when the density is 1 everywhere.) ⇒

9. Find the centroid of one loop (say the loop along the x-axis) of the three-leaf rose $r = \cos(3\theta)$. (Recall that the centroid is the center of mass when the density is 1 everywhere, and that the mass in this case is the same as the area, which was the subject of exercise 11 in section 15.2.) The computations of the integrals for the moments M_x and M_y are elementary but quite long; Sage can help. ⇒

10. Find the center of mass of a two dimensional object that occupies the region $0 \le x \le \pi$, $0 \le y \le \sin x$, with density $\sigma = 1$. ⇒

11. A two-dimensional object has shape given by $r = 1 + \cos\theta$ and density $\sigma(r,\theta) = 2 + \cos\theta$. Set up the three integrals required to compute the center of mass. ⇒

12. A two-dimensional object has shape given by $r = \cos\theta$ and density $\sigma(r,\theta) = r + 1$. Set up the three integrals required to compute the center of mass. ⇒

13. A two-dimensional object sits inside $r = 1 + \cos\theta$ and outside $r = \cos\theta$, and has density 1 everywhere. Set up the integrals required to compute the center of mass. ⇒

15.4 Surface Area

We next seek to compute the area of a surface above (or below) a region in the x-y plane. How might we approximate this? We start, as usual, by dividing the region into a grid of small rectangles. We want to approximate the area of the surface above one of these small rectangles. The area is very close to the area of the tangent plane above the small rectangle. If the tangent plane just happened to be horizontal, of course the area would simply be the area of the rectangle. For a typical plane, however, the area is the area of a parallelogram, as indicated in figure 15.4.1. Note that the area of the parallelogram is obviously larger the more "tilted" the tangent plane. In the interactive figure you can see that viewed from above the four parallelograms exactly cover a rectangular region in the x-y plane.

Now recall a curious fact: the area of a parallelogram can be computed as the cross product of two vectors (page 314). We simply need to acquire two vectors, parallel to the sides of the parallelogram and with lengths to match. But this is easy: in the x direction we use the tangent vector we already know, namely $\langle 1, 0, f_x\rangle$ and multiply by Δx to shrink it to the right size: $\langle \Delta x, 0, f_x \Delta x\rangle$. In the y direction we do the same thing and

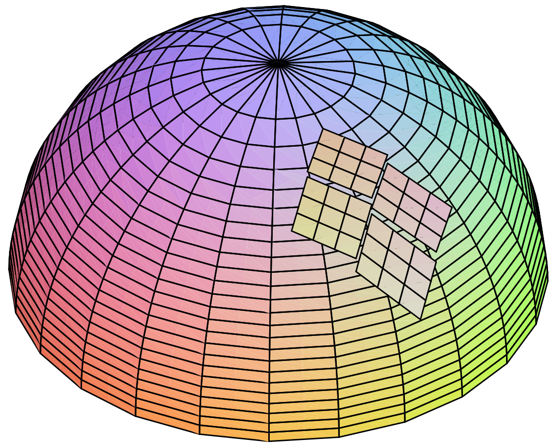

Figure 15.4.1 Small parallelograms at points of tangency. (AP)

get $\langle 0, \Delta y, f_y \Delta y\rangle$. The cross product of these vectors is $\langle f_x, f_y, -1\rangle\,\Delta x\,\Delta y$ with length $\sqrt{f_x^2+f_y^2+1}\,\Delta x\,\Delta y$, the area of the parallelogram. Now we add these up and take the limit, to produce the integral

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\sqrt{f_x^2+f_y^2+1}\,dy\,dx.$$

As before, the limits need not be constant.

EXAMPLE 15.4.1 We find the area of the hemisphere $z=\sqrt{1-x^2-y^2}$. We compute the derivatives

$$f_x=\frac{-x}{\sqrt{1-x^2-y^2}} \qquad f_y=\frac{-y}{\sqrt{1-x^2-y^2}},$$

and then the area is

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}+1}\,dy\,dx.$$

This is a bit on the messy side, but we can use polar coordinates:

$$\int_0^{2\pi}\int_0^1\sqrt{\frac{1}{1-r^2}}\,r\,dr\,d\theta.$$

This integral is improper, since the function is undefined at the limit 1. We therefore compute

$$\lim_{a\to 1^-}\int_0^a\sqrt{\frac{1}{1-r^2}}\,r\,dr=\lim_{a\to 1^-}-\sqrt{1-a^2}+1=1,$$

using the substitution $u = 1 - r^2$. Then the area is

$$\int_0^{2\pi} 1\,d\theta = 2\pi.$$

You may recall that the area of a sphere of radius r is $4\pi r^2$, so half the area of a unit sphere is $(1/2)4\pi = 2\pi$, in agreement with our answer. (Alternately, we can view this calculation as proving that the formula for the area of a sphere is correct.) □

Exercises 15.4.

