4

Systems of Linear Equations; Matrices

Introduction

Systems of linear equations can be used to solve resource allocation problems in business and economics (see Problems 73 and 76 in Section 4.3 on production schedules for boats and leases for airplanes). Such systems can involve many equations in many variables. So after reviewing methods for solving two linear equations in two variables, we use matrices and matrix operations to develop procedures that are suitable for solving linear systems of any size. We also discuss Wassily Leontief's Nobel prizewinning application of matrices to economic planning for industrialized countries.

4.1 Review: Systems of Linear Equations in Two Variables

- Systems of Linear Equations in Two Variables
- Graphing
- Substitution
- Elimination by Addition
- Applications

Systems of Linear Equations in Two Variables

To establish basic concepts, let's consider the following simple example: If 2 adult tickets and 1 child ticket cost \$32, and if 1 adult ticket and 3 child tickets cost \$36, what is the price of each?

Let: x = price of adult ticket
y = price of child ticket

$$\text{Then:}\quad \begin{aligned} 2x + y &= 32 \\ x + 3y &= 36 \end{aligned}$$

Now we have a system of two linear equations in two variables. It is easy to find ordered pairs (x, y) that satisfy one or the other of these equations. For example, the ordered pair $(16, 0)$ satisfies the first equation but not the second, and the ordered pair $(24, 4)$ satisfies the second but not the first. To solve this system, we must find all ordered pairs of real numbers that satisfy both equations at the same time. In general, we have the following definition:

DEFINITION Systems of Two Linear Equations in Two Variables

Given the **linear system**

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned}$$

where a, b, c, d, h, and k are real constants, a pair of numbers $x = x_0$ and $y = y_0$ [also written as an ordered pair (x_0, y_0)] is a **solution** of this system if each equation is satisfied by the pair. The set of all such ordered pairs is called the **solution set** for the system. To **solve** a system is to find its solution set.

We will consider three methods of solving such systems: *graphing, substitution,* and *elimination by addition*. Each method has its advantages, depending on the situation.

Graphing

Recall that the graph of a line is a graph of all the ordered pairs that satisfy the equation of the line. To solve the ticket problem by graphing, we graph both equations in the same coordinate system. The coordinates of any points that the graphs have in common must be solutions to the system since they satisfy both equations.

EXAMPLE 1 Solving a System by Graphing Solve the ticket problem by graphing:

$$\begin{aligned} 2x + y &= 32 \\ x + 3y &= 36 \end{aligned}$$

SOLUTION An easy way to find two distinct points on the first line is to find the x and y intercepts. Substitute $y = 0$ to find the x intercept $(2x = 32$, so $x = 16)$, and substitute $x = 0$ to find the y intercept $(y = 32)$. Then draw the line through

$(16, 0)$ and $(0, 32)$. After graphing both lines in the same coordinate system (Fig. 1), estimate the coordinates of the intersection point:

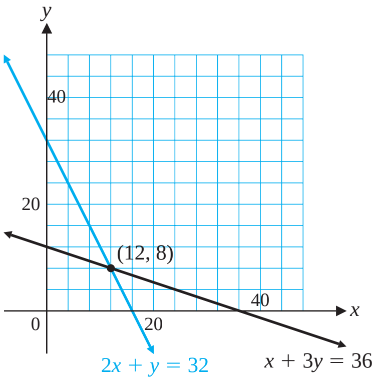


Figure 1

$x = \$12$ Adult ticket
$y = \$8$ Child ticket

CHECK

$$\begin{aligned} 2x + y &= 32 & x + 3y &= 36 \\ 2(12) + 8 &\overset{?}{=} 32 & 12 + 3(8) &\overset{?}{=} 36 \\ 32 &\overset{\checkmark}{=} 32 & 36 &\overset{\checkmark}{=} 36 \end{aligned}$$

Check that $(12, 8)$ satisfies each of the original equations.

Matched Problem 1 Solve by graphing and check:

$$\begin{aligned} 2x - y &= -3 \\ x + 2y &= -4 \end{aligned}$$

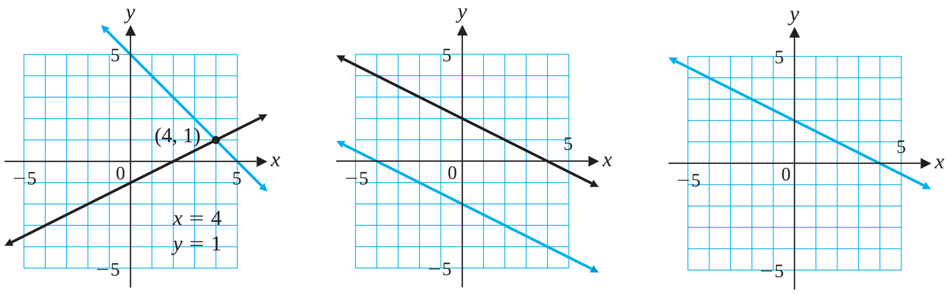
It is clear that Example 1 has exactly one solution since the lines have exactly one point in common. In general, lines in a rectangular coordinate system are related to each other in one of the three ways illustrated in the next example.

EXAMPLE 2 Solving a System by Graphing Solve each of the following systems by graphing:

(A) $x - 2y = 2$, $x + y = 5$ (B) $x + 2y = -4$, $2x + 4y = 8$ (C) $2x + 4y = 8$, $x + 2y = 4$

SOLUTION

(A) (B) (C)



Intersection at one point only—exactly one solution

Lines are parallel (each has slope $-\frac{1}{2}$)—no solutions

Lines coincide—infinite number of solutions

Matched Problem 2 Solve each of the following systems by graphing:

(A) $x + y = 4$, $2x - y = 2$ (B) $6x - 3y = 9$, $2x - y = 3$ (C) $2x - y = 4$, $6x - 3y = -18$

We introduce some terms that describe the different types of solutions to systems of equations.

DEFINITION Systems of Linear Equations: Basic Terms

A system of linear equations is **consistent** if it has one or more solutions and **inconsistent** if no solutions exist. Furthermore, a consistent system is said to be **independent** if it has exactly one solution (often referred to as the **unique solution**) and **dependent** if it has more than one solution. Two systems of equations are **equivalent** if they have the same solution set.

Referring to the three systems in Example 2, the system in part (A) is consistent and independent with the unique solution $x = 4, y = 1$. The system in part (B) is inconsistent. And the system in part (C) is consistent and dependent with an infinite number of solutions (all points on the two coinciding lines).

CAUTION Given a system of equations, do not confuse the *number of variables* with the *number of solutions*. The systems of Example 2 involve two variables, x and y. A solution to such a system is a *pair* of numbers, one for x and one for y. So the system in Example 2A has two variables, but exactly one solution, namely $x = 4$, $y = 1$. ▲

Explore and Discuss 1

No; no

Can a consistent and dependent system have exactly two solutions? Exactly three solutions? Explain.

By graphing a system of two linear equations in two variables, we gain useful information about the solution set of the system. In general, any two lines in a coordinate plane must intersect in exactly one point, be parallel, or coincide (have identical graphs). So the systems in Example 2 illustrate the only three possible types of solutions for systems of two linear equations in two variables. These ideas are summarized in Theorem 1.

THEOREM 1 Possible Solutions to a Linear System

The linear system

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned}$$

must have

(A) Exactly one solution Consistent and independent

or

(B) No solution Inconsistent

or

(C) Infinitely many solutions Consistent and dependent

There are no other possibilities.

In the past, one drawback to solving systems by graphing was the inaccuracy of hand-drawn graphs. Graphing calculators have changed that. Graphical solutions on a graphing calculator provide an accurate approximation of the solution to a system of linear equations in two variables. Example 3 demonstrates this.

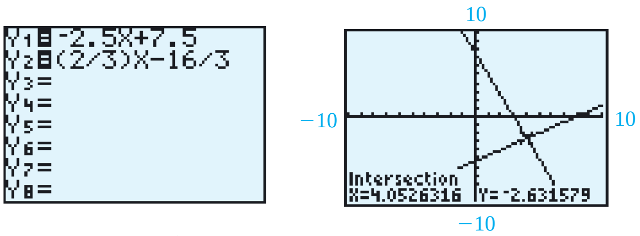
EXAMPLE 3 Solving a System Using a Graphing Calculator Solve to two decimal places using graphical approximation techniques on a graphing calculator:

$$\begin{aligned} 5x + 2y &= 15 \\ 2x - 3y &= 16 \end{aligned}$$

SOLUTION First, solve each equation for y:

$$\begin{aligned} 5x + 2y &= 15 \\ 2y &= -5x + 15 \\ y &= -2.5x + 7.5 \end{aligned} \qquad \begin{aligned} 2x - 3y &= 16 \\ -3y &= -2x + 16 \\ y &= \frac{2}{3}x - \frac{16}{3} \end{aligned}$$

Next, enter each equation in the graphing calculator (Fig. 2A), graph in an appropriate viewing window, and approximate the intersection point (Fig. 2B).



```
Y1■-2.5X+7.5
Y2■(2/3)X-16/3
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=
```

```
Intersection
X=4.0526316  Y=-2.631579
```

Figure 2 (A) Equation definitions (B) Intersection point

Rounding the values in Figure 2B to two decimal places, we see that the solution is $x = 4.05$ and $y = -2.63$, or $(4.05, -2.63)$.

CHECK

$$\begin{aligned} 5x + 2y &= 15 \\ 5(4.05) + 2(-2.63) &\overset{?}{=} 15 \\ 14.99 &\overset{\checkmark}{\approx} 15 \end{aligned} \qquad \begin{aligned} 2x - 3y &= 16 \\ 2(4.05) - 3(-2.63) &\overset{?}{=} 16 \\ 15.99 &\overset{\checkmark}{\approx} 16 \end{aligned}$$

The checks are sufficiently close but, due to rounding, not exact.

Matched Problem 3 Solve to two decimal places using graphical approximation techniques on a graphing calculator:

$$\begin{aligned} 2x - 5y &= -25 \\ 4x + 3y &= 5 \end{aligned}$$

Graphical methods help us to visualize a system and its solutions, reveal relationships that might otherwise be hidden, and, with the assistance of a graphing calculator, provide accurate approximations to solutions.

Substitution

Now we review an algebraic method that is easy to use and provides exact solutions to a system of two equations in two variables, provided that solutions exist. In this method, first we choose one of two equations in a system and solve for one variable in terms of the other. (We make a choice that avoids fractions, if possible.) Then we **substitute** the result into the other equation and solve the resulting linear equation in one variable. Finally, we substitute this result back into the results of the first step to find the second variable.

EXAMPLE 4 Solving a System by Substitution Solve by substitution:

$$\begin{aligned} 5x + y &= 4 \\ 2x - 3y &= 5 \end{aligned}$$

SOLUTION Solve either equation for one variable in terms of the other; then substitute into the remaining equation. In this problem, we avoid fractions by choosing the first equation and solving for y in terms of x:

$$\begin{aligned} 5x + y &= 4 && \text{Solve the first equation for } y \text{ in terms of } x. \\ y &= \underbrace{4 - 5x} && \text{Substitute into the second equation.} \end{aligned}$$

$$\begin{aligned} 2x - 3y &= 5 && \text{Second equation} \\ 2x - 3(\mathbf{4 - 5x}) &= 5 && \text{Solve for } x. \\ 2x - 12 + 15x &= 5 \\ 17x &= 17 \\ \boldsymbol{x} &= \mathbf{1} \end{aligned}$$

Now, replace x with 1 in $y = 4 - 5x$ to find y:

$$\begin{aligned} y &= 4 - 5x \\ y &= 4 - 5(\mathbf{1}) \\ \boldsymbol{y} &= \mathbf{-1} \end{aligned}$$

The solution is $x = 1$, $y = -1$ or $(1, -1)$.

CHECK

$$\begin{aligned} 5x + \quad y &= 4 & \qquad 2x - \quad 3y &= 5 \\ 5(1) + (-1) &\overset{?}{=} 4 & \qquad 2(1) - 3(-1) &\overset{?}{=} 5 \\ 4 &\overset{\checkmark}{=} 4 & \qquad 5 &\overset{\checkmark}{=} 5 \end{aligned}$$

Matched Problem 4 Solve by substitution:

$$\begin{aligned} 3x + 2y &= -2 \\ 2x - \quad y &= -6 \end{aligned}$$

Explore and Discuss 2 Return to Example 2 and solve each system by substitution. Based on your results, describe how you can recognize a dependent system or an inconsistent system when using substitution.

Elimination by Addition

The methods of graphing and substitution both work well for systems involving two variables. However, neither is easily extended to larger systems. Now we turn to **elimination by addition**. This is probably the most important method of solution. It readily generalizes to larger systems and forms the basis for computer-based solution methods.

To solve an equation such as $2x - 5 = 3$, we perform operations on the equation until we reach an equivalent equation whose solution is obvious (see Appendix A, Section A.7).

$$\begin{aligned} 2x - 5 &= 3 && \text{Add 5 to both sides.} \\ 2x &= 8 && \text{Divide both sides by 2.} \\ x &= 4 \end{aligned}$$

Theorem 2 indicates that we can solve systems of linear equations in a similar manner.

THEOREM 2 Operations That Produce Equivalent Systems

A system of linear equations is transformed into an equivalent system if

(A) Two equations are interchanged.

(B) An equation is multiplied by a nonzero constant.

(C) A constant multiple of one equation is added to another equation.

Any one of the three operations in Theorem 2 can be used to produce an equivalent system, but the operations in parts (B) and (C) will be of most use to us now. Part (A) becomes useful when we apply the theorem to larger systems. The use of Theorem 2 is best illustrated by examples.

EXAMPLE 5 Solving a System Using Elimination by Addition Solve the following system using elimination by addition:

$$\begin{aligned} 3x - 2y &= 8 \\ 2x + 5y &= -1 \end{aligned}$$

SOLUTION We use Theorem 2 to eliminate one of the variables, obtaining a system with an obvious solution:

$$\begin{aligned} 3x - 2y &= 8 \\ 2x + 5y &= -1 \end{aligned}$$

Multiply the top equation by 5 and the bottom equation by 2 (Theorem 2B).

$$\boxed{\begin{aligned} \mathbf{5}(3x - 2y) &= \mathbf{5}(8) \\ \mathbf{2}(2x + 5y) &= \mathbf{2}(-1) \end{aligned}}$$

$$\begin{aligned} 15x - 10y &= 40 \\ 4x + 10y &= -2 \\ \hline 19x \qquad &= 38 \end{aligned}$$

Add the top equation to the bottom equation (Theorem 2C), eliminating the y terms.

Divide both sides by 19, which is the same as multiplying the equation by $\frac{1}{19}$ (Theorem 2B).

$$\mathbf{x = 2}$$

This equation paired with either of the two original equations produces a system equivalent to the original system.

Knowing that $x = 2$, we substitute this number back into either of the two original equations (we choose the second) to solve for y:

$$\begin{aligned} 2(\mathbf{2}) + 5y &= -1 \\ 5y &= -5 \\ \mathbf{y} &= \mathbf{-1} \end{aligned}$$

The solution is $x = 2, y = -1$ or $(2, -1)$.