1. Find the area of the surface of a right circular cone of height h and base radius a. ⇒

2. Find the area of the portion of the plane $z = mx$ inside the cylinder $x^2 + y^2 = a^2$. ⇒

3. Find the area of the portion of the plane $x + y + z = 1$ in the first octant. ⇒

4. Find the area of the upper half of the cone $x^2 + y^2 = z^2$ inside the cylinder $x^2 + y^2 - 2x = 0$. ⇒

5. Find the area of the upper half of the cone $x^2 + y^2 = z^2$ above the interior of one loop of $r = \cos(2\theta)$. ⇒

6. Find the area of the upper hemisphere of $x^2 + y^2 + z^2 = 1$ above the interior of one loop of $r = \cos(2\theta)$. ⇒

7. The plane $ax + by + cz = d$ cuts a triangle in the first octant provided that a, b, c and d are all positive. Find the area of this triangle. ⇒

8. Find the area of the portion of the cone $x^2 + y^2 = 3z^2$ lying above the xy plane and inside the cylinder $x^2 + y^2 = 4y$. ⇒

15.5 Triple Integrals

It will come as no surprise that we can also do triple integrals—integrals over a three-dimensional region. The simplest application allows us to compute volumes in an alternate way.

To approximate a volume in three dimensions, we can divide the three-dimensional region into small rectangular boxes, each $\Delta x \times \Delta y \times \Delta z$ with volume $\Delta x \Delta y \Delta z$. Then we add them all up and take the limit, to get an integral:

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} dz\,dy\,dx.$$

If the limits are constant, we are simply computing the volume of a rectangular box.

EXAMPLE 15.5.1 We use an integral to compute the volume of the box with opposite corners at $(0, 0, 0)$ and $(1, 2, 3)$.

$$\int_0^1 \int_0^2 \int_0^3 dz\,dy\,dx = \int_0^1 \int_0^2 z\big|_0^3\,dy\,dx = \int_0^1 \int_0^2 3\,dy\,dx = \int_0^1 3y\big|_0^2\,dx = \int_0^1 6\,dx = 6.$$

□

Of course, this is more interesting and useful when the limits are not constant.

EXAMPLE 15.5.2 Find the volume of the tetrahedron with corners at $(0,0,0)$, $(0,3,0)$, $(2,3,0)$, and $(2,3,5)$.

The whole problem comes down to correctly describing the region by inequalities: $0 \le x \le 2$, $3x/2 \le y \le 3$, $0 \le z \le 5x/2$. The lower y limit comes from the equation of the line $y = 3x/2$ that forms one edge of the tetrahedron in the x-y plane; the upper z limit comes from the equation of the plane $z = 5x/2$ that forms the "upper" side of the tetrahedron; see figure 15.5.1. Now the volume is

$$\begin{aligned}
\int_0^2 \int_{3x/2}^3 \int_0^{5x/2} dz\,dy\,dx &= \int_0^2 \int_{3x/2}^3 z\Big|_0^{5x/2}\,dy\,dx \\
&= \int_0^2 \int_{3x/2}^3 \frac{5x}{2}\,dy\,dx \\
&= \int_0^2 \frac{5x}{2}y\Big|_{3x/2}^3\,dx \\
&= \int_0^2 \frac{15x}{2} - \frac{15x^2}{4}\,dx \\
&= \frac{15x^2}{4} - \frac{15x^3}{12}\Big|_0^2 \\
&= 15 - 10 = 5.
\end{aligned}$$

□

Pretty much just the way we did for two dimensions we can use triple integration to compute mass, center of mass, and various average quantities.

EXAMPLE 15.5.3 Suppose the temperature at a point is given by $T = xyz$. Find the average temperature in the cube with opposite corners at $(0,0,0)$ and $(2,2,2)$.

In two dimensions we add up the temperature at "each" point and divide by the area; here we add up the temperatures and divide by the volume, 8:

$$\begin{aligned}
\frac{1}{8}\int_0^2 \int_0^2 \int_0^2 xyz\,dz\,dy\,dx &= \frac{1}{8}\int_0^2 \int_0^2 \frac{xyz^2}{2}\Big|_0^2\,dy\,dx = \frac{1}{16}\int_0^2 \int_0^2 xy\,dy\,dx \\
&= \frac{1}{4}\int_0^2 \frac{xy^2}{2}\Big|_0^2\,dx = \frac{1}{8}\int_0^2 4x\,dx = \frac{1}{2}\,\frac{x^2}{2}\Big|_0^2 = 1.
\end{aligned}$$

□

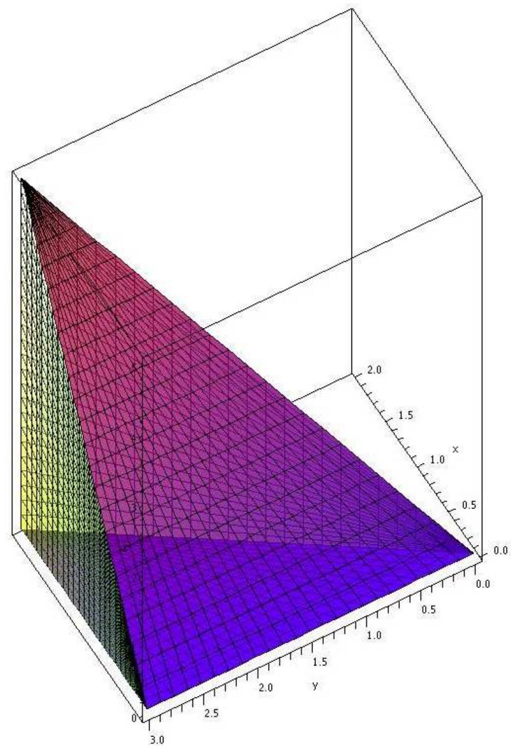

Figure 15.5.1 A tetrahedron. (AP)

EXAMPLE 15.5.4 Suppose the density of an object is given by xz, and the object occupies the tetrahedron with corners $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, and $(0,1,1)$. Find the mass and center of mass of the object.

As usual, the mass is the integral of density over the region:

$$M = \int_0^1 \int_x^1 \int_0^{y-x} xz\,dz\,dy\,dx = \int_0^1 \int_x^1 \frac{x(y-x)^2}{2}\,dy\,dx = \frac{1}{2}\int_0^1 \frac{x(1-x)^3}{3}\,dx$$
$$= \frac{1}{6}\int_0^1 x - 3x^2 + 3x^3 - x^4\,dx = \frac{1}{120}.$$

We compute moments as before, except now there is a third moment:

$$M_{xy} = \int_0^1 \int_x^1 \int_0^{y-x} xz^2\,dz\,dy\,dx = \frac{1}{360},$$
$$M_{xz} = \int_0^1 \int_x^1 \int_0^{y-x} xyz\,dz\,dy\,dx = \frac{1}{144},$$
$$M_{yz} = \int_0^1 \int_x^1 \int_0^{y-x} x^2z\,dz\,dy\,dx = \frac{1}{360}.$$

Finally, the coordinates of the center of mass are $\bar{x} = M_{yz}/M = 1/3$, $\bar{y} = M_{xz}/M = 5/6$, and $\bar{z} = M_{xy}/M = 1/3$. $\square$

Exercises 15.5.

1. Evaluate $\displaystyle\int_0^1 \int_0^x \int_0^{x+y} 2x + y - 1\,dz\,dy\,dx$. ⇒

2. Evaluate $\displaystyle\int_0^2 \int_{-1}^{x^2} \int_1^{y} xyz\,dz\,dy\,dx$. ⇒

3. Evaluate $\displaystyle\int_0^1 \int_0^x \int_0^{\ln y} e^{x+y+z}\,dz\,dy\,dx$. ⇒

4. Evaluate $\displaystyle\int_0^{\pi/2} \int_0^{\sin\theta} \int_0^{r\cos\theta} r^2\,dz\,dr\,d\theta$. ⇒

5. Evaluate $\displaystyle\int_0^{\pi} \int_0^{\sin\theta} \int_0^{r\sin\theta} r\cos^2\theta\,dz\,dr\,d\theta$. ⇒

6. Evaluate $\displaystyle\int_0^1 \int_0^{y^2} \int_0^{x+y} x\,dz\,dx\,dy$. ⇒

7. Evaluate $\displaystyle\int_1^2 \int_y^{y^2} \int_0^{\ln(y+z)} e^x\,dx\,dz\,dy$. ⇒

8. Compute $\displaystyle\int_0^{\pi} \int_0^{\pi/2} \int_0^1 z\sin x + z\cos y\,dz\,dy\,dx$. ⇒

9. For each of the integrals in the previous exercises, give a description of the volume (both algebraic and geometric) that is the domain of integration.

10. Compute $\displaystyle\int\int\int x + y + z\,dV$ over the region $x^2 + y^2 + z^2 \le 1$ in the first octant. ⇒

11. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one corner. ⇒

12. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one edge. ⇒

13. An object occupies the volume of the upper hemisphere of $x^2 + y^2 + z^2 = 4$ and has density z at (x, y, z). Find the center of mass. ⇒

14. An object occupies the volume of the pyramid with corners at $(1, 1, 0)$, $(1, -1, 0)$, $(-1, -1, 0)$, $(-1, 1, 0)$, and $(0, 0, 2)$ and has density $x^2 + y^2$ at (x, y, z). Find the center of mass. ⇒

15. Verify the moments M_{xy}, M_{xz}, and M_{yz} of example 15.5.4 by evaluating the integrals.

16. Find the region E for which $\displaystyle\iiint_E (1 - x^2 - y^2 - z^2)\,dV$ is a maximum.

15.6 Cylindrical and Spherical Coordinates

We have seen that sometimes double integrals are simplified by doing them in polar coordinates; not surprisingly, triple integrals are sometimes simpler in cylindrical coordinates or spherical coordinates. To set up integrals in polar coordinates, we had to understand the shape and area of a typical small region into which the region of integration was divided. We need to do the same thing here, for three dimensional regions.

The cylindrical coordinate system is the simplest, since it is just the polar coordinate system plus a z coordinate. A typical small unit of volume is the shape shown in figure 15.2.1 "fattened up" in the z direction, so its volume is $r\Delta r\Delta\theta\Delta z$, or in the limit, $r\,dr\,d\theta\,dz$.

EXAMPLE 15.6.1 Find the volume under $z=\sqrt{4-r^2}$ above the quarter circle inside $x^2+y^2=4$ in the first quadrant.

We could of course do this with a double integral, but we'll use a triple integral:

$$\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 \sqrt{4-r^2}\,r\,dr\,d\theta = \frac{4\pi}{3}.$$

Compare this to example 15.2.1. □

EXAMPLE 15.6.2 An object occupies the space inside both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$, and has density x^2 at (x,y,z). Find the total mass.