CHECK

$$\begin{array}{rcl} 3x - 2y &=& 8 \\ 3(2) - 2(-1) &\overset{?}{=}& 8 \\ 8 &\overset{\checkmark}{=}& 8 \end{array} \qquad \begin{array}{rcl} 2x + 5y &=& -1 \\ 2(2) + 5(-1) &\overset{?}{=}& -1 \\ -1 &\overset{\checkmark}{=}& -1 \end{array}$$

Matched Problem 5 Solve the following system using elimination by addition:

$$\begin{aligned} 5x - 2y &= 12 \\ 2x + 3y &= 1 \end{aligned}$$

Let's see what happens in the elimination process when a system has either no solution or infinitely many solutions. Consider the following system:

$$\begin{aligned} 2x + 6y &= -3 \\ x + 3y &= 2 \end{aligned}$$

Multiplying the second equation by -2 and adding, we obtain

$$\begin{aligned} 2x + 6y &= -3 \\ \underline{-2x - 6y} &\underline{= -4} \\ 0 &= -7 \quad \text{Not possible} \end{aligned}$$

We have obtained a contradiction. The assumption that the original system has solutions must be false. So the system has no solutions, and its solution set is the empty set. The graphs of the equations are parallel lines, and the system is inconsistent.

Now consider the system

$$\begin{aligned} x - \tfrac{1}{2}y &= 4 \\ -2x + y &= -8 \end{aligned}$$

If we multiply the top equation by 2 and add the result to the bottom equation, we obtain

$$\begin{aligned} 2x - y &= 8 \\ \underline{-2x + y} &\underline{= -8} \\ 0 &= 0 \end{aligned}$$

Obtaining $0 = 0$ implies that the equations are equivalent; that is, their graphs coincide and the system is dependent. If we let $x = k$, where k is any real number, and solve either equation for y, we obtain $y = 2k - 8$. So $(k, 2k - 8)$ is a solution to this system for any real number k. The variable k is called a **parameter** and replacing k with a real number produces a **particular solution** to the system. For example, some particular solutions to this system are

$k = -1$	$k = 2$	$k = 5$	$k = 9.4$
$(-1, -10)$	$(2, -4)$	$(5, 2)$	$(9.4, 10.8)$

Applications

Many real-world problems are solved readily by constructing a mathematical model consisting of two linear equations in two variables and applying the solution methods that we have discussed. We shall examine two applications in detail.

EXAMPLE 6 Diet Jasmine wants to use milk and orange juice to increase the amount of calcium and vitamin A in her daily diet. An ounce of milk contains 37 milligrams of calcium and 57 micrograms* of vitamin A. An ounce of orange juice contains 5 milligrams of calcium and 65 micrograms of vitamin A. How many ounces of milk and orange juice should Jasmine drink each day to provide exactly 500 milligrams of calcium and 1,200 micrograms of vitamin A?

SOLUTION The first step in solving an application problem is to introduce the proper variables. Often, the question asked in the problem will guide you in this decision. Reading the last sentence in Example 6, we see that we must determine a certain number of ounces of milk and orange juice. So we introduce variables to represent these unknown quantities:

x = number of ounces of milk
y = number of ounces of orange juice

---

*A microgram (μg) is one millionth (10^{-6}) of a gram.

Next, we summarize the given information using a table. It is convenient to organize the table so that the quantities represented by the variables correspond to columns in the table (rather than to rows) as shown.

	Milk	**Orange Juice**	**Total Needed**
Calcium	37 mg/oz	5 mg/oz	500 mg
Vitamin A	57 μg/oz	65 μg/oz	1,200 μg

Now we use the information in the table to form equations involving x and y:

$$\begin{array}{ccccc} \left(\begin{array}{c}\text{calcium in } x \text{ oz}\\ \text{of milk}\end{array}\right) & + & \left(\begin{array}{c}\text{calcium in } y \text{ oz}\\ \text{of orange juice}\end{array}\right) & = & \left(\begin{array}{c}\text{total calcium}\\ \text{needed (mg)}\end{array}\right) \\ 37x & + & 5y & = & 500 \end{array}$$

$$\begin{array}{ccccc} \left(\begin{array}{c}\text{vitamin A in } x \text{ oz}\\ \text{of milk}\end{array}\right) & + & \left(\begin{array}{c}\text{vitamin A in } y \text{ oz}\\ \text{of orange juice}\end{array}\right) & = & \left(\begin{array}{c}\text{total vitamin A}\\ \text{needed } (\mu\text{g})\end{array}\right) \\ 57x & + & 65y & = & 1{,}200 \end{array}$$

So we have the following model to solve:

$$\begin{aligned} 37x + 5y &= 500 \\ 57x + 65y &= 1{,}200 \end{aligned}$$

We can multiply the first equation by -13 and use elimination by addition:

$$\begin{array}{rcl} -481x - 65y &=& -6{,}500 \\ 57x + 65y &=& 1{,}200 \\ \hline -424x \qquad &=& -5{,}300 \\ x &=& 12.5 \end{array} \qquad \begin{array}{rcl} 37(12.5) + 5y &=& 500 \\ 5y &=& 37.5 \\ y &=& 7.5 \end{array}$$

Drinking 12.5 ounces of milk and 7.5 ounces of orange juice each day will provide Jasmine with the required amounts of calcium and vitamin A.

CHECK

$$\begin{array}{rcl} 37x + 5y &=& 500 \\ 37(12.5) + 5(7.5) &\overset{?}{=}& 500 \\ 500 &\overset{\checkmark}{=}& 500 \end{array} \qquad \begin{array}{rcl} 57x + 65y &=& 1{,}200 \\ 57(12.5) + 65(7.5) &\overset{?}{=}& 1{,}200 \\ 1{,}200 &\overset{\checkmark}{=}& 1{,}200 \end{array}$$

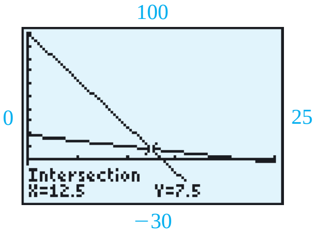


Figure 3

$y_1 = (500 - 37x)/5$

$y_2 = (1{,}200 - 57x)/65$

Figure 3 illustrates a solution to Example 6 using graphical approximation techniques.

Matched Problem 6 Dennis wants to use cottage cheese and yogurt to increase the amount of protein and calcium in his daily diet. An ounce of cottage cheese contains 3 grams of protein and 15 milligrams of calcium. An ounce of yogurt contains 1 gram of protein and 41 milligrams of calcium. How many ounces of cottage cheese and yogurt should Dennis eat each day to provide exactly 62 grams of protein and 760 milligrams of calcium?

In a free market economy, the price of a product is determined by the relationship between supply and demand. Suppliers are more willing to supply a product at higher prices. So when the price is high, the supply is high. If the relationship between price and supply is linear, then the graph of the price–supply equation is a line with positive slope. On the other hand, consumers of a product are generally less willing to buy a product at higher prices. So when the price is high, demand is low. If the relationship between price and demand is linear, the graph of the price–demand equation is a line with negative slope. In a free competitive market, the price of a product tends to move toward an **equilibrium price**, in which the supply and demand are equal; that common value of the supply and demand is the **equilibrium quantity**. To find the equilibrium price, we solve the system consisting of the price–supply and price–demand equations.

EXAMPLE 7 Supply and Demand At a price of \$1.88 per pound, the supply for cherries in a large city is 16,000 pounds, and the demand is 10,600 pounds. When the price drops to \$1.46 per pound, the supply decreases to 10,000 pounds, and the demand increases to 12,700 pounds. Assume that the price–supply and price–demand equations are linear.

(A) Find the price–supply equation.

(B) Find the price–demand equation.

(C) Find the supply and demand at a price of \$2.09 per pound.

(D) Find the supply and demand at a price of \$1.32 per pound.

(E) Use the substitution method to find the equilibrium price and equilibrium demand.

SOLUTION

(A) Let p be the price per pound, and let x be the quantity in thousands of pounds. Then $(16, 1.88)$ and $(10, 1.46)$ are solutions of the price–supply equation. Use the point–slope form for the equation of a line, $y - y_1 = m(x - x_1)$, to obtain the price–supply equation:

$$
\begin{aligned}
p - 1.88 &= \frac{1.46 - 1.88}{10 - 16}(x - 16) && \text{Simplify.}\\
p - 1.88 &= 0.07(x - 16) && \text{Solve for } p.\\
p &= 0.07x + 0.76 && \text{Price–supply equation}
\end{aligned}
$$

(B) Again, let p be the price per pound, and let x be the quantity in thousands of pounds. Then $(10.6, 1.88)$ and $(12.7, 1.46)$ are solutions of the price–demand equation.

$$
\begin{aligned}
p - 1.88 &= \frac{1.46 - 1.88}{12.7 - 10.6}(x - 10.6) && \text{Simplify.}\\
p - 1.88 &= -0.2(x - 10.6) && \text{Solve for } p.\\
p &= -0.2x + 4 && \text{Price–demand equation}
\end{aligned}
$$

(C) Substitute $p = 2.09$ into the price–supply equation, and also into the price–demand equation, and solve for x:

Price–supply equation	Price–demand equation
$p = 0.07x + 0.76$	$p = -0.2x + 4$
$2.09 = 0.07x + 0.76$	$2.09 = -0.2x + 4$
$x = 19$	$x = 9.55$

At a price of \$2.09 per pound, the supply is 19,000 pounds of cherries and the demand is 9,550 pounds. (The supply is greater than the demand, so the price will tend to come down.)

(D) Substitute $p = 1.32$ in each equation and solve for x:

Price–supply equation	Price–demand equation
$p = 0.07x + 0.76$	$p = -0.2x + 4$
$1.32 = 0.07x + 0.76$	$1.32 = -0.2x + 4$
$x = 8$	$x = 13.4$

At a price of \$1.32 per pound, the supply is 8,000 pounds of cherries, and the demand is 13,400 pounds. (The demand is greater than the supply, so the price will tend to go up.)

(E) We solve the linear system

$$
\begin{aligned}
p &= 0.07x + 0.76 && \text{Price–supply equation}\\
p &= -0.2x + 4 && \text{Price–demand equation}
\end{aligned}
$$

using substitution (substitute $p = -0.2x + 4$ in the first equation):

$$\begin{aligned} -0.2x + 4 &= 0.07x + 0.76 \\ -0.27x &= -3.24 \\ x &= 12 \text{ thousand pounds} \end{aligned} \qquad \text{Equilibrium quantity}$$

Now substitute $x = 12$ into the price–demand equation:

$$\begin{aligned} p &= -0.2(12) + 4 \\ p &= \$1.60 \text{ per pound} \end{aligned} \qquad \text{Equilibrium price}$$

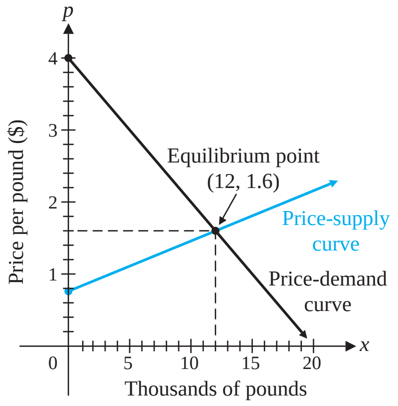


Figure 4

The results are interpreted graphically in Figure 4 (it is customary to refer to the graphs of price–supply and price–demand equations as “curves” even when they are lines). Note that if the price is above the equilibrium price of \$1.60 per pound, the supply will exceed the demand and the price will come down. If the price is below the equilibrium price of \$1.60 per pound, the demand will exceed the supply and the price will go up. So the price will stabilize at \$1.60 per pound. At this equilibrium price, suppliers will supply 12,000 pounds of cherries and consumers will purchase 12,000 pounds.

Matched Problem 7 Find the equilibrium quantity and equilibrium price, and graph the following price–supply and price–demand equations:

$$\begin{aligned} p &= 0.08q + 0.66 && \text{Price–supply equation} \\ p &= -0.1q + 3 && \text{Price–demand equation} \end{aligned}$$

Exercises 4.1

Skills Warm-up Exercises

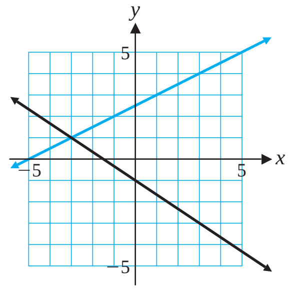
W *In Problems 1–6, find the x and y coordinates of the intersection of the given lines. (If necessary, review Section 1.2).*

1. $y = 5x + 7$ and the y axis $(0, 7)$
2. $y = 5x + 7$ and the x axis. $(-7/5, 0)$
3. $3x + 4y = 72$ and the x axis $(24, 0)$
4. $3x + 4y = 72$ and the y axis $(0, 18)$
5. $6x - 5y = 120$ and $x = 5$ $(5, -18)$
6. $6x - 5y = 120$ and $y = 3$ $(22.5, 3)$

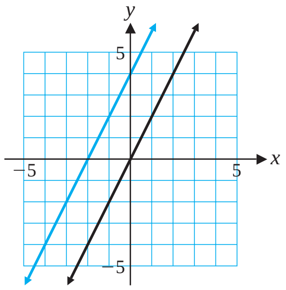
In Problems 7 and 8, find an equation in point–slope form, $y - y_1 = m(x - x_1)$, of the line through the given points.

7. $(2, 7)$ and $(4, -5)$.
8. $(3, 20)$ and $(-5, 4)$.

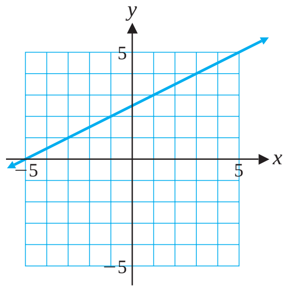
A *Match each system in Problems 9–12 with one of the following graphs, and use the graph to solve the system.*



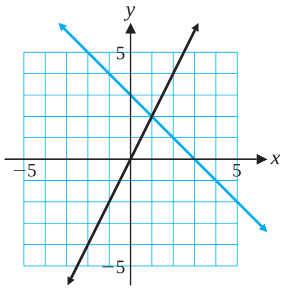
(A)



(B)



(C)



(D)

9. $\begin{aligned} -4x + 2y &= 8 \\ 2x - y &= 0 \end{aligned}$
(B); no solution

10. $\begin{aligned} x + y &= 3 \\ 2x - y &= 0 \end{aligned}$
(D); $x = 1, y = 2$

11. $\begin{aligned} -x + 2y &= 5 \\ 2x + 3y &= -3 \end{aligned}$
(A); $x = -3, y = 1$

12. $\begin{aligned} 2x - 4y &= -10 \\ -x + 2y &= 5 \end{aligned}$
(C); infinitely many solutions

Solve Problems 13–16 by graphing.

13. $\begin{aligned} 3x - y &= 2 \\ x + 2y &= 10 \end{aligned}$
$x = 2, y = 4$

14. $\begin{aligned} 3x - 2y &= 12 \\ 7x + 2y &= 8 \end{aligned}$
$x = 2, y = -3$

15. $\begin{aligned} m + 2n &= 4 \\ 2m + 4n &= -8 \end{aligned}$
No solution (parallel lines)

16. $\begin{aligned} 3u + 5v &= 15 \\ 6u + 10v &= -30 \end{aligned}$
No solution (parallel lines)

Solve Problems 17–20 using substitution.

17. $\begin{aligned} y &= 2x - 3 \\ x + 2y &= 14 \end{aligned}$
$x = 4, y = 5$

18. $\begin{aligned} y &= x - 4 \\ x + 3y &= 12 \end{aligned}$
$x = 6, y = 2$

*Answer located in Additional Instructor’s Answers section.