We set this up in cylindrical coordinates, recalling that $x=r\cos\theta$:

$$\begin{aligned}\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3\cos^2(\theta)\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^1 2\sqrt{4-r^2}\,r^3\cos^2(\theta)\,dr\,d\theta\\ &= \int_0^{2\pi}\left(\frac{128}{15}-\frac{22}{5}\sqrt{3}\right)\cos^2(\theta)\,d\theta\\ &= \left(\frac{128}{15}-\frac{22}{5}\sqrt{3}\right)\pi\end{aligned}$$

□

Spherical coordinates are somewhat more difficult to understand. The small volume we want will be defined by $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$, as pictured in figure 15.6.1. To gain a better understanding, see the Java applet. The small volume is nearly box shaped, with 4 flat sides and two sides formed from bits of concentric spheres. When $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$ are all very small, the volume of this little region will be nearly the volume we get by treating it as a box. One dimension of the box is simply $\Delta\rho$, the change in distance from the origin. The other two dimensions are the lengths of small circular arcs, so they are $r\Delta\alpha$ for some suitable r and α, just as in the polar coordinates case.

The easiest of these to understand is the arc corresponding to a change in ϕ, which is nearly identical to the derivation for polar coordinates, as shown in the left graph in figure 15.6.2. In that graph we are looking "face on" at the side of the box we are interested in, so the small angle pictured is precisely $\Delta\phi$, the vertical axis really is the z axis, but the horizontal axis is *not* a real axis—it is just some line through the origin in the x-y plane.

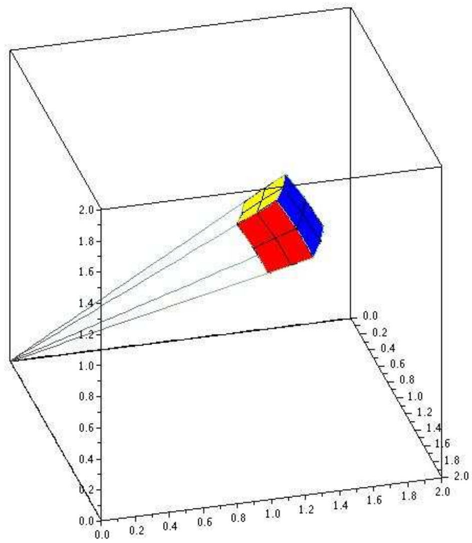

Figure 15.6.1 A small unit of volume for spherical coordinates. (AP)

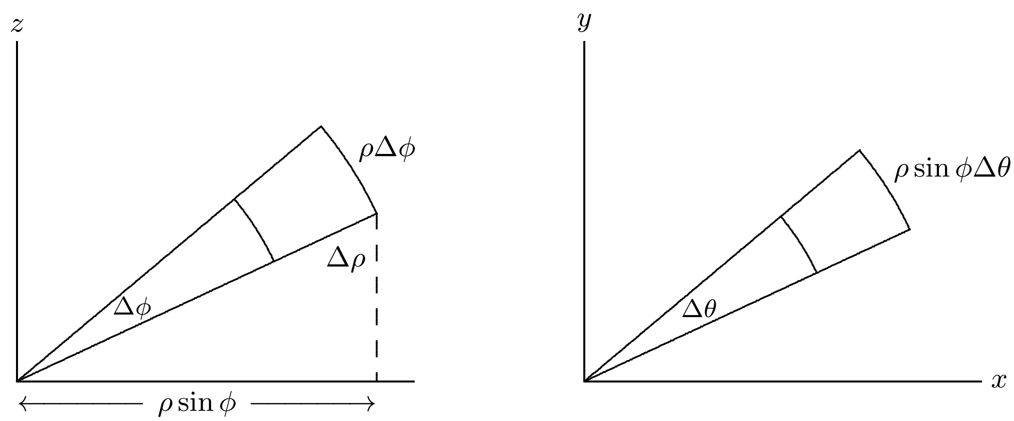

Figure 15.6.2 Setting up integration in spherical coordinates.

Because the other arc is governed by θ, we need to imagine looking straight down the z axis, so that the apparent angle we see is $\Delta\theta$. In this view, the axes really are the x and y axes. In this graph, the apparent distance from the origin is not ρ but $\rho\sin\phi$, as indicated in the left graph.

The upshot is that the volume of the little box is approximately $\Delta\rho(\rho\Delta\phi)(\rho\sin\phi\Delta\theta) = \rho^2\sin\phi\Delta\rho\Delta\phi\Delta\theta$, or in the limit $\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

EXAMPLE 15.6.3 Suppose the temperature at (x, y, z) is $T = 1/(1 + x^2 + y^2 + z^2)$. Find the average temperature in the unit sphere centered at the origin.

In two dimensions we add up the temperature at "each" point and divide by the area; here we add up the temperatures and divide by the volume, $(4/3)\pi$:

$$\frac{3}{4\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}\frac{1}{1+x^2+y^2+z^2}\,dz\,dy\,dx$$

This looks quite messy; since everything in the problem is closely related to a sphere, we'll convert to spherical coordinates.

$$\frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^1\frac{1}{1+\rho^2}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{3}{4\pi}(4\pi-\pi^2)=3-\frac{3\pi}{4}.$$

□

Exercises 15.6.

1. Evaluate $\displaystyle\int_0^1\int_0^x\int_0^{\sqrt{x^2+y^2}}\frac{(x^2+y^2)^{3/2}}{x^2+y^2+z^2}\,dz\,dy\,dx$. ⇒

2. Evaluate $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$. ⇒

3. Evaluate $\displaystyle\int\int\int x^2\,dV$ over the interior of the cylinder $x^2+y^2=1$ between $z=0$ and $z=5$. ⇒

4. Evaluate $\displaystyle\int\int\int xy\,dV$ over the interior of the cylinder $x^2+y^2=1$ between $z=0$ and $z=5$. ⇒

5. Evaluate $\displaystyle\int\int\int z\,dV$ over the region above the x-y plane, inside $x^2+y^2-2x=0$ and under $x^2+y^2+z^2=4$. ⇒

6. Evaluate $\displaystyle\int\int\int yz\,dV$ over the region in the first octant, inside $x^2+y^2-2x=0$ and under $x^2+y^2+z^2=4$. ⇒

7. Evaluate $\displaystyle\int\int\int x^2+y^2\,dV$ over the interior of $x^2+y^2+z^2=4$. ⇒

8. Evaluate $\displaystyle\int\int\int \sqrt{x^2+y^2}\,dV$ over the interior of $x^2+y^2+z^2=4$. ⇒

9. Compute $\displaystyle\int\int\int x+y+z\,dV$ over the region inside $x^2+y^2+z^2=1$ in the first octant. ⇒

10. Find the mass of a right circular cone of height h and base radius a if the density k is proportional to the distance from the base. ⇒

11. Find the mass of a right circular cone of height h and base radius a if the density k is proportional to the distance from its axis of symmetry. ⇒

12. An object occupies the region inside the unit sphere at the origin, and has density equal to the distance from the x-axis. Find the mass. ⇒

13. An object occupies the region inside the unit sphere at the origin, and has density equal to the square of the distance from the origin. Find the mass. $\Rightarrow$

14. An object occupies the region between the unit sphere at the origin and a sphere of radius 2 with center at the origin, and has density equal to the distance from the origin. Find the mass. $\Rightarrow$

15. An object occupies the region in the first octant bounded by the cones $\phi = \pi/4$ and $\phi = \arctan 2$, and the sphere $\rho = \sqrt{6}$, and has density k proportional to the distance from the origin. Find the mass. $\Rightarrow$

15.7 Change of Variables

One of the most useful techniques for evaluating integrals is substitution, both "u-substitution" and trigonometric substitution, in which we change the variable to something more convenient. As we have seen, sometimes changing from rectangular coordinates to another coordinate system is helpful, and this too changes the variables. This is certainly a more complicated change, since instead of changing one variable for another we change an entire suite of variables, but as it turns out it is really very similar to the kinds of change of variables we already know as substitution.

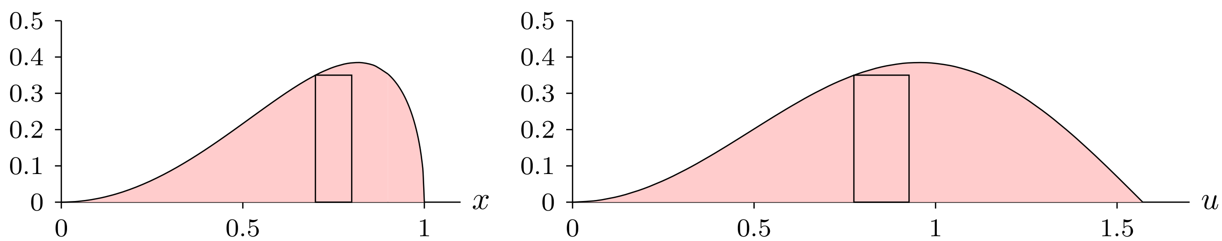

Figure 15.7.1 Single change of variable.

Let's examine the single variable case again, from a slightly different perspective than we have previously used. Suppose we start with the problem

$$\int_0^1 x^2\sqrt{1-x^2}\,dx;$$

this computes the area in the left graph of figure 15.7.1. We use the substitution $x = \sin u$ to transform the function from $x^2\sqrt{1-x^2}$ to $\sin^2 u\sqrt{1-\sin^2 u}$, and we also convert dx to $\cos u\,du$. Finally, we convert the limits 0 and 1 to 0 and $\pi/2$. This transforms the integral:

$$\int_0^1 x^2\sqrt{1-x^2}\,dx = \int_0^{\pi/2} \sin^2 u\sqrt{1-\sin^2 u}\cos u\,du.$$

We want to notice that there are three different conversions: the main function, the differential dx, and the interval of integration. The function is converted to $\sin^2 u\sqrt{1-\sin^2 u}$, shown in the right-hand graph of figure 15.7.1. It is evident that the two curves pictured there have the same y-values in the same order, but the horizontal scale has been changed. Even though the heights are the same, the two integrals

$$\int_0^1 x^2\sqrt{1-x^2}\,dx \qquad \text{and} \qquad \int_0^{\pi/2} \sin^2 u\sqrt{1-\sin^2 u}\,du$$

are not the same; clearly the right hand area is larger. One way to understand the problem is to note that if both areas are approximated using, say, ten subintervals, that the approximating rectangles on the right are wider than their counterparts on the left, as indicated. In the picture, the width of the rectangle on the left is $\Delta x = 0.1$, between 0.7 and 0.8. The rectangle on the right is situated between the corresponding values $\arcsin(0.7)$ and $\arcsin(0.8)$ so that $\Delta u = \arcsin(0.8) - \arcsin(0.7)$. To make the widths match, and the areas therefore the same, we can multiply Δu by a correction factor; in this case the correction factor is approximately $\cos u = \cos(\arcsin(0.7))$, which we compute when we convert dx to $\cos u\,du$.