7. $y - 7 = -6(x - 2)$ **8.** $y - 20 = 2(x - 3)$

19. $2x + y = 6$
$x - y = -3$
$x = 1, y = 4$

20. $3x - y = 7$
$2x + 3y = 1$
$x = 2, y = -1$

Solve Problems 21–24 using elimination by addition.

21. $3u - 2v = 12$
$7u + 2v = 8$
$u = 2, v = -3$

22. $2x - 3y = -8$
$5x + 3y = 1$
$x = -1, y = 2$

23. $2m - n = 10$
$m - 2n = -4$
$m = 8, n = 6$

24. $2x + 3y = 1$
$3x - y = 7$
$x = 2, y = -1$

B *Solve Problems 25–34 using substitution or elimination by addition.*

25. $9x - 3y = 24$
$11x + 2y = 1$
$x = 1, y = -5$

26. $4x + 3y = 26$
$3x - 11y = -7$
$x = 5, y = 2$

27. $2x - 3y = -2$
$-4x + 6y = 7$
No solution (inconsistent)

28. $3x - 6y = -9$
$-2x + 4y = 12$
No solution (parallel lines)

29. $3x + 8y = 4$
$15x + 10y = -10$
$x = -\frac{4}{3}, y = 1$

30. $7m + 12n = -1$
$5m - 3n = 7$
$m = 1, n = -\frac{2}{3}$

31. $-6x + 10y = -30$
$3x - 5y = 15$
Infinitely many solutions (dependent)

32. $2x + 4y = -8$
$x + 2y = 4$
No solution (parallel lines)

33. $x + y = 1$
$0.3x - 0.4y = 0$
$x = \frac{4}{7}, y = \frac{3}{7}$

34. $x + y = 1$
$0.5x - 0.4y = 0$
$x = \frac{4}{9}, y = \frac{5}{9}$

In Problems 35–42, solve the system. Note that each solution can be found mentally, without the use of a calculator or pencil-and-paper calculation; try to visualize the graphs of both lines.

35. $x + 0y = 7$
$0x + y = 3$
$x = 7, y = 3$

36. $x + 0y = -4$
$0x + y = 9$
$x = -4, y = 9$

37. $5x + 0y = 4$
$0x + 3y = -2$
$x = \frac{4}{5}, y = -\frac{2}{3}$

38. $6x + 0y = 7$
$0x + 4y = 9$
$x = \frac{7}{6}, y = \frac{9}{4}$

39. $x + y = 0$
$x - y = 0$
$x = 0, y = 0$

40. $-2x + y = 0$
$5x - y = 0$
$x = 0, y = 0$

41. $x - 2y = 4$
$0x + y = 5$
$x = 14, y = 5$

42. $x + 3y = 9$
$0x + y = -2$
$x = 15, y = -2$

43. In a free competitive market, if the supply of a good is greater than the demand, will the price tend to go up or come down? Price tends to come down.

44. In a free competitive market, if the demand for a good is greater than the supply, will the price tend to go up or come down? Price tends to go up.

Problems 45–48 are concerned with the linear system

$$\begin{aligned} y &= mx + b \\ y &= nx + c \end{aligned}$$

where m, b, n, and c are nonzero constants.

45. If the system has a unique solution, discuss the relationships among the four constants.

46. If the system has no solution, discuss the relationships among the four constants.

47. If the system has an infinite number of solutions, discuss the relationships among the four constants.

48. If $m = 0$, how many solutions does the system have?

In Problems 49–56, use a graphing calculator to find the solution to each system. Round any approximate solutions to three decimal places.

49. $y = 2x - 9$
$y = 3x + 5$
$(-14, -37)$

50. $y = -3x + 3$
$y = 5x + 8$
$x = -0.625, y = 4.875$

51. $y = 2x + 1$
$y = 2x + 7$
No solution

52. $y = -3x + 6$
$y = -3x + 9$
No solution (parallel lines)

53. $3x - 2y = 15$
$4x + 3y = 13$
$(4.176, -1.235)$

54. $3x - 7y = -20$
$2x + 5y = 8$
$x = -1.517, y = 2.207$

55. $-2.4x + 3.5y = 0.1$
$-1.7x + 2.6y = -0.2$
$(-3.310, -2.241)$

56. $4.2x + 5.4y = -12.9$
$6.4x + 3.7y = -4.5$
$x = 1.232, y = -3.347$

C *In Problems 57–62, graph the equations in the same coordinate system. Find the coordinates of any points where two or more lines intersect and discuss the nature of the solution set.*

57. $x - 2y = -6$
$2x + y = 8$
$x + 2y = -2$*

58. $x + y = 3$
$x + 3y = 15$
$3x - y = 5$*

59. $x + y = 1$
$x - 2y = -8$
$3x + y = -3$*

60. $x - y = 6$
$x - 2y = 8$
$x + 4y = -4$*

61. $4x - 3y = -24$
$2x + 3y = 12$
$8x - 6y = 24$*

62. $2x + 3y = 18$
$2x - 6y = -6$
$4x + 6y = -24$*

63. The coefficients of the three systems given below are similar. One might guess that the solution sets to the three systems would be nearly identical. Develop evidence for or against this guess by considering graphs of the systems and solutions obtained using substitution or elimination by addition.

(A) $5x + 4y = 4$
$11x + 9y = 4$
$(20, -24)$

(B) $5x + 4y = 4$
$11x + 8y = 4$
$(-4, 6)$

(C) $5x + 4y = 4$
$10x + 8y = 4$ No solution

64. Repeat Problem 63 for the following systems:

(A) $6x - 5y = 10$
$-13x + 11y = -20$
$(10, 10)$

(B) $6x - 5y = 10$
$-13x + 10y = -20$
$(0, -2)$

(C) $6x - 5y = 10$
$-12x + 10y = -20$ Infinitely many solutions

Applications

65. Supply and demand for T-shirts. Suppose that the supply and demand equations for printed T-shirts for a particular week are

$$p = 0.7q + 3 \quad \text{Price–supply equation}$$
$$p = -1.7q + 15 \quad \text{Price–demand equation}$$

where p is the price in dollars and q is the quantity in hundreds.

(A) Find the supply and demand (to the nearest unit) if T-shirts are \$4 each. Discuss the stability of the T-shirt market at this price level. Supply: 143 T-shirts; demand: 647 T-shirts

(B) Find the supply and demand (to the nearest unit) if T-shirts are \$9 each. Discuss the stability of the T-shirt market at this price level. Supply: 857 T-shirts; demand: 353 T-shirts

(C) Find the equilibrium price and quantity. Equilibrium price = \$6.50; equilibrium quantity = 500 T-shirts

(D) Graph the two equations in the same coordinate system and identify the equilibrium point, supply curve, and demand curve.*

66. Supply and demand for baseball caps. Suppose that the supply and demand for printed baseball caps for a particular week are

$$p = 0.4q + 3.2 \quad \text{Price–supply equation}$$
$$p = -1.9q + 17 \quad \text{Price–demand equation}$$

where p is the price in dollars and q is the quantity in hundreds.

(A) Find the supply and demand (to the nearest unit) if baseball caps are \$4 each. Discuss the stability of the baseball cap market at this price level. Supply: 200 baseball caps; demand: 684 baseball caps

(B) Find the supply and demand (to the nearest unit) if baseball caps are \$9 each. Discuss the stability of the baseball cap market at this price level. Supply: 1,450 baseball caps; demand: 421 baseball caps

(C) Find the equilibrium price and quantity. Equilibrium price = \$5.60; equilibrium quantity = 600 baseball caps

(D) Graph the two equations in the same coordinate system and identify the equilibrium point, supply curve, and demand curve.*

67. Supply and demand for soybeans. At \$4.80 per bushel, the annual supply for soybeans in the Midwest is 1.9 billion bushels, and the annual demand is 2.0 billion bushels. When the price increases to \$5.10 per bushel, the annual supply increases to 2.1 billion bushels, and the annual demand decreases to 1.8 billion bushels. Assume that the price–supply and price–demand equations are linear. (*Source:* U.S. Census Bureau)

(A) Find the price–supply equation. $p = 1.5x + 1.95$

(B) Find the price–demand equation. $p = -1.5x + 7.8$

(C) Find the equilibrium price and quantity. Equilibrium price: \$4.875; equilibrium quantity: 1.95 billion bushels

(D) Graph the two equations in the same coordinate system and identify the equilibrium point, supply curve, and demand curve.*

68. Supply and demand for corn. At \$2.13 per bushel, the annual supply for corn in the Midwest is 8.9 billion bushels and the annual demand is 6.5 billion bushels. When the price falls to \$1.50 per bushel, the annual supply decreases to 8.2 billion bushels and the annual demand increases to 7.4 billion bushels. Assume that the price–supply and price–demand equations are linear. (*Source:* U.S. Census Bureau)

(A) Find the price–supply equation. $p = 0.9x - 5.88$

(B) Find the price–demand equation. $p = -0.7x + 6.68$

(C) Find the equilibrium price and quantity. Equilibrium = \$1.185 per bushel; equilibrium quantity = 7.85 billion bushels

(D) Graph the two equations in the same coordinate system and identify the equilibrium point, supply curve, and demand curve.*

69. Break-even analysis. A small plant manufactures riding lawn mowers. The plant has fixed costs (leases, insurance, etc.) of \$48,000 per day and variable costs (labor, materials, etc.) of \$1,400 per unit produced. The mowers are sold for \$1,800 each. So the cost and revenue equations are

$$y = 48{,}000 + 1{,}400x \quad \text{Cost equation}$$
$$y = 1{,}800x \quad \text{Revenue equation}$$

where x is the total number of mowers produced and sold each day. The daily costs and revenue are in dollars.

(A) How many units must be manufactured and sold each day for the company to break even? 120 mowers

(B) Graph both equations in the same coordinate system and show the break-even point. Interpret the regions between the lines to the left and to the right of the break-even point.*

70. Break-even analysis. Repeat Problem 69 with the cost and revenue equations

$$y = 65{,}000 + 1{,}100x \quad \text{Cost equation}$$
$$y = 1{,}600x \quad \text{Revenue equation*}$$

71. Break-even analysis. A company markets exercise DVDs that sell for \$19.95, including shipping and handling. The monthly fixed costs (advertising, rent, etc.) are \$24,000 and the variable costs (materials, shipping, etc) are \$7.45 per DVD.

(A) Find the cost equation and the revenue equation. $C = 24{,}000 + 7.45x$; $R = 19.95x$

(B) How many DVDs must be sold each month for the company to break even? 1,920

(C) Graph the cost and revenue equations in the same coordinate system and show the break-even point. Interpret the regions between the lines to the left and to the right of the break-even point.*

72. Break-even analysis. Repeat Problem 71 if the monthly fixed costs increase to \$27,200, the variable costs increase to \$9.15, and the company raises the selling price of the DVDs to \$21.95.*

73. Delivery charges. United Express, a national package delivery service, charges a base price for overnight delivery of packages weighing 1 pound or less and a surcharge for each additional pound (or fraction thereof). A customer is billed \$27.75 for shipping a 5-pound package and \$64.50 for a 20-pound package. Find the base price and the surcharge for each additional pound. Base price = \$17.95; surcharge = \$2.45/lb

74. Delivery charges. Refer to Problem 73. Federated Shipping, a competing overnight delivery service, informs the customer in Problem 73 that they would ship the 5-pound package for \$29.95 and the 20-pound package for \$59.20.

(A) If Federated Shipping computes its cost in the same manner as United Express, find the base price and the surcharge for Federated Shipping. Base price $= \$22.15$; surcharge $= \$1.95/\text{lb}$

✎ (B) Devise a simple rule that the customer can use to choose the cheaper of the two services for each package shipped. Justify your answer.*

75. Coffee blends. A coffee company uses Colombian and Brazilian coffee beans to produce two blends, robust and mild. A pound of the robust blend requires 12 ounces of Colombian beans and 4 ounces of Brazilian beans. A pound of the mild blend requires 6 ounces of Colombian beans and 10 ounces of Brazilian beans. Coffee is shipped in 132-pound burlap bags. The company has 50 bags of Colombian beans and 40 bags of Brazilian beans on hand. How many pounds of each blend should the company produce in order to use all the available beans? 5,720 lb robust blend; 6,160 lb mild blend

76. Coffee blends. Refer to Problem 75.

(A) If the company decides to discontinue production of the robust blend and produce only the mild blend, how many pounds of the mild blend can the company produce? How many beans of each type will the company use? Are there any beans that are not used?*

(B) Repeat part (A) if the company decides to discontinue production of the mild blend and produce only the robust blend.*

77. Animal diet. Animals in an experiment are to be kept under a strict diet. Each animal should receive 20 grams of protein and 6 grams of fat. The laboratory technician is able to purchase two food mixes: Mix A has 10% protein and 6% fat; mix B has 20% protein and 2% fat. How many grams of each mix should be used to obtain the right diet for one animal? Mix A: 80 g; mix B: 60 g

78. Fertilizer. A fruit grower uses two types of fertilizer in an orange grove, brand A and brand B. Each bag of brand A contains 8 pounds of nitrogen and 4 pounds of phosphoric acid. Each bag of brand B contains 7 pounds of nitrogen and 6 pounds of phosphoric acid. Tests indicate that the grove needs 720 pounds of nitrogen and 500 pounds of phosphoric acid. How many bags of each brand should be used to provide the required amounts of nitrogen and phosphoric acid? Brand A: 41 bags; Brand B: 56 bags

79. Electronics. A supplier for the electronics industry manufactures keyboards and screens for graphing calculators at plants in Mexico and Taiwan. The hourly production rates at each plant are given in the table. How many hours should each plant be operated to exactly fill an order for 4,000 keyboards and 4,000 screens?

Plant	Keyboards	Screens
Mexico	40	32
Taiwan	20	32

Operate the Mexico plant for 75 hours and the Taiwan plant for 50 hours.

80. Sausage. A company produces Italian sausages and bratwursts at plants in Green Bay and Sheboygan. The hourly production rates at each plant are given in the table. How many hours should each plant operate to exactly fill an order for 62,250 Italian sausages and 76,500 bratwursts?

Plant	Italian Sausage	Bratwurst
Green Bay	800	800
Sheboygan	500	1,000

Operate the Green Bay plant for 60 hours and the Sheboygan plant for 28.5 hours.

81. Physics. An object dropped off the top of a tall building falls vertically with constant acceleration. If s is the distance of the object above the ground (in feet) t seconds after its release, then s and t are related by an equation of the form $s = a + bt^2$, where a and b are constants. Suppose the object is 180 feet above the ground 1 second after its release and 132 feet above the ground 2 seconds after its release.