Now let's move to functions of two variables. Suppose we want to convert an integral

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} f(x,y)\,dy\,dx$$

to use new variables u and v. In the single variable case, there's typically just one reason to want to change the variable: to make the function "nicer" so that we can find an antiderivative. In the two variable case, there is a second potential reason: the two-dimensional region over which we need to integrate is somehow unpleasant, and we want the region in terms of u and v to be nicer—to be a rectangle, for example. Ideally, of course, the new function and the new region will be no worse than the originals, and at least one of them will be better; this doesn't always pan out.

As before, there are three parts to the conversion: the function itself must be rewritten in terms of u and v, $dy\,dx$ must be converted to $du\,dv$, and the old region must be converted to the new region. We will develop the necessary techniques by considering a particular example, and we will use an example we already know how to do by other means.

Consider

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dy\,dx.$$

The limits correspond to integrating over the top half of a circular disk, and we recognize that the function will simplify in polar coordinates, so we would normally convert to polar coordinates:

$$\int_0^\pi \int_0^1 \sqrt{r^2}\, r\, dr\, d\theta = \frac{\pi}{3}.$$

But let's instead approach this as a substitution problem, starting with $x = r\cos\theta$, $y = r\sin\theta$. This pair of equations describes a function from "r-θ space" to "x-y space", and because it involves familiar concepts, it is not too hard to understand what it does. In figure 15.7.2 we have indicated geometrically a bit about how this function behaves. The four dots labeled a–d in the r-θ plane correspond to the three dots in the x-y plane; dots a and b both go to the origin because $r = 0$. The horizontal arrow in the r-θ plane has $r = 1$ everywhere and θ ranges from 0 to π, so the corresponding points $x = r\cos\theta$, $y = r\sin\theta$ start at $(1, 0)$ and follow the unit circle counter-clockwise. Finally, the vertical arrow has $\theta = \pi/4$ and r ranges from 0 to 1, so it maps to the straight arrow in the x-y plane. Extrapolating from these few examples, it's not hard to see that every vertical line in the r-θ plane is transformed to a line through the origin in the x-y plane, and every horizontal line in the r-θ plane is transformed to a circle with center at the origin in the x-y plane. Since we are interested in integrating over the half-disk in the x-y plane, we will integrate over the rectangle $[0, \pi] \times [0, 1]$ in the r-θ plane, because we now see that the points in this rectangle are sent precisely to the upper half disk by $x = r\cos\theta$ and $y = r\sin\theta$.

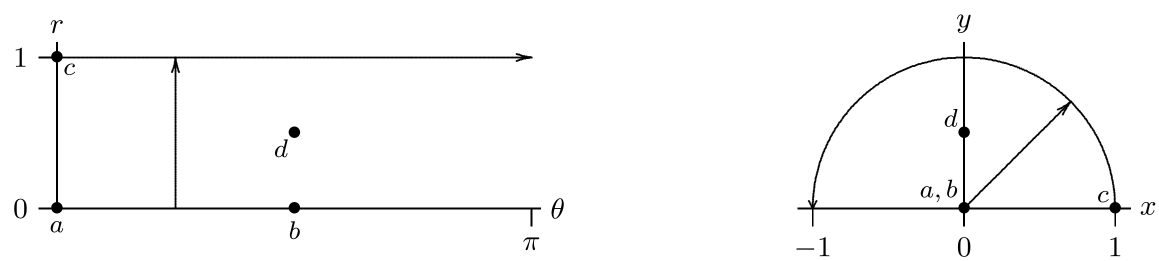

Figure 15.7.2 Double change of variable.

At this point we are two-thirds done with the task: we know the r-θ limits of integration, and we can easily convert the function to the new variables:

$$\sqrt{x^2 + y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r\sqrt{\cos^2\theta + \sin^2\theta} = r. \tag{15.7.1}$$

The final, and most difficult, task is to figure out what replaces $dx\,dy$. (Of course, we actually know the answer, because we are in effect converting to polar coordinates. What we really want is a series of steps that gets to that right answer but that will also work for other substitutions that are not so familiar.)

Let's take a step back and remember how integration arises from approximation. When we approximate the integral in the x-y plane, we are computing the volumes of tall thin boxes, in this case boxes that are $\Delta x \times \Delta y \times \sqrt{x^2+y^2}$. We are aiming to come up with an integral in the r-θ plane that looks like this:

$$\int_0^{\pi}\int_0^1 r(?)\,dr\,d\theta. \tag{15.7.2}$$

What we're missing is exactly the right quantity to replace the "?" so that we get the correct answer. Of course, this integral is also the result of an approximation, in which we add up volumes of boxes that are $\Delta r \times \Delta\theta \times$ height; the problem is that the height that will give us the correct answer is not simply r. Or put another way, we can think of the correct height as r, but the area of the base $\Delta r\Delta\theta$ as being wrong. The height r comes from equation 15.7.1, which is to say, it is precisely the same as the corresponding height in the x-y version of the integral. The problem is that the area of the base $\Delta x \times \Delta y$ is not the same as the area of the base $\Delta r \times \Delta\theta$. We can think of the "?" in the integral as a correction factor that is needed so that $?\,dr\,d\theta = dx\,dy$.

So let's think about what that little base $\Delta r \times \Delta\theta$ corresponds to. We know that each bit of horizontal line in the r-θ plane corresponds to a bit of circular arc in the x-y plane, and each bit of vertical line in the r-θ plane corresponds to a bit of "radial line" in the x-y plane. In figure 15.7.3 we show a typical rectangle in the r-θ plane and its corresponding area in the x-y plane.

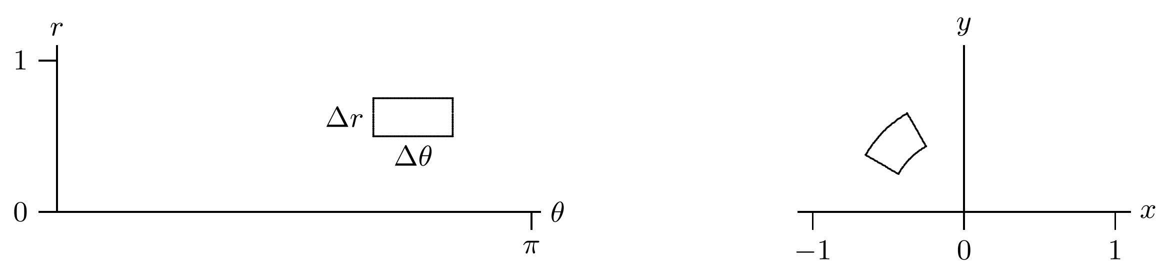

Figure 15.7.3 Corresponding areas. (AP)

In this case, the region in the x-y plane is approximately a rectangle with dimensions $\Delta r \times r\Delta\theta$, but in general the corner angles will not be right angles, so the region will typically be (almost) a parallelogram. We need to compute the area of this parallelogram. We know a neat way to do this: compute the length of a certain cross product (page 314). If we can determine an appropriate two vectors we'll be nearly done.

Fortunately, we've really done this before. The sides of the region in the x-y plane are formed by temporarily fixing either r or θ and letting the other variable range over a small interval. In figure 15.7.4, for example, the upper right edge of the region is formed by fixing $\theta = 2\pi/3$ and letting r run from 0.5 to 0.75. In other words, we have a vector function $\mathbf{v}(r) = \langle r\cos\theta_0, r\sin\theta_0, 0\rangle$, and we are interested in a restricted set of values for r. A vector tangent to this path is given by the derivative $\mathbf{v}'(r) = \langle \cos\theta_0, \sin\theta_0, 0\rangle$, and a small tangent vector, with length approximately equal to the side of the region, is $\langle \cos\theta_0, \sin\theta_0, 0\rangle\, dr$. Likewise, if we fix $r = r_0 = 0.5$, we get the vector function $\mathbf{w}(\theta) = \langle r_0\cos\theta, r_0\sin\theta, 0\rangle$ with derivative $\mathbf{w}'(\theta) = \langle -r_0\sin\theta, r_0\cos\theta, 0\rangle$ and a small tangent vector $\langle -r_0\sin\theta_0, r_0\cos\theta_0, 0\rangle\, d\theta$ when $\theta = \theta_0$ (at the corner we're focusing on). These vectors are shown in figure 15.7.4, with the actual region outlined by a dotted boundary. Of course, since both Δr and $\Delta\theta$ are quite large, the parallelogram is not a particularly good approximation to the true area.

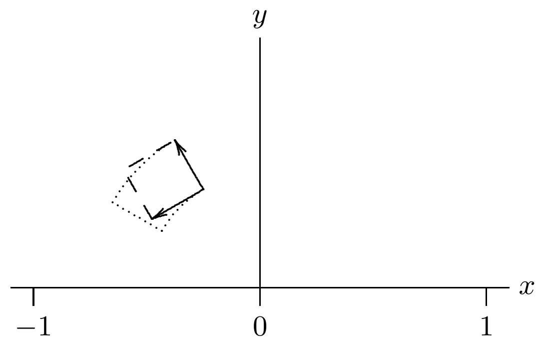

Figure 15.7.4 The approximating parallelogram.

The area of this parallelogram is the length of the cross product:

$$\begin{aligned}
\langle -r_0\sin\theta_0, r_0\cos\theta_0, 0\rangle\, d\theta \times \langle \cos\theta_0, \sin\theta_0, 0\rangle\, dr &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r_0\sin\theta_0 & r_0\cos\theta_0 & 0 \\ \cos\theta_0 & \sin\theta_0 & 0 \end{vmatrix}\, d\theta\, dr \\
&= \langle 0, 0, -r_0\sin^2\theta_0 - r_0\cos^2\theta_0\rangle\, d\theta\, dr \\
&= \langle 0, 0, -r_0\rangle\, d\theta\, dr.
\end{aligned}$$

The length of this vector is $r_0\, dr\, d\theta$. So in general, for any values of r and θ, the area in the x-y plane corresponding to a small rectangle anchored at (θ, r) in the r-θ plane is approximately $r\, dr\, d\theta$. In other words, "r" replaces the "?" in equation 15.7.2.