(A) Find the constants a and b. $a = 196, b = -16$

(B) How tall is the building? 196 ft

(C) How long does the object fall? 3.5 sec

82. Physics. Repeat Problem 81 if the object is 240 feet above the ground after 1 second and 192 feet above the ground after 2 seconds. **(A)** $a = 256, b = -16$ **(B)** 256 ft **(C)** 4 sec

83. Earthquakes. An earthquake emits a primary wave and a secondary wave. Near the surface of the Earth the primary wave travels at 5 miles per second and the secondary wave at 3 miles per second. From the time lag between the two waves arriving at a given receiving station, it is possible to estimate the distance to the quake. Suppose a station measured a time difference of 16 seconds between the arrival of the two waves. How long did each wave travel, and how far was the earthquake from the station? 40 sec, 24 sec, 120 mi

84. Sound waves. A ship using sound-sensing devices above and below water recorded a surface explosion 6 seconds sooner by its underwater device than its above-water device. Sound travels in air at 1,100 feet per second and in seawater at 5,000 feet per second. How long did it take each sound wave to reach the ship? How far was the explosion from the ship? $\frac{22}{13}$ sec, $\frac{100}{13}$ sec, 8,462 ft

85. Psychology. People approach certain situations with "mixed emotions." For example, public speaking often brings forth the positive response of recognition and the negative response of failure. Which dominates? J. S. Brown, in an experiment on approach and avoidance, trained rats by feeding them from a goal box. The rats received mild electric shocks from the same goal box. This established an approach—avoidance conflict relative to the goal box. Using an appropriate apparatus, Brown arrived at the following relationships:

$$p = -\tfrac{1}{5}d + 70 \quad \text{Approach equation}$$

$$p = -\tfrac{4}{3}d + 230 \quad \text{Avoidance equation}$$

where $30 \le d \le 172.5$. The approach equation gives the pull (in grams) toward the food goal box when the rat is placed d centimeters away from it. The avoidance equation gives the pull (in grams) away from the shock goal box when the rat is placed d centimeters from it.

(A) Graph the approach equation and the avoidance equation in the same coordinate system.*

(B) Find the value of d for the point of intersection of these two equations. $d = 141$ cm (approx.)

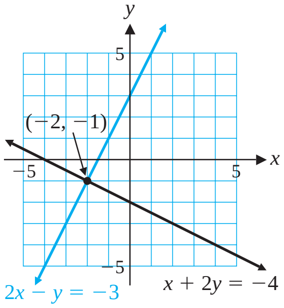
(C) What do you think the rat would do when placed the distance d from the box found in part (B)? Vacillate

(*Source: Journal of Comparative and Physiological Psychology,* 41:450–465.)

Answers to Matched Problems

1. $x = -2, y = -1$

$$\begin{aligned} 2x - \quad y &= -3 \\ 2(-2) - (-1) &\overset{?}{=} -3 \\ -3 &\overset{\checkmark}{=} -3 \\ x + \quad 2y &= -4 \\ (-2) + 2(-1) &\overset{?}{=} -4 \\ -4 &\overset{\checkmark}{=} -4 \end{aligned}$$



2. (A) $x = 2, y = 2$

(B) Infinitely many solutions

(C) No solution

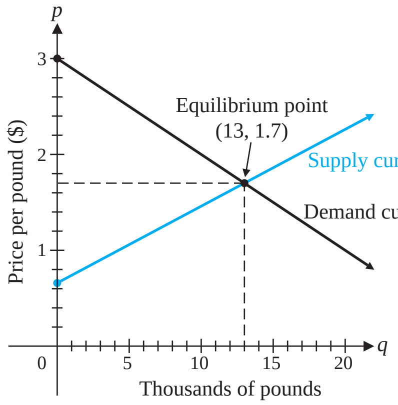
3. $x = -1.92, y = 4.23$

4. $x = -2, y = 2$

5. $x = 2, y = -1$

6. 16.5 oz of cottage cheese, 12.5 oz of yogurt

7. Equilibrium quantity $=$ 13 thousand pounds; equilibrium price $=$ \$1.70 per pound



4.2 Systems of Linear Equations and Augmented Matrices

- Matrices
- Solving Linear Systems Using Augmented Matrices
- Summary

Most linear systems of any consequence involve large numbers of equations and variables. It is impractical to try to solve such systems by hand. In the past, these complex systems could be solved only on large computers. Now there are a wide array of approaches to solving linear systems, ranging from graphing calculators to software and spreadsheets. In the rest of this chapter, we develop several *matrix methods* for solving systems with the understanding that these methods are generally used with a graphing calculator. It is important to keep in mind that we are not presenting these techniques as efficient methods for solving linear systems by hand. Instead, we emphasize formulation of mathematical models and interpretation of the results—two activities that graphing calculators cannot perform for you.

Matrices

In solving systems of equations using elimination by addition, the coefficients of the variables and the constant terms played a central role. The process can be made more efficient for generalization and computer work by the introduction of a mathematical form called a *matrix*. A **matrix** is a rectangular array of numbers written within brackets. Two examples are

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 7 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 5 & 12 \\ 0 & 1 & 8 \\ -3 & 10 & 9 \\ -6 & 0 & -1 \end{bmatrix} \tag{1}$$

Each number in a matrix is called an **element** of the matrix. Matrix A has 6 elements arranged in 2 rows and 3 columns. Matrix B has 12 elements arranged in 4 rows and 3 columns. If a matrix has m rows and n columns, it is called an $\boldsymbol{m \times n}$ **matrix** (read "m by n matrix"). The expression $m \times n$ is called the **size** of the matrix, and the numbers m and n are called the **dimensions** of the matrix. It is important to note that the number of rows is always given first. Referring to equations (1), A is a 2×3 matrix and B is a 4×3 matrix. A matrix with n rows and n columns is called a **square matrix of order n**. A matrix with only 1 column is called a **column matrix**, and a matrix with only 1 row is called a **row matrix**.

$$\overset{3 \times 3}{\begin{bmatrix} 0.5 & 0.2 & 1.0 \\ 0.0 & 0.3 & 0.5 \\ 0.7 & 0.0 & 0.2 \end{bmatrix}} \qquad \overset{4 \times 1}{\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}} \qquad \overset{1 \times 4}{\begin{bmatrix} 2 & \frac{1}{2} & 0 & -\frac{2}{3} \end{bmatrix}}$$

Square matrix of order 3 Column matrix Row matrix

The **position** of an element in a matrix is given by the row and column containing the element. This is usually denoted using **double subscript notation** a_{ij}, where i is the row and j is the column containing the element a_{ij}, as illustrated below:

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 7 & 0 & -2 \end{bmatrix} \qquad \begin{matrix} a_{11} = 1, & a_{12} = -4, & a_{13} = 5 \\ a_{21} = 7, & a_{22} = 0, & a_{23} = -2 \end{matrix}$$

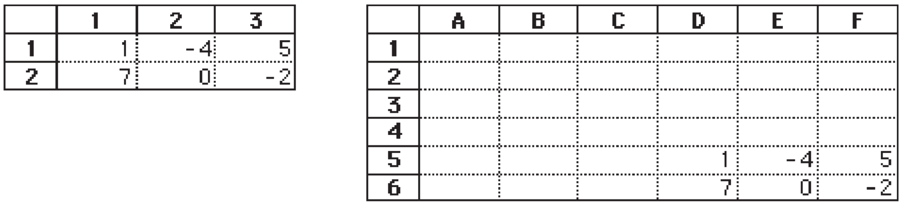
Note that a_{12} is read "a sub one two" (*not* "a sub twelve"). The elements $a_{11} = 1$ and $a_{22} = 0$ make up the *principal diagonal* of A. In general, the **principal diagonal** of a matrix A consists of the elements $a_{11}, a_{22}, a_{33}, \ldots$.

Remark—Most graphing calculators are capable of storing and manipulating matrices. Figure 1 shows matrix A displayed in the editing screen of a graphing calculator. The size of the matrix is given at the top of the screen. The position and value of the currently selected element is given at the bottom. Note that a comma is used in the notation for the position. This is common practice on many graphing calculators but not in mathematical literature. In a spreadsheet, matrices are referred to by their location (upper left corner to lower right corner), using either row and column numbers (Fig. 2A) or row numbers and column letters (Fig. 2B).

```
MATRIX[A] 2 ×3
[1      -4      5    ]
[7      0      -2    ]

1,2=-4
```

Figure 1 **Matrix notation on a graphing calculator**



	1	2	3
1	1	-4	5
2	7	0	-2

	A	B	C	D	E	F
1						
2						
3						
4						
5				1	-4	5
6				7	0	-2

(A) Location of matrix A: R1C1:R2C3

(B) Location of matrix A: D5:F6

Figure 2 **Matrix notation in a spreadsheet**

Matrices serve as a shorthand for solving systems of linear equations. Associated with the system

$$\begin{aligned} 2x - 3y &= 5 \\ x + 2y &= -3 \end{aligned} \tag{2}$$

are its **coefficient matrix**, **constant matrix**, and **augmented matrix**:

$$\overset{\text{Coefficient matrix}}{\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}} \qquad \overset{\text{Constant matrix}}{\begin{bmatrix} 5 \\ -3 \end{bmatrix}} \qquad \overset{\text{Augmented matrix}}{\left[\begin{array}{cc|c} 2 & -3 & 5 \\ 1 & 2 & -3 \end{array}\right]}$$

Note that the augmented matrix is just the coefficient matrix, augmented by the constant matrix. The vertical bar is included only as a visual aid to separate the coefficients from the constant terms. The augmented matrix contains all of the essential information about the linear system—everything but the names of the variables.

For ease of generalization to the larger systems in later sections, we will change the notation for the variables in system (2) to a subscript form. That is, in place of x and y, we use x_1 and x_2, respectively, and system (2) is rewritten as

$$\begin{aligned} 2x_1 - 3x_2 &= 5 \\ x_1 + 2x_2 &= -3 \end{aligned}$$

In general, associated with each linear system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= k_1 \\ a_{21}x_1 + a_{22}x_2 &= k_2 \end{aligned} \tag{3}$$

where x_1 and x_2 are variables, is the *augmented matrix* of the system:

$$\begin{array}{cc} \begin{array}{ccc} \text{Column 1 } (C_1) & \text{Column 2 } (C_2) & \text{Column 3 } (C_3) \\ \downarrow & \downarrow & \downarrow \end{array} & \\ \left[\begin{array}{cc|c} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \end{array}\right] & \begin{array}{l} \leftarrow \text{Row 1 } (R_1) \\ \leftarrow \text{Row 2 } (R_2) \end{array} \end{array}$$

This matrix contains the essential parts of system (3). Our objective is to learn how to manipulate augmented matrices in order to solve system (3), if a solution exists. The manipulative process is closely related to the elimination process discussed in Section 4.1.

Recall that two linear systems are said to be equivalent if they have the same solution set. In Theorem 2, Section 4.1, we used the operations listed below to transform linear systems into equivalent systems:

(A) Two equations are interchanged.

(B) An equation is multiplied by a nonzero constant.

(C) A constant multiple of one equation is added to another equation.

Paralleling the earlier discussion, we say that two augmented matrices are **row equivalent**, denoted by the symbol ~ placed between the two matrices, if they are augmented matrices of equivalent systems of equations. How do we transform augmented matrices into row-equivalent matrices? We use Theorem 1, which is a direct consequence of the operations listed in Section 4.1.

THEOREM 1 Operations That Produce Row-Equivalent Matrices

An augmented matrix is transformed into a row-equivalent matrix by performing any of the following **row operations**:

(A) Two rows are interchanged $(R_i \leftrightarrow R_j)$.

(B) A row is multiplied by a nonzero constant $(kR_i \rightarrow R_i)$.

(C) A constant multiple of one row is added to another row $(kR_j + R_i \rightarrow R_i)$.

Note: The arrow → means "replaces."

Solving Linear Systems Using Augmented Matrices

We illustrate the use of Theorem 1 by several examples.

EXAMPLE 1 Solving a System Using Augmented Matrix Methods Solve using augmented matrix methods:

$$\begin{aligned} 3x_1 + 4x_2 &= 1 \\ x_1 - 2x_2 &= 7 \end{aligned} \tag{4}$$

SOLUTION We start by writing the augmented matrix corresponding to system (4):

$$\left[\begin{array}{rr|r} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \tag{5}$$

Our objective is to use row operations from Theorem 1 to try to transform matrix (5) into the form

$$\left[\begin{array}{rr|r} 1 & 0 & m \\ 0 & 1 & n \end{array}\right] \tag{6}$$

where m and n are real numbers. Then the solution to system (4) will be obvious, since matrix (6) will be the augmented matrix of the following system (a row in an augmented matrix always corresponds to an equation in a linear system):

$$\begin{aligned} x_1 &= m & \qquad x_1 + 0x_2 &= m \\ x_2 &= n & \qquad 0x_1 + x_2 &= n \end{aligned}$$

Now we use row operations to transform matrix (5) into form (6).

Step 1 To get a 1 in the upper left corner, we interchange R_1 and R_2 (Theorem 1A):

$$\left[\begin{array}{rr|r} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{rr|r} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right]$$

Step 2 To get a 0 in the lower left corner, we multiply R_1 by (-3) and add to R_2 (Theorem 1C)—this changes R_2 but not R_1. Some people find it useful to write $(-3R_1)$ outside the matrix to help reduce errors in arithmetic, as shown:

$$\begin{array}{c} \left[\begin{array}{rr|r} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right] \\ \begin{array}{rrr} -3 & 6 & -21 \end{array} \end{array} \underbrace{(-3)R_1}_{} \underset{\sim}{+} R_2 \to R_2 \left[\begin{array}{rr|r} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right]$$

Step 3 To get a 1 in the second row, second column, we multiply R_2 by $\frac{1}{10}$ (Theorem 1B):

$$\left[\begin{array}{rr|r} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right] \tfrac{1}{10}R_2 \underset{\sim}{\to} R_2 \left[\begin{array}{rr|r} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

Step 4 To get a 0 in the first row, second column, we multiply R_2 by 2 and add the result to R_1 (Theorem 1C)—this changes R_1 but not R_2:

$$\begin{array}{c} \begin{array}{rrr} 0 & 2 & -4 \end{array} \\ \left[\begin{array}{rr|r} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right] \end{array} \overbrace{2R_2}^{} + \underset{\sim}{R_1} \to R_1 \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

We have accomplished our objective! The last matrix is the augmented matrix for the system

$$\begin{aligned} x_1 &= 3 & \qquad x_1 + 0x_2 &= 3 \\ x_2 &= -2 & \qquad 0x_1 + x_2 &= -2 \end{aligned} \tag{7}$$

Since system (7) is equivalent to system (4), our starting system, we have solved system (4); that is, $x_1 = 3$ and $x_2 = -2$.

CHECK

$$\begin{aligned} 3x_1 + 4x_2 &= 1 & \qquad x_1 - 2x_2 &= 7 \\ 3(3) + 4(-2) &\overset{?}{=} 1 & \qquad 3 - 2(-2) &\overset{?}{=} 7 \\ 1 &\overset{\checkmark}{=} 1 & \qquad 7 &\overset{\checkmark}{=} 7 \end{aligned}$$

The preceding process may be written more compactly as follows:

Step 1: Need a 1 here.

$$\left[\begin{array}{cc|c}3&4&1\\1&-2&7\end{array}\right] \quad R_1 \leftrightarrow R_2$$

Step 2: Need a 0 here.

$$\sim\left[\begin{array}{cc|c}1&-2&7\\3&4&1\end{array}\right] \quad \underbrace{(-3)R_1}_{\vdots} + R_3 \to R_2$$
$$\begin{array}{cc|c}-3&6&-21\end{array}$$

Step 3: Need 1 here.

$$\sim\left[\begin{array}{cc|c}1&-2&7\\0&10&-20\end{array}\right] \quad \tfrac{1}{10}R_2 \to R_2$$

Step 4: Need a 0 here.