In general, a substitution will start with equations $x = f(u,v)$ and $y = g(u,v)$. Again, it will be straightforward to convert the function being integrated. Converting the limits will require, as above, an understanding of just how the functions f and g transform the u-v plane into the x-y plane. Finally, the small vectors we need to approximate an area will be $\langle f_u, g_u, 0\rangle\, du$ and $\langle f_v, g_v, 0\rangle\, dv$. The cross product of these is $\langle 0, 0, f_u g_v - g_u f_v\rangle\, du\, dv$ with length $|f_u g_v - g_u f_v|\, du\, dv$. The quantity $|f_u g_v - g_u f_v|$ is usually denoted

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = |f_u g_v - g_u f_v|$$

and called the **Jacobian**. Note that this is the absolute value of the two by two determinant

$$\begin{vmatrix} f_u & g_u \\ f_v & g_v \end{vmatrix},$$

which may be easier to remember. (Confusingly, the matrix, the determinant of the matrix, and the absolute value of the determinant are all called the Jacobian by various authors.)

Because there are two things to worry about, namely, the form of the function and the region of integration, transformations in two (or more) variables are quite tricky to discover.

EXAMPLE 15.7.1 Integrate $x^2 - xy + y^2$ over the region $x^2 - xy + y^2 \le 2$.

The equation $x^2 - xy + y^2 = 2$ describes an ellipse as in figure 15.7.5; the region of integration is the interior of the ellipse. We will use the transformation $x = \sqrt{2}u - \sqrt{2/3}v$, $y = \sqrt{2}u + \sqrt{2/3}v$. Substituting into the function itself we get

$$x^2 - xy + y^2 = 2u^2 + 2v^2.$$

The boundary of the ellipse is $x^2 - xy + y^2 = 2$, so the boundary of the corresponding region in the u-v plane is $2u^2 + 2v^2 = 2$ or $u^2 + v^2 = 1$, the unit circle, so this substitution makes the region of integration simpler.

Next, we compute the Jacobian, using $f = \sqrt{2}u - \sqrt{2/3}v$ and $g = \sqrt{2}u + \sqrt{2/3}v$:

$$f_u g_v - g_u f_v = \sqrt{2}\sqrt{2/3} + \sqrt{2}\sqrt{2/3} = \frac{4}{\sqrt{3}}.$$

Hence the new integral is

$$\iint_R (2u^2 + 2v^2)\frac{4}{\sqrt{3}}\,du\,dv,$$

where R is the interior of the unit circle. This is still not an easy integral, but it is easily transformed to polar coordinates, and then easily integrated. □

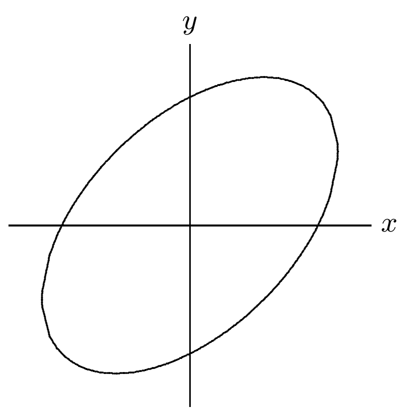

Figure 15.7.5 $x^2 - xy + y^2 = 2$

There is a similar change of variables formula for triple integrals, though it is a bit more difficult to derive. Suppose we use three substitution functions, $x = f(u,v,w)$, $y = g(u,v,w)$, and $z = h(u,v,w)$. The Jacobian determinant is now

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} f_u & g_u & h_u \\ f_v & g_v & h_v \\ f_w & g_w & h_w \end{vmatrix}.$$

Then the integral is transformed in a similar fashion:

$$\iiint_R F(x,y,z)\,dV = \iiint_S F(f(u,v,w), g(u,v,w), h(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw,$$

where of course the region S in uvw space corresponds to the region R in xyz space.

Exercises 15.7.

1. Complete example 15.7.1 by converting to polar coordinates and evaluating the integral. ⇒
2. Evaluate $\iint xy\,dx\,dy$ over the square with corners $(0,0)$, $(1,1)$, $(2,0)$, and $(1,-1)$ in two ways: directly, and using $x = (u+v)/2$, $y = (u-v)/2$. ⇒
3. Evaluate $\iint x^2 + y^2\,dx\,dy$ over the square with corners $(-1,0)$, $(0,1)$, $(1,0)$, and $(0,-1)$ in two ways: directly, and using $x = (u+v)/2$, $y = (u-v)/2$. ⇒
4. Evaluate $\iint (x+y)e^{x-y}\,dx\,dy$ over the triangle with corners $(0,0)$, $(-1,1)$, and $(1,1)$ in two ways: directly, and using $x = (u+v)/2$, $y = (u-v)/2$. ⇒
5. Evaluate $\iint y(x-y)\,dx\,dy$ over the parallelogram with corners $(0,0)$, $(3,3)$, $(7,3)$, and $(4,0)$ in two ways: directly, and using $x = u+v$, $y = u$. ⇒
6. Evaluate $\iint \sqrt{x^2+y^2}\,dx\,dy$ over the triangle with corners $(0,0)$, $(4,4)$, and $(4,0)$ using $x = u$, $y = uv$. ⇒
7. Evaluate $\iint y\sin(xy)\,dx\,dy$ over the region bounded by $xy = 1$, $xy = 4$, $y = 1$, and $y = 4$ using $x = u/v$, $y = v$. ⇒
8. Evaluate $\iint \sin(9x^2+4y^2)\,dA$, over the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$. ⇒
9. Compute the Jacobian for the substitutions $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. ⇒

10. Evaluate $\iiint\limits_E dV$ where E is the solid enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

using the transformation $x = au$, $y = bv$, and $z = cw$. $\Rightarrow$