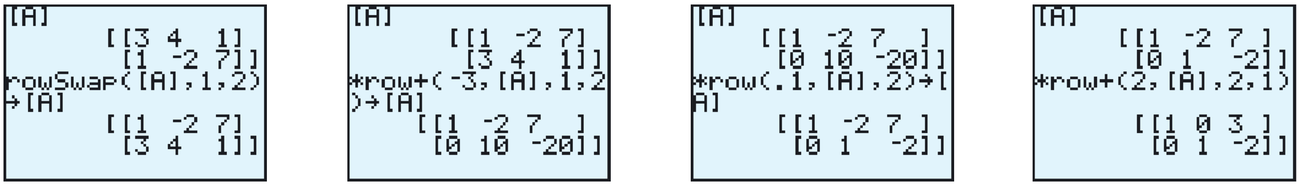
$$\begin{array}{cc|c}0&2&-4\end{array}$$
$$\sim\left[\begin{array}{cc|c}1&-2&7\\0&1&-2\end{array}\right] \quad \underbrace{2R_2}_{\vdots} + R_1 \to R_1$$
$$\sim\left[\begin{array}{cc|c}1&0&3\\0&1&-2\end{array}\right]$$

Therefore, $x_1 = 3$ and $x_2 = -2$.

Matched Problem 1 Solve using augmented matrix methods:

$$\begin{aligned}2x_1 - x_2 &= -7\\ x_1 + 2x_2 &= 4\end{aligned}$$

Many graphing calculators can perform row operations. Figure 3 shows the results of performing the row operations used in the solution of Example 1. Consult your manual for the details of performing row operations on your graphing calculator.



```
[A]
      [[3 4  1]
       [1 -2 7]]
rowSwap([A],1,2)
→[A]
      [[1 -2 7]
       [3 4  1]]
```

(A) $R_1 \leftrightarrow R_2$

```
[A]
      [[1 -2 7]
       [3 4  1]]
*row+(-3,[A],1,2
)→[A]
     [[1 -2 7  ]
      [0 10 -20]]
```

(B) $(-3)R_1 + R_2 \to R_2$

```
[A]
    [[1 -2 7  ]
     [0 10 -20]]
*row(.1,[A],2)→[
A]
     [[1 -2 7 ]
      [0 1  -2]]
```

(C) $\frac{1}{10}R_2 \to R_2$

```
[A]
     [[1 -2 7 ]
      [0 1  -2]]
*row+(2,[A],2,1)

     [[1 0 3 ]
      [0 1 -2]]
```

(D) $2R_2 + R_1 \to R_1$

Figure 3 **Row operations on a graphing calculator**

Explore and Discuss 1 The summary following the solution of Example 1 shows five augmented matrices. Write the linear system that each matrix represents, solve each system graphically, and discuss the relationships among these solutions.

EXAMPLE 2 Solving a System Using Augmented Matrix Methods Solve using augmented matrix methods:

$$\begin{aligned}2x_1 - 3x_2 &= 6\\ 3x_1 + 4x_2 &= \frac{1}{2}\end{aligned}$$

SOLUTION

Step 1: Need a 1 here.

$$\left[\begin{array}{cc|c} 2 & -3 & 6 \\ 3 & 4 & \frac{1}{2} \end{array}\right] \qquad \tfrac{1}{2}R_1 \to R_1$$

Step 2: Need a 0 here.

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 3 \\ 3 & 4 & \frac{1}{2} \\ -3 & \frac{9}{2} & -9 \end{array}\right] \qquad \underbrace{(-3)R_1}_{} + R_2 \to R_2$$

Step 3: Need a 1 here.

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 3 \\ 0 & \frac{17}{2} & -\frac{17}{2} \end{array}\right] \qquad \tfrac{2}{17}R_2 \to R_2$$

Step 4: Need a 0 here.

$$\begin{array}{ccc} 0 & \frac{3}{2} & -\frac{3}{2} \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 3 \\ 0 & 1 & -1 \end{array}\right] \qquad \underbrace{\tfrac{3}{2}R_2}_{} + R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -1 \end{array}\right]$$

So, $x_1 = \frac{3}{2}$ and $x_2 = -1$. The check is left for you.

Matched Problem 2 Solve using augmented matrix methods:

$$\begin{aligned} 5x_1 - 2x_2 &= 11 \\ 2x_1 + 3x_2 &= \frac{5}{2} \end{aligned}$$

EXAMPLE 3 Solving a System Using Augmented Matrix Methods Solve using augmented matrix methods:

$$\begin{aligned} 2x_1 - x_2 &= 4 \\ -6x_1 + 3x_2 &= -12 \end{aligned} \tag{8}$$

SOLUTION

$$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ -6 & 3 & -12 \end{array}\right] \qquad \begin{array}{l} \tfrac{1}{2}R_1 \to R_1 \text{ (to get a 1 in the upper left corner)} \\ \tfrac{1}{3}R_2 \to R_2 \text{ (this simplifies } R_2\text{)} \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 2 \\ -2 & 1 & -4 \\ 2 & -1 & 4 \end{array}\right] \qquad \underbrace{2R_1}_{} + R_2 \to R_2 \text{ (to get a 0 in the lower left corner)}$$

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 0 \end{array}\right]$$

The last matrix corresponds to the system

$$\begin{aligned} x_1 - \frac{1}{2}x_2 &= 2 \qquad x_1 - \frac{1}{2}x_2 = 2 \\ 0 &= 0 \qquad 0x_1 + 0x_2 = 0 \end{aligned} \tag{9}$$

This system is equivalent to the original system. Geometrically, the graphs of the two original equations coincide, and there are infinitely many solutions. In general, if we end up with a row of zeros in an augmented matrix for a two-equation, two-variable system, the system is dependent, and there are infinitely many solutions.

We represent the infinitely many solutions using the same method that was used in Section 4.1; that is, by introducing a parameter. We start by solving

$x_1 - \frac{1}{2}x_2 = 2$, the first equation in system (9), for either variable in terms of the other. We choose to solve for x_1 in terms of x_2 because it is easier:

$$x_1 = \frac{1}{2}x_2 + 2 \tag{10}$$

Now we introduce a parameter t (we can use other letters, such as k, s, p, q, and so on, to represent a parameter also). If we let $x_2 = t$, then for any real number t,

$$\begin{aligned} x_1 &= \frac{1}{2}t + 2 \\ x_2 &= t \end{aligned} \tag{11}$$

represents a solution of system (8). Using ordered-pair notation, we write: For any real number t,

$$\left(\frac{1}{2}t + 2, t\right) \tag{12}$$

is a solution of system (8). More formally, we write

$$\text{solution set} = \left\{\left(\frac{1}{2}t + 2, t\right) \middle| t \in R\right\} \tag{13}$$

Typically we use the less formal notations (11) or (12) to represent the solution set for problems of this type.

CHECK The following is a check that system (11) provides a solution to system (8) for any real number t:

$$\begin{aligned} 2x_1 - x_2 &= 4 \\ 2\left(\frac{1}{2}t + 2\right) - t &\overset{?}{=} 4 \\ t + 4 - t &\overset{?}{=} 4 \\ 4 &\overset{\checkmark}{=} 4 \end{aligned} \qquad \begin{aligned} -6x_1 + 3x_2 &= -12 \\ -6\left(\frac{1}{2}t + 2\right) + 3t &\overset{?}{=} -12 \\ -3t - 12 + 3t &\overset{?}{=} -12 \\ -12 &\overset{\checkmark}{=} -12 \end{aligned}$$

Matched Problem 3 Solve using augmented matrix methods:

$$\begin{aligned} -2x_1 + 6x_2 &= 6 \\ 3x_1 - 9x_2 &= -9 \end{aligned}$$

Explore and Discuss 2 The solution of Example 3 involved three augmented matrices. Write the linear system that each matrix represents, solve each system graphically, and discuss the relationships among these solutions.

EXAMPLE 4 Solving a System Using Augmented Matrix Methods

Solve using augmented matrix methods:

$$\begin{aligned} 2x_1 + 6x_2 &= -3 \\ x_1 + 3x_2 &= 2 \end{aligned}$$

SOLUTION

$$\left[\begin{array}{cc|c} 2 & 6 & -3 \\ 1 & 3 & 2 \end{array}\right] \qquad R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 6 & -3 \end{array}\right] \qquad \underbrace{(-2)R_1}_{\downarrow} + R_2 \to R_2$$

$$\begin{array}{ccc} -2 & -6 & -4 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & -7 \end{array}\right] \qquad R_2 \text{ implies the contradiction } 0 = -7.$$

This is the augmented matrix of the system

$$\begin{aligned} x_1 + 3x_2 &= 2 \qquad & x_1 + 3x_2 &= 2 \\ 0 &= -7 \qquad & 0x_1 + 0x_2 &= -7 \end{aligned}$$

The second equation is not satisfied by any ordered pair of real numbers. As we saw in Section 4.1, the original system is inconsistent and has no solution. If in a row of an augmented matrix we obtain all zeros to the left of the vertical bar and a nonzero number to the right, the system is inconsistent and there are no solutions.

Matched Problem 4 Solve using augmented matrix methods:

$$\begin{aligned} 2x_1 - x_2 &= 3 \\ 4x_1 - 2x_2 &= -1 \end{aligned}$$

Summary

Examples 2, 3, and 4 illustrate the three possible solution types for a system of two linear equations in two variables, as discussed in Theorem 1, Section 4.1. Examining the final matrix form in each of these solutions leads to the following summary.

SUMMARY

Possible Final Matrix Forms for a System of Two Linear Equations in Two Variables

Form 1: Exactly one solution (consistent and independent)	**Form 2: Infinitely many solutions (consistent and dependent)**	**Form 3: No solution (inconsistent)**
$\left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right]$	$\left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & 0 \end{array}\right]$	$\left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & p \end{array}\right]$

m, n, p are real numbers; $p \neq 0$

The process of solving systems of equations described in this section is referred to as **Gauss–Jordan elimination**.We formalize this method in the next section so that it will apply to systems of any size, including systems where the number of equations and the number of variables are not the same.

Exercises 4.2

Skills Warm-up Exercises

W *Problems 1–14 refer to the following matrices: (If necessary, review the terminology at the beginning of section 4.2.)*

$$A = \begin{bmatrix} 2 & -4 & 0 \\ 6 & 1 & -5 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 9 & 0 \\ -4 & 8 & 7 \\ 2 & 4 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -3 & 0 \end{bmatrix} \quad D = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$$

1. How many elements are there in A? In C? 6; 3

2. How many elements are there in B? In D? 9; 2

3. What is the size of B? Of D? 3×3; 2×1

4. What is the size of A? Of C? 2×3; 1×3

5. Which of the matrices is a column matrix? *D*

6. Which of the matrices is a row matrix? *C*

7. Which of the matrices is a square matrix? *B*

8. Which of the matrices does not contain the element 0? *D*

*Answer located in Additional Instructor's Answers section.

A **9.** List the elements on the principal diagonal of A. 2, 1

10. List the elements on the principal diagonal of B. $-1, 8, 0$

11. For matrix B, list the elements b_{31}, b_{22}, b_{13}. 2, 8, 0

12. For matrix A, list the elements a_{21}, a_{12}. $6, -4$

13. For matrix C, find $c_{11} + c_{12} + c_{13}$. -1

14. For matrix D, find $d_{11} + d_{21}$. 3

In Problems 15–18, write the coefficient matrix and the augmented matrix of the given system of linear equations.

15. $\begin{aligned} 3x_1 + 5x_2 &= 8 \\ 2x_1 - 4x_2 &= -7 \end{aligned}$ *

16. $\begin{aligned} -8x_1 + 3x_2 &= 10 \\ 6x_1 + 5x_2 &= 13 \end{aligned}$ *

17. $\begin{aligned} x_1 + 4x_2 &= 15 \\ 6x_1 \qquad &= 18 \end{aligned}$ *

18. $\begin{aligned} 5x_1 - x_2 &= 10 \\ 3x_2 &= 21 \end{aligned}$ *

In Problems 19–22, write the system of linear equations that is represented by the given augmented matrix. Assume that the variables are x_1 and x_2.

19. $\left[\begin{array}{cc|c} 2 & 5 & 7 \\ 1 & 4 & 9 \end{array}\right]$ *

20. $\left[\begin{array}{cc|c} 0 & 3 & 15 \\ -8 & 2 & 25 \end{array}\right]$ *

21. $\left[\begin{array}{cc|c} 4 & 0 & -10 \\ 0 & 8 & 40 \end{array}\right]$ *

22. $\left[\begin{array}{cc|c} 1 & -2 & 12 \\ 0 & 1 & 6 \end{array}\right]$ *

Perform the row operations indicated in Problems 23–34 on the following matrix:

$$\left[\begin{array}{cc|c} 1 & -3 & 2 \\ 4 & -6 & -8 \end{array}\right]$$

23. $R_1 \leftrightarrow R_2$ *

24. $\frac{1}{2}R_2 \to R_2$ *

25. $-4R_1 \to R_1$ *

26. $-2R_1 \to R_1$ *

27. $2R_2 \to R_2$ *

28. $-1R_2 \to R_2$ *

29. $(-4)R_1 + R_2 \to R_2$ *

30. $\left(-\frac{1}{2}\right)R_2 + R_1 \to R_1$ *

31. $(-2)R_1 + R_2 \to R_2$ *

32. $(-3)R_1 + R_2 \to R_2$ *

33. $(-1)R_1 + R_2 \to R_2$ *

34. $R_1 + R_2 \to R_2$ *

Each of the matrices in Problems 35–42 is the result of performing a single row operation on the matrix A shown below. Identify the row operation.

$$A = \left[\begin{array}{cc|c} -1 & 2 & -3 \\ 6 & -3 & 12 \end{array}\right]$$

35. $\left[\begin{array}{cc|c} -1 & 2 & -3 \\ 2 & -1 & 4 \end{array}\right]$
$\frac{1}{3}R_2 \to R_2$

36. $\left[\begin{array}{cc|c} -2 & 4 & -6 \\ 6 & -3 & 12 \end{array}\right]$
$2R_1 \to R_1$

37. $\left[\begin{array}{cc|c} -1 & 2 & -3 \\ 0 & 9 & -6 \end{array}\right]$
$6R_1 + R_2 \to R_2$

38. $\left[\begin{array}{cc|c} 3 & 0 & 5 \\ 6 & -3 & 12 \end{array}\right]$
$\frac{2}{3}R_2 + R_1 \to R_1$

39. $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 6 & -3 & 12 \end{array}\right]$
$\frac{1}{3}R_2 + R_1 \to R_1$

40. $\left[\begin{array}{cc|c} -1 & 2 & -3 \\ 2 & 5 & 0 \end{array}\right]$
$4R_1 + R_2 \to R_2$

41. $\left[\begin{array}{cc|c} 6 & -3 & 12 \\ -1 & 2 & -3 \end{array}\right]$
$R_1 \leftrightarrow R_2$

42. $\left[\begin{array}{cc|c} -1 & 2 & -3 \\ 0 & 9 & -6 \end{array}\right]$
$6R_1 + R_2 \to R_2$

✎ *Solve Problems 43–46 using augmented matrix methods. Graph each solution set. Discuss the differences between the graph of an equation in the system and the graph of the system's solution set.*

43. $\begin{aligned} 3x_1 - 2x_2 &= 6 \\ 4x_1 - 3x_2 &= 6 \end{aligned}$ *

44. $\begin{aligned} x_1 - 2x_2 &= 5 \\ -2x_1 + 4x_2 &= -10 \end{aligned}$ *

45. $\begin{aligned} 3x_1 - 2x_2 &= -3 \\ -6x_1 + 4x_2 &= 6 \end{aligned}$ *

46. $\begin{aligned} x_1 - 2x_2 &= 1 \\ -2x_1 + 5x_2 &= 2 \end{aligned}$ *

✎ *Solve Problems 47 and 48 using augmented matrix methods. Write the linear system represented by each augmented matrix in your solution, and solve each of these systems graphically. Discuss the relationships among the solutions of these systems.*

47. $\begin{aligned} x_1 + x_2 &= 5 \\ x_1 - x_2 &= 1 \end{aligned}$ *

48. $\begin{aligned} x_1 - x_2 &= 2 \\ x_1 + x_2 &= 6 \end{aligned}$ *

Each of the matrices in Problems 49–54 is the final matrix form for a system of two linear equations in the variables x_1 and x_2. Write the solution of the system.

49. $\left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 6 \end{array}\right]$
$x_1 = -4, x_2 = 6$

50. $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -5 \end{array}\right]$
$x_1 = 3, x_2 = -5$

51. $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & 4 \end{array}\right]$
No solution

52. $\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 0 & -9 \end{array}\right]$
No solution

53. $\left[\begin{array}{cc|c} 1 & -2 & 15 \\ 0 & 0 & 0 \end{array}\right]$
$x_1 = 2t + 15, x_2 = t$ for any real number t

54. $\left[\begin{array}{cc|c} 1 & 5 & 10 \\ 0 & 0 & 0 \end{array}\right]$
$x_1 = -5t + 10, x_2 = t$ for any real number t

B *Solve Problems 55–74 using augmented matrix methods.*

55. $\begin{aligned} x_1 - 2x_2 &= 1 \\ 2x_1 - x_2 &= 5 \end{aligned}$
$x_1 = 3, x_2 = 1$

56. $\begin{aligned} x_1 + 3x_2 &= 1 \\ 3x_1 - 2x_2 &= 14 \end{aligned}$
$x_1 = 4, x_2 = -1$

57. $\begin{aligned} x_1 - 4x_2 &= -2 \\ -2x_1 + x_2 &= -3 \end{aligned}$
$x_1 = 2, x_2 = 1$

58. $\begin{aligned} x_1 - 3x_2 &= -5 \\ -3x_1 - x_2 &= 5 \end{aligned}$
$x_1 = -2, x_2 = 1$

59. $\begin{aligned} 3x_1 - x_2 &= 2 \\ x_1 + 2x_2 &= 10 \end{aligned}$
$x_1 = 2, x_2 = 4$

60. $\begin{aligned} 2x_1 + x_2 &= 0 \\ x_1 - 2x_2 &= -5 \end{aligned}$
$x_1 = -1, x_2 = 2$

61. $\begin{aligned} x_1 + 2x_2 &= 4 \\ 2x_1 + 4x_2 &= -8 \end{aligned}$
No solution

62. $\begin{aligned} 2x_1 - 3x_2 &= -2 \\ -4x_1 + 6x_2 &= 7 \end{aligned}$
No solution

63. $\begin{aligned} 2x_1 + x_2 &= 6 \\ x_1 - x_2 &= -3 \end{aligned}$
$x_1 = 1, x_2 = 4$

64. $\begin{aligned} 3x_1 - x_2 &= -5 \\ x_1 + 3x_2 &= 5 \end{aligned}$
$x_1 = -1, x_2 = 2$

65. $\begin{aligned} 3x_1 - 6x_2 &= -9 \\ -2x_1 + 4x_2 &= 6 \end{aligned}$

66. $\begin{aligned} 2x_1 - 4x_2 &= -2 \\ -3x_1 + 6x_2 &= 3 \end{aligned}$

65. Infinitely many solutions: $x_2 = s$, $x_1 = 2s - 3$ for any real number s

66. Infinitely many solutions: $x_2 = s$, $x_1 = 2s - 1$ for any real number s

67. Infinitely many solutions; $x_2 = s$, $x_1 = \frac{1}{2}s + \frac{1}{2}$ for any real number s

67. $4x_1 - 2x_2 = 2$
$-6x_1 + 3x_2 = -3$

68. $-6x_1 + 2x_2 = 4$
$3x_1 - x_2 = -2$

69. $2x_1 + x_2 = 1$
$4x_1 - x_2 = -7$
$x_1 = -1, x_2 = 3$

70. $2x_1 - x_2 = -8$
$2x_1 + x_2 = 8$
$x_1 = 0, x_2 = 8$

71. $4x_1 - 6x_2 = 8$
$-6x_1 + 9x_2 = -10$
No solution

72. $2x_1 - 4x_2 = -4$
$-3x_1 + 6x_2 = 4$
No solution

73. $-4x_1 + 6x_2 = -8$
$6x_1 - 9x_2 = 12$

74. $-2x_1 + 4x_2 = 4$
$3x_1 - 6x_2 = -6$

C *Solve Problems 75–80 using augmented matrix methods.*

75. $3x_1 - x_2 = 7$
$2x_1 + 3x_2 = 1$
$x_1 = 2, x_2 = -1$

76. $2x_1 - 3x_2 = -8$
$5x_1 + 3x_2 = 1$
$x_1 = -1, x_2 = 2$

77. $3x_1 + 2x_2 = 4$
$2x_1 - x_2 = 5$
$x_1 = 2, x_2 = -1$

78. $4x_1 + 3x_2 = 26$
$3x_1 - 11x_2 = -7$
$x_1 = 5, x_2 = 2$

79. $0.2x_1 - 0.5x_2 = 0.07$
$0.8x_1 - 0.3x_2 = 0.79$
$x_1 = 1.1, x_2 = 0.3$

80. $0.3x_1 - 0.6x_2 = 0.18$
$0.5x_1 - 0.2x_2 = 0.54$
$x_1 = 1.2, x_2 = 0.3$

73. Infinitely many solutions: $x_2 = t$, $x_1 = \frac{3}{2}t + 2$ for any real number t

68. Infinitely many solutions: $x_2 = s$, $x_1 = \frac{1}{3}s - \frac{2}{3}$ for any real number s

Solve Problems 81–84 using augmented matrix methods. Use a graphing calculator to perform the row operations.

81. $0.8x_1 + 2.88x_2 = 4$
$1.25x_1 + 4.34x_2 = 5$
$x_1 = -23.125, x_2 = 7.8125$

82. $2.7x_1 - 15.12x_2 = 27$
$3.25x_1 - 18.52x_2 = 33$
$x_1 = 1.25, x_2 = -1.5625$

83. $4.8x_1 - 40.32x_2 = 295.2$
$-3.75x_1 + 28.7x_2 = -211.2$
$x_1 = 3.225, x_2 = -6.9375$

84. $5.7x_1 - 8.55x_2 = -35.91$
$4.5x_1 + 5.73x_2 = 76.17$
$x_1 = 6.2625, x_2 = 8.375$

Answers to Matched Problems

1. $x_1 = -2, x_2 = 3$

2. $x_1 = 2, x_2 = -\frac{1}{2}$

3. The system is dependent. For t any real number, a solution is $x_1 = 3t - 3$, $x_2 = t$.

4. Inconsistent—no solution

74. Infinitely many solutions: $x_2 = s$, $x_1 = 2s - 2$ for any real number s

4.3 Gauss–Jordan Elimination

- Reduced Matrices
- Solving Systems by Gauss–Jordan Elimination
- Application

Now that you have had some experience with row operations on simple augmented matrices, we consider systems involving more than two variables. We will not require a system to have the same number of equations as variables. Just as for systems of two linear equations in two variables, any linear system, regardless of the number of equations or number of variables, has either

1. Exactly one solution (consistent and independent), or
2. Infinitely many solutions (consistent and dependent), or
3. No solution (inconsistent).

Reduced Matrices

In the preceding section we used row operations to transform the augmented matrix for a system of two equations in two variables,

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \end{array}\right] \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 &= k_1 \\ a_{21}x_1 + a_{22}x_2 &= k_2 \end{aligned}$$

into one of the following simplified forms:

$$\begin{array}{ccc} \text{Form 1} & \text{Form 2} & \text{Form 3} \\ \left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right] & \left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & 0 \end{array}\right] & \left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & p \end{array}\right] \end{array} \qquad (1)$$

where m, n, and p are real numbers, $p \neq 0$. Each of these reduced forms represents a system that has a different type of solution set, and no two of these forms are row equivalent.

For large linear systems, it is not practical to list all such simplified forms; there are too many of them. Instead, we give a general definition of a simplified form called a **reduced matrix**, which can be applied to all matrices and systems, regardless of size.

DEFINITION Reduced Form

A matrix is said to be in **reduced row echelon form**, or, more simply, in **reduced form**, if

1. Each row consisting entirely of zeros is below any row having at least one nonzero element.
2. The leftmost nonzero element in each row is 1.
3. All other elements in the column containing the leftmost 1 of a given row are zeros.
4. The leftmost 1 in any row is to the right of the leftmost 1 in the row above.

The following matrices are in reduced form. Check each one carefully to convince yourself that the conditions in the definition are met.

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

EXAMPLE 1 Reduced Forms The following matrices are not in reduced form. Indicate which condition in the definition is violated for each matrix. State the row operation(s) required to transform the matrix into reduced form, and find the reduced form.

(A) $\left[\begin{array}{cc|c} 0 & 1 & -2 \\ 1 & 0 & 3 \end{array}\right]$ (B) $\left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

(C) $\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{array}\right]$ (D) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right]$

SOLUTION

(A) Condition 4 is violated: The leftmost 1 in row 2 is not to the right of the leftmost 1 in row 1. Perform the row operation $R_1 \leftrightarrow R_2$ to obtain

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

(B) Condition 3 is violated: The column containing the leftmost 1 in row 2 has a nonzero element above the 1. Perform the row operation $2R_2 + R_1 \rightarrow R_1$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

(C) Condition 1 is violated: The second row contains all zeros and is not below any row having at least one nonzero element. Perform the row operation $R_2 \leftrightarrow R_3$ to obtain

$$\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

(D) Condition 2 is violated: The leftmost nonzero element in row 2 is not a 1. Perform the row operation $\frac{1}{2}R_2 \rightarrow R_2$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -5 \end{array}\right]$$

Matched Problem 1 The matrices below are not in reduced form. Indicate which condition in the definition is violated for each matrix. State the row operation(s) required to transform the matrix into reduced form, and find the reduced form.

(A) $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 3 & -6 \end{array}\right]$

(B) $\left[\begin{array}{ccc|c} 1 & 5 & 4 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(C) $\left[\begin{array}{ccc|c} 0 & 1 & 0 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$

(D) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right]$

Solving Systems by Gauss–Jordan Elimination

We are now ready to outline the Gauss–Jordan method for solving systems of linear equations. The method systematically transforms an augmented matrix into a reduced form. The system corresponding to a reduced augmented matrix is called a **reduced system**. As we shall see, reduced systems are easy to solve.

The Gauss–Jordan elimination method is named after the German mathematician Carl Friedrich Gauss (1777–1885) and the German geodesist Wilhelm Jordan (1842–1899). Gauss, one of the greatest mathematicians of all time, used a method of solving systems of equations in his astronomical work that was later generalized by Jordan to solve problems in large-scale surveying.

EXAMPLE 2 Solving a System Using Gauss–Jordan Elimination Solve by Gauss–Jordan elimination:

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= 3 \\ 3x_1 + x_2 - x_3 &= 7 \\ x_1 - 3x_2 + 2x_3 &= 0 \end{aligned}$$

SOLUTION Write the augmented matrix and follow the steps indicated at the right.

Need a 1 here.

$$\left[\begin{array}{rrr|r} 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \\ 1 & -3 & 2 & 0 \end{array}\right] \begin{array}{l} R_1 \leftrightarrow R_3 \\ \\ \\ \end{array}$$

Step 1 Choose the leftmost nonzero column and get a 1 at the top.

Need 0's here.

$$\sim \left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 3 & 1 & -1 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right] \begin{array}{l} \\ (-3)R_1 + R_2 \rightarrow R_2 \\ (-2)R_1 + R_3 \rightarrow R_3 \end{array}$$

Step 2 Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.

Need a 1 here.

$$\sim \left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 0 & 4 & -3 & 3 \end{array}\right] \begin{array}{l} \\ 0.1R_2 \rightarrow R_2 \\ \\ \end{array}$$

Step 3 Repeat step 1 with the *submatrix* formed by (mentally) deleting the top row.

Need 0's here.

$$\sim \left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 4 & -3 & 3 \end{array}\right] \begin{array}{l} 3R_2 + R_1 \rightarrow R_1 \\ \\ (-4)R_2 + R_3 \rightarrow R_3 \end{array}$$

Step 4 Repeat step 2 with the *entire matrix.*

Need a 1 here.

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -0.1 & 2.1 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & -0.2 & 0.2 \end{array}\right] \begin{array}{l} \\ (-5)R_3 \rightarrow R_3 \\ \\ \end{array}$$

Step 3 Repeat step 1 with the *submatrix* formed by (mentally) deleting the top rows.

Need 0's here.

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -0.1 & 2.1 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} 0.1R_3 + R_1 \rightarrow R_1 \\ 0.7R_3 + R_2 \rightarrow R_2 \\ \\ \end{array}$$

Step 4 Repeat step 2 with the *entire matrix.*

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The matrix is now in reduced form, and we can solve the corresponding reduced system.

$$\begin{aligned} x_1 \qquad\qquad &= 2 \\ x_2 \qquad &= 0 \\ x_3 &= -1 \end{aligned}$$

The solution to this system is $x_1 = 2, x_2 = 0, x_3 = -1$. You should check this solution in the original system.

Matched Problem 2 Solve by Gauss–Jordan elimination:

$$\begin{aligned} 3x_1 + x_2 - 2x_3 &= 2 \\ x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 - 3x_3 &= 3 \end{aligned}$$

PROCEDURE Gauss–Jordan Elimination

Step 1 Choose the leftmost nonzero column and use appropriate row operations to get a 1 at the top.

Step 2 Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.

(Continued)

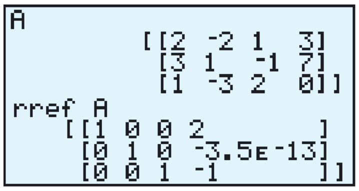
Step 3 Repeat step 1 with the **submatrix** formed by (mentally) deleting the row used in step 2 and all rows above this row.

Step 4 Repeat step 2 with the **entire matrix,** including the rows deleted mentally. Continue this process until the entire matrix is in reduced form.

Note: If at any point in this process we obtain a row with all zeros to the left of the vertical line and a nonzero number to the right, we can stop before we find the reduced form since we will have a contradiction: $0 = n, n \neq 0$. We can then conclude that the system has no solution.

Remarks

1. Even though each matrix has a unique reduced form, the sequence of steps presented here for transforming a matrix into a reduced form is not unique. For example, it is possible to use row operations in such a way that computations involving fractions are minimized. But we emphasize again that we are not interested in the most efficient hand methods for transforming small matrices into reduced forms. Our main interest is in giving you a little experience with a method that is suitable for solving large-scale systems on a graphing calculator or computer.
2. Most graphing calculators have the ability to find reduced forms. Figure 1 illustrates the solution of Example 2 on a TI-86 graphing calculator using the rref command (rref is an acronym for reduced row echelon form). Notice that in row 2 and column 4 of the reduced form the graphing calculator has displayed the very small number –3.5E – 13, instead of the exact value 0. This is a common occurrence on a graphing calculator and causes no problems. Just replace any very small numbers displayed in scientific notation with 0.



```
A
        [[2 -2 1  3]
         [3 1  -1 7]
         [1 -3 2  0]]
rref A
    [[1 0 0 2        ]
     [0 1 0 -3.5E-13]
     [0 0 1 -1       ]]
```

Figure 1 **Gauss–Jordan elimination on a graphing calculator**

EXAMPLE 3 Solving a System Using Gauss–Jordan Elimination Solve by Gauss–Jordan elimination:

$$\begin{aligned} 2x_1 - 4x_2 + x_3 &= -4 \\ 4x_1 - 8x_2 + 7x_3 &= 2 \\ -2x_1 + 4x_2 - 3x_3 &= 5 \end{aligned}$$

SOLUTION

$$\left[\begin{array}{rrr|r} 2 & -4 & 1 & -4 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right] \quad 0.5R_1 \rightarrow R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & -2 & 0.5 & -2 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right] \quad \begin{array}{l} (-4)R_1 + R_2 \rightarrow R_2 \\ 2R_1 + R_3 \rightarrow R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & -2 & 1 \end{array}\right] \quad 0.2R_2 \rightarrow R_2$$

Note that column 3 is the leftmost nonzero column in this submatrix.

$$\sim \left[\begin{array}{rrr|r} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{array}\right] \quad \begin{array}{l} (-0.5)R_2 + R_1 \rightarrow R_1 \\ 2R_2 + R_3 \rightarrow R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

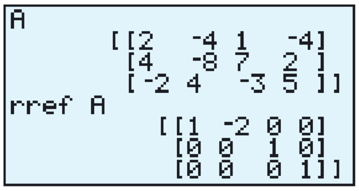
We stop the Gauss–Jordan elimination, even though the matrix is not in reduced form, since the last row produces a contradiction.

The system has no solution.

Matched Problem 3 Solve by Gauss–Jordan elimination:

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= -8 \\ 4x_1 - 8x_2 + 3x_3 &= 4 \\ -2x_1 + 4x_2 + x_3 &= 11 \end{aligned}$$

CAUTION Figure 2 shows the solution to Example 3 on a graphing calculator with a built-in reduced-form routine. Notice that the graphing calculator does not stop when a contradiction first occurs, but continues on to find the reduced form. Nevertheless, the last row in the reduced form still produces a contradiction. Do not confuse this type of reduced form with one that represents a consistent system (see Fig. 1). ▲



```
A
        [[2   ⁻4 1   ⁻4]
         [4   ⁻8 7   2 ]
         [⁻2 4   ⁻3 5 ]]
rref A
            [[1  ⁻2 0 0]
             [0 0   1 0]
             [0 0   0 1]]
```

Figure 2 **Recognizing contradictions on a graphing calculator**

EXAMPLE 4 Solving a System Using Gauss–Jordan Elimination Solve by Gauss–Jordan elimination:

$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 &= 15 \\ 2x_1 + 4x_2 - 6x_3 &= 10 \\ -2x_1 - 3x_2 + 4x_3 &= -6 \end{aligned}$$

SOLUTION

$$\left[\begin{array}{rrr|r} 3 & 6 & -9 & 15 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right] \qquad \frac{1}{3}R_1 \to R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right] \qquad \begin{array}{l} (-2)R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 \end{array}\right] \qquad R_2 \leftrightarrow R_3$$

Note that we must interchange rows 2 and 3 to obtain a nonzero entry at the top of the second column of this submatrix.

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & -3 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad (-2)R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 1 & -3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix is now in reduced form. Write the corresponding reduced system and solve.

$$\begin{aligned} x_1 + x_3 &= -3 \\ x_2 - 2x_3 &= 4 \end{aligned}$$

We discard the equation corresponding to the third (all zero) row in the reduced form, since it is satisfied by all values of x_1, x_2, and x_3.

Note that the leftmost variable in each equation appears in one and only one equation. We solve for the leftmost variables x_1 and x_2 in terms of the remaining variable, x_3 :

$$\begin{aligned} x_1 &= -x_3 - 3 \\ x_2 &= 2x_3 + 4 \end{aligned}$$

If we let $x_3 = t$, then for any real number t,

$$\begin{aligned} x_1 &= -t - 3 \\ x_2 &= 2t + 4 \\ x_3 &= t \end{aligned}$$

You should check that $(-t - 3, 2t + 4, t)$ is a solution of the original system for any real number t. Some particular solutions are

$t = 0$	$t = -2$	$t = 3.5$
$(-3, 4, 0)$	$(-1, 0, -2)$	$(-6.5, 11, 3.5)$

In general,

If the number of leftmost 1's in a reduced augmented coefficient matrix is less than the number of variables in the system and there are no contradictions, then the system is dependent and has infinitely many solutions.

Describing the solution set to this type of system is not difficult. In a reduced system, the **leftmost variables** correspond to the leftmost 1's in the corresponding reduced augmented matrix. The definition of reduced form for an augmented matrix ensures that each leftmost variable in the corresponding reduced system appears in one and only one equation of the system. Solving for each leftmost variable in terms of the remaining variables and writing a general solution to the system is usually easy. Example 5 illustrates a slightly more involved case.

Matched Problem 4 Solve by Gauss–Jordan elimination:

$$\begin{aligned} 2x_1 - 2x_2 - 4x_3 &= -2 \\ 3x_1 - 3x_2 - 6x_3 &= -3 \\ -2x_1 + 3x_2 + x_3 &= 7 \end{aligned}$$

Explore and Discuss 1 Explain why the definition of reduced form ensures that each leftmost variable in a reduced system appears in one and only one equation and no equation contains more than one leftmost variable. Discuss methods for determining whether a consistent system is independent or dependent by examining the reduced form.

EXAMPLE 5 Solving a System Using Gauss–Jordan Elimination Solve by Gauss–Jordan elimination:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 - x_5 &= 1 \\ 2x_1 + 4x_2 + 8x_3 + 3x_4 - 4x_5 &= 2 \\ x_1 + 3x_2 + 7x_3 \qquad\quad + 3x_5 &= -2 \end{aligned}$$

SOLUTION

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 1 & -1 & 1 \\ 2 & 4 & 8 & 3 & -4 & 2 \\ 1 & 3 & 7 & 0 & 3 & -2 \end{array}\right] \quad \begin{array}{l} (-2)R_1 + R_2 \rightarrow R_2 \\ (-1)R_1 + R_3 \rightarrow R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 4 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 3 & -1 & 4 & -3 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 4 & 1 & -1 & 1 \\ 0 & 1 & 3 & -1 & 4 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right] \quad (-2)R_2 + R_1 \rightarrow R_1$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & -9 & 7 \\ 0 & 1 & 3 & -1 & 4 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right] \quad \begin{array}{l} (-3)R_3 + R_1 \rightarrow R_1 \\ R_3 + R_2 \rightarrow R_2 \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -3 & 7 \\ 0 & 1 & 3 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right] \quad \text{Matrix is in reduced form.}$$

$$\begin{aligned} x_1 \quad - 2x_3 \quad - 3x_5 &= 7 \\ x_2 + 3x_3 \quad + 2x_5 &= -3 \\ x_4 - 2x_5 &= 0 \end{aligned}$$

Solve for the leftmost variables x_1, x_2, and x_4 in terms of the remaining variables x_3 and x_5:

$$\begin{aligned} x_1 &= 2x_3 + 3x_5 + 7 \\ x_2 &= -3x_3 - 2x_5 - 3 \\ x_4 &= 2x_5 \end{aligned}$$

If we let $x_3 = s$ and $x_5 = t$, then for any real numbers s and t,

$$\begin{aligned} x_1 &= 2s + 3t + 7 \\ x_2 &= -3s - 2t - 3 \\ x_3 &= s \\ x_4 &= 2t \\ x_5 &= t \end{aligned}$$

is a solution. The check is left for you.

Matched Problem 5 Solve by Gauss–Jordan elimination:

$$\begin{aligned} x_1 - x_2 + 2x_3 \qquad - 2x_5 &= 3 \\ -2x_1 + 2x_2 - 4x_3 - x_4 + x_5 &= -5 \\ 3x_1 - 3x_2 + 7x_3 + x_4 - 4x_5 &= 6 \end{aligned}$$

Application

Dependent systems of linear equations provide an excellent opportunity to discuss mathematical modeling in more detail. The process of using mathematics to solve real-world problems can be broken down into three steps (Fig. 3):

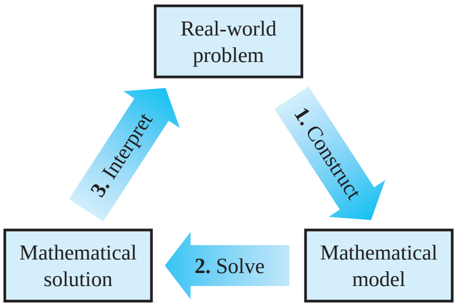


Figure 3

Step 1 *Construct* a mathematical model whose solution will provide information about the real-world problem.

Step 2 *Solve* the mathematical model.

Step 3 *Interpret* the solution to the mathematical model in terms of the original real-world problem.

In more complex problems, this cycle may have to be repeated several times to obtain the required information about the real-world problem.

EXAMPLE 6 Purchasing A company that rents small moving trucks wants to purchase 25 trucks with a combined capacity of 28,000 cubic feet. Three different types of trucks are available: a 10-foot truck with a capacity of 350 cubic feet, a 14-foot truck with a capacity of 700 cubic feet, and a 24-foot truck with a capacity of 1,400 cubic feet. How many of each type of truck should the company purchase?

SOLUTION The question in this example indicates that the relevant variables are the number of each type of truck:

x_1 = number of 10-foot trucks
x_2 = number of 14-foot trucks
x_3 = number of 24-foot trucks

Next we form the mathematical model:

$$\begin{aligned} x_1 + x_2 + x_3 &= 25 && \text{Total number of trucks} \\ 350x_1 + 700x_2 + 1{,}400x_3 &= 28{,}000 && \text{Total capacity} \end{aligned} \tag{2}$$

Now we form the augmented matrix of the system and solve by Gauss–Jordan elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 350 & 700 & 1{,}400 & 28{,}000 \end{array}\right] \quad \tfrac{1}{350}R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 1 & 2 & 4 & 80 \end{array}\right] \quad -R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 0 & 1 & 3 & 55 \end{array}\right] \quad -R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & -30 \\ 0 & 1 & 3 & 55 \end{array}\right] \quad \text{Matrix is in reduced form.}$$

$$\begin{aligned} x_1 - 2x_3 &= -30 \quad \text{or} \quad x_1 = 2x_3 - 30 \\ x_2 + 3x_3 &= 55 \quad \text{or} \quad x_2 = -3x_3 + 55 \end{aligned}$$

Let $x_3 = t$. Then for t any real number,

$$\begin{aligned} x_1 &= 2t - 30 \\ x_2 &= -3t + 55 \\ x_3 &= t \end{aligned} \tag{3}$$

is a solution to mathematical model (2).

Now we must interpret this solution in terms of the original problem. Since the variables x_1, x_2, and x_3 represent numbers of trucks, they must be nonnegative real numbers. And since we can't purchase a fractional number of trucks, each must be a nonnegative whole number. Since $t = x_3$, it follows that t must also be a nonnegative whole number. The first and second equations in model (3) place additional restrictions on the values that t can assume:

$$x_1 = 2t - 30 \ge 0 \quad \text{implies that} \quad t \ge 15$$

$$x_2 = -3t + 55 \ge 0 \quad \text{implies that} \quad t \le \frac{55}{3} = 18\tfrac{1}{3}$$

So the only possible values of t that will produce meaningful solutions to the original problem are 15, 16, 17, and 18. That is, the only combinations of 25 trucks that will result in a combined capacity of 28,000 cubic feet are $x_1 = 2t - 30$ 10-foot trucks, $x_2 = -3t + 55$ 14-foot trucks, and $x_3 = t$ 24-foot trucks, where $t = 15, 16, 17$, or 18. A table is a convenient way to display these solutions:

	10-Foot Truck	**14-Foot Truck**	**24-Foot Truck**
t	x_1	x_2	x_3
15	0	10	15
16	2	7	16
17	4	4	17
18	6	1	18

Matched Problem 6 A company that rents small moving trucks wants to purchase 16 trucks with a combined capacity of 19,200 cubic feet. Three different types of trucks are available: a cargo van with a capacity of 300 cubic feet, a 15-foot truck with a capacity of 900 cubic feet, and a 24-foot truck with a capacity of 1,500 cubic feet. How many of each type of truck should the company purchase?

Explore and Discuss 2 Refer to Example 6. The rental company charges \$19.95 per day for a 10-foot truck, \$29.95 per day for a 14-foot truck, and \$39.95 per day for a 24-foot truck. Which of the four possible choices in the table would produce the largest daily income from truck rentals?

10 14-foot trucks and
15 24-foot trucks

Exercises 4.3

W

Skills Warm-up Exercises

In Problems 1–4, write the augmented matrix of the system of linear equations. (If necessary, review the terminology of Section 4.2.)

1. $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 12 \\ x_1 + 7x_2 - 5x_3 &= 15 \end{aligned}$*

2. $\begin{aligned} 4x_1 + x_2 &= 8 \\ 3x_1 - 5x_2 &= 6 \\ x_1 + 9x_2 &= 4 \end{aligned}$*

3. $\begin{aligned} x_1 \quad\quad + 6x_3 &= 2 \\ x_2 - x_3 &= 5 \\ x_1 + 3x_2 \quad\quad &= 7 \end{aligned}$*

4. $\begin{aligned} 3x_1 + 4x_2 \quad\quad &= 10 \\ x_1 \quad\quad + 5x_3 &= 15 \\ -x_2 + x_3 &= 20 \end{aligned}$*

In Problems 5–8, write the system of linear equations that is represented by the augmented matrix. Assume that the variables are $x_1, x_2, \ldots$.

5. $\left[\begin{array}{rr|r} 1 & -3 & 4 \\ 3 & 2 & 5 \\ -1 & 6 & 3 \end{array}\right]$*

6. $\left[\begin{array}{rrr|r} -1 & 5 & 2 & 8 \\ 4 & 0 & -3 & 7 \end{array}\right]$*

7. $\left[\begin{array}{rrrr|r} 5 & -2 & 0 & 8 & 4 \end{array}\right]$*

8. $\left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 3 \\ 0 & 2 & 1 & -5 \end{array}\right]$*

*Answer located in Additional Instructor's Answers section.

A *In Problems 9–18, if a matrix is in reduced form, say so. If not, explain why and indicate the row operation(s) necessary to transform the matrix into reduced form.*

9. $\left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$*

10. $\left[\begin{array}{rr|r} 0 & 1 & 2 \\ 1 & 0 & -1 \end{array}\right]$*

11. $\left[\begin{array}{rrr|r} 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 4 \end{array}\right]$*

12. $\left[\begin{array}{rrr|r} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$*

13. $\left[\begin{array}{rrr|r} 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$*

14. $\left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$*

15. $\left[\begin{array}{rrr|r} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$*

16. $\left[\begin{array}{rrr|r} 1 & 0 & -1 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$*

17. $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$ * **18.** $\left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$ *

Write the linear system corresponding to each reduced augmented matrix in Problems 19–28 and write the solution of the system.

19. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $x_1 = -2, x_2 = 3, x_3 = 0$

20. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$ $x_1 = -2, x_2 = 0, x_3 = 1, x_4 = 3$

21. $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = 2t + 3, x_2 = -t - 5, x_3 = t$ for t any real number

22. $\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = 2t - 3, x_2 = t, x_3 = 5$ for t any real number

23. $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$ No solution

24. $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = 5, x_2 = -3$

25. $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 5 \\ 0 & 1 & 2 & -7 \end{array}\right]$ $x_1 = 3t + 5, x_2 = -2t - 7, x_3 = t$ for t any real number

26. $\left[\begin{array}{ccc|c} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 6 \end{array}\right]$ $x_1 = -t - 4, x_2 = t + 6, x_3 = t$ for t any real number

27. $\left[\begin{array}{cccc|c} 1 & -2 & 0 & -3 & -5 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right]$ $x_1 = 2s + 3t - 5, x_2 = s, x_3 = -3t + 2, x_4 = t$ for s and t any real numbers

28. $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 4 \\ 0 & 1 & -1 & 2 & -1 \end{array}\right]$ $x_1 = 2s - 3t + 4, x_2 = s - 2t - 1, x_3 = s, x_4 = t$ for s and t any real numbers

29. In which of Problems 19, 21, 23, 25, and 27 is the number of leftmost ones equal to the number of variables? 19

30. In which of Problems 20, 22, 24, 26, and 28 is the number of leftmost ones equal to the number of variables? 20, 24

31. In which of Problems 19, 21, 23, 25, and 27 is the number of leftmost ones less than the number of variables? 21, 25, 27

32. In which of Problems 20, 22, 24, 26, and 28 is the number of leftmost ones less than the number of variables? 22, 26, 28

In Problems 33–38, discuss the validity of each statement about linear systems. If the statement is always true, explain why. If not, give a counterexample.

33. If the number of leftmost ones is equal to the number of variables, then the system has exactly one solution. False

34. If the number of leftmost ones is less than the number of variables, then the system has infinitely many solutions. False

35. If the number of leftmost ones is equal to the number of variables and the system is consistent, then the system has exactly one solution. True

36. If the number of leftmost ones is less than the number of variables and the system is consistent, then the system has infinitely many solutions. True

37. The number of equations is less than or equal to the number of variables. False

38. The number of leftmost ones is less than or equal to the number of equations. True

B *Use row operations to change each matrix in Problems 39–46 to reduced form.*

39. $\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & 3 \end{array}\right]$ * **40.** $\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 2 & -4 \end{array}\right]$ *

41. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 2 & 3 & 4 & 25 \end{array}\right]$ * **42.** $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 3 & 5 & 7 & 30 \end{array}\right]$ *

43. $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & -6 \end{array}\right]$ * **44.** $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -2 & 2 \end{array}\right]$ *

45. $\left[\begin{array}{ccc|c} 1 & 2 & -2 & -1 \\ 0 & 3 & -6 & 1 \\ 0 & -1 & 2 & -\frac{1}{3} \end{array}\right]$ * **46.** $\left[\begin{array}{ccc|c} 0 & -2 & 8 & 1 \\ 2 & -2 & 6 & -4 \\ 0 & -1 & 4 & \frac{1}{2} \end{array}\right]$ *

Solve Problems 47–62 using Gauss–Jordan elimination.

47. $$\begin{aligned} 2x_1 + 4x_2 - 10x_3 &= -2 \\ 3x_1 + 9x_2 - 21x_3 &= 0 \\ x_1 + 5x_2 - 12x_3 &= 1 \end{aligned}$$ $x_1 = -2, x_2 = 3, x_3 = 1$

48. $$\begin{aligned} 3x_1 + 5x_2 - x_3 &= -7 \\ x_1 + x_2 + x_3 &= -1 \\ 2x_1 \qquad + 11x_3 &= 7 \end{aligned}$$ $x_1 = -2, x_2 = 0, x_3 = 1$

49. $$\begin{aligned} 3x_1 + 8x_2 - x_3 &= -18 \\ 2x_1 + x_2 + 5x_3 &= 8 \\ 2x_1 + 4x_2 + 2x_3 &= -4 \end{aligned}$$ $x_1 = 0, x_2 = -2, x_3 = 2$

50. $$\begin{aligned} 2x_1 + 6x_2 + 15x_3 &= -12 \\ 4x_1 + 7x_2 + 13x_3 &= -10 \\ 3x_1 + 6x_2 + 12x_3 &= -9 \end{aligned}$$ $x_1 = -3, x_2 = 4, x_3 = -2$

51. $$\begin{aligned} 2x_1 - x_2 - 3x_3 &= 8 \\ x_1 - 2x_2 \qquad &= 7 \end{aligned}$$ $x_1 = 2t + 3, x_2 = t - 2, x_3 = t$ for t any real number

52. $$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= 10 \\ 3x_1 + 3x_2 - 3x_3 &= 6 \end{aligned}$$ $x_1 = -t - 1, x_2 = 2t + 3, x_3 = t,$ for t any real number

53. $$\begin{aligned} 2x_1 - x_2 &= 0 \\ 3x_1 + 2x_2 &= 7 \\ x_1 - x_2 &= -1 \end{aligned}$$ $x_1 = 1, x_2 = 2$

54. $$\begin{aligned} 2x_1 - x_2 &= 0 \\ 3x_1 + 2x_2 &= 7 \\ x_1 - x_2 &= -2 \end{aligned}$$ No solution

55. $$\begin{aligned} 3x_1 - 4x_2 - x_3 &= 1 \\ 2x_1 - 3x_2 + x_3 &= 1 \\ x_1 - 2x_2 + 3x_3 &= 2 \end{aligned}$$ No solution

56.
$$\begin{aligned} 3x_1 + 7x_2 - x_3 &= 11\\ x_1 + 2x_2 - x_3 &= 3\\ 2x_1 + 4x_2 - 2x_3 &= 10 \end{aligned}$$
No solution

57.
$$\begin{aligned} 3x_1 - 2x_2 + x_3 &= -7\\ 2x_1 + x_2 - 4x_3 &= 0\\ x_1 + x_2 - 3x_3 &= 1 \end{aligned}$$
$x_1 = t - 1, x_2 = 2t + 2, x_3 = t$ for t any real number

58.
$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 21\\ x_1 - x_2 - 5x_3 &= -2\\ 2x_1 + x_2 - x_3 &= 11 \end{aligned}$$
Infinitely many solutions: $x_1 = 2t + 3, x_2 = -3t + 5, x_3 = t$, for t any real number

59.
$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2\\ -3x_1 - 6x_2 + 3x_3 &= -3 \end{aligned}$$
$x_1 = -2s + t + 1, x_2 = s, x_3 = t$ for s and t any real numbers

60.
$$\begin{aligned} 3x_1 - 9x_2 + 12x_3 &= 6\\ -2x_1 + 6x_2 - 8x_3 &= -4 \end{aligned}$$
Infinitely many solutions: $x_1 = 3t - 4s + 2, x_2 = t, x_3 = s$ for s and t any real numbers

61.
$$\begin{aligned} 4x_1 - x_2 + 2x_3 &= 3\\ -4x_1 + x_2 - 3x_3 &= -10\\ 8x_1 - 2x_2 + 9x_3 &= -1 \end{aligned}$$
No solution

62.
$$\begin{aligned} 4x_1 - 2x_2 + 2x_3 &= 5\\ -6x_1 + 3x_2 - 3x_3 &= -2\\ 10x_1 - 5x_2 + 9x_3 &= 4 \end{aligned}$$
No solution

63. Consider a consistent system of three linear equations in three variables. Discuss the nature of the system and its solution set if the reduced form of the augmented coefficient matrix has

(A) One leftmost 1* (B) Two leftmost 1's*

(C) Three leftmost 1's* (D) Four leftmost 1's Impossible

64. Consider a system of three linear equations in three variables. Give examples of two reduced forms that are not row-equivalent if the system is

(A) Consistent and dependent* (B) Inconsistent*

C *Solve Problems 65–70 using Gauss–Jordan elimination.*

65.
$$\begin{aligned} x_1 + 2x_2 - 4x_3 - x_4 &= 7\\ 2x_1 + 5x_2 - 9x_3 - 4x_4 &= 16\\ x_1 + 5x_2 - 7x_3 - 7x_4 &= 13 \end{aligned}$$
$x_1 = 2s - 3t + 3$, $x_2 = s + 2t + 2, x_3 = s$, $x_4 = t$ for s and t any real numbers

66.
$$\begin{aligned} 2x_1 + 4x_2 + 5x_3 + 4x_4 &= 8\\ x_1 + 2x_2 + 2x_3 + x_4 &= 3 \end{aligned}$$
$x_1 = -2t + 3s - 1, x_2 = t$, $x_3 = -2s + 2, x_4 = s$ for s and t any real numbers

67.
$$\begin{aligned} x_1 - x_2 + 3x_3 - 2x_4 &= 1\\ -2x_1 + 4x_2 - 3x_3 + x_4 &= 0.5\\ 3x_1 - x_2 + 10x_3 - 4x_4 &= 2.9\\ 4x_1 - 3x_2 + 8x_3 - 2x_4 &= 0.6 \end{aligned}$$
$x_1 = -0.5, x_2 = 0.2$, $x_3 = 0.3, x_4 = -0.4$

68.
$$\begin{aligned} x_1 + x_2 + 4x_3 + x_4 &= 1.3\\ -x_1 + x_2 - x_3 &= 1.1\\ 2x_1 + x_3 + 3x_4 &= -4.4\\ 2x_1 + 5x_2 + 11x_3 + 3x_4 &= 5.6 \end{aligned}$$
$x_1 = -1.2, x_2 = 0.6$, $x_3 = 0.7, x_4 = -0.9$

69.
$$\begin{aligned} x_1 - 2x_2 + x_3 + x_4 + 2x_5 &= 2\\ -2x_1 + 4x_2 + 2x_3 + 2x_4 - 2x_5 &= 0\\ 3x_1 - 6x_2 + x_3 + x_4 + 5x_5 &= 4\\ -x_1 + 2x_2 + 3x_3 + x_4 + x_5 &= 3 \end{aligned}$$
*

70.
$$\begin{aligned} x_1 - 3x_2 + x_3 + x_4 + 2x_5 &= 2\\ -x_1 + 5x_2 + 2x_3 + 2x_4 - 2x_5 &= 0\\ 2x_1 - 6x_2 + 2x_3 + 2x_4 + 4x_5 &= 4\\ -x_1 + 3x_2 - x_3 + x_5 &= -3 \end{aligned}$$
*

71. Find a, b, and c so that the graph of the quadratic equation $y = ax^2 + bx + c$ passes through the points $(-2, 9)$, $(1, -9)$, and $(4, 9)$. $a = 2, b = -4, c = -7$

72. Find a, b, and c so that the graph of the quadratic equation $y = ax^2 + bx + c$ passes through the points $(-1, -5)$, $(2, 7)$, and $(5, 1)$. $a = -1, b = 5, c = 1$

Applications

Construct a mathematical model for each of the following problems. (The answers in the back of the book include both the mathematical model and the interpretation of its solution.) Use Gauss–Jordan elimination to solve the model and then interpret the solution.

73. Boat production. A small manufacturing plant makes three types of inflatable boats: one-person, two-person, and four-person models. Each boat requires the services of three departments, as listed in the table. The cutting, assembly, and packaging departments have available a maximum of 380, 330, and 120 labor-hours per week, respectively.

Department	One-Person Boat	Two-Person Boat	Four-Person Boat
Cutting	0.5 hr	1.0 hr	1.5 hr
Assembly	0.6 hr	0.9 hr	1.2 hr
Packaging	0.2 hr	0.3 hr	0.5 hr

(A) How many boats of each type must be produced each week for the plant to operate at full capacity?*

(B) How is the production schedule in part (A) affected if the packaging department is no longer used?*

(C) How is the production schedule in part (A) affected if the four-person boat is no longer produced?*

74. Production scheduling. Repeat Problem 73 assuming that the cutting, assembly, and packaging departments have available a maximum of 350, 330, and 115 labor-hours per week, respectively.*

75. Tank car leases. A chemical manufacturer wants to lease a fleet of 24 railroad tank cars with a combined carrying capacity of 520,000 gallons. Tank cars with three different carrying capacities are available: 8,000 gallons, 16,000 gallons, and 24,000 gallons. How many of each type of tank car should be leased?*

76. Airplane leases. A corporation wants to lease a fleet of 12 airplanes with a combined carrying capacity of 220 passengers. The three available types of planes carry 10, 15, and 20 passengers, respectively. How many of each type of plane should be leased?*

77. Tank car leases. Refer to Problem 75. The cost of leasing an 8,000-gallon tank car is \$450 per month, a 16,000-gallon tank car is \$650 per month, and a 24,000-gallon tank car is \$1,150 per month. Which of the solutions to Problem 75 would minimize the monthly leasing cost?*

78. Airplane leases. Refer to Problem 76. The cost of leasing a 10-passenger airplane is $8,000 per month, a 15-passenger airplane is $14,000 per month, and a 20-passenger airplane is $16,000 per month. Which of the solutions to Problem 76 would minimize the monthly leasing cost?*

79. Income tax. A corporation has a taxable income of $7,650,000. At this income level, the federal income tax rate is 50%, the state tax rate is 20%, and the local tax rate is 10%. If each tax rate is applied to the total taxable income, the resulting tax liability for the corporation would be 80% of taxable income. However, it is customary to deduct taxes paid to one agency before computing taxes for the other agencies. Assume that the federal taxes are based on the income that remains after the state and local taxes are deducted, and that state and local taxes are computed in a similar manner. What is the tax liability of the corporation (as a percentage of taxable income) if these deductions are taken into consideration?*

80. Income tax. Repeat Problem 79 if local taxes are not allowed as a deduction for federal and state taxes.*

81. Taxable income. As a result of several mergers and acquisitions, stock in four companies has been distributed among the companies. Each row of the following table gives the percentage of stock in the four companies that a particular company owns and the annual net income of each company (in millions of dollars):

	Percentage of Stock Owned in Company				Annual Net Income
Company	**A**	**B**	**C**	**D**	**Million $**
A	71	8	3	7	3.2
B	12	81	11	13	2.6
C	11	9	72	8	3.8
D	6	2	14	72	4.4

So company *A* holds 71% of its own stock, 8% of the stock in company *B*, 3% of the stock in company *C*, etc. For the purpose of assessing a state tax on corporate income, the taxable income of each company is defined to be its share of its own annual net income plus its share of the taxable income of each of the other companies, as determined by the percentages in the table. What is the taxable income of each company (to the nearest thousand dollars)?*

82. Taxable income. Repeat Problem 81 if tax law is changed so that the taxable income of a company is defined to be all of its own annual net income plus its share of the taxable income of each of the other companies.*

83. Nutrition. A dietitian in a hospital is to arrange a special diet composed of three basic foods. The diet is to include exactly 340 units of calcium, 180 units of iron, and 220 units of vitamin A. The number of units per ounce of each special ingredient for each of the foods is indicated in the table.

	Units per Ounce		
	Food A	**Food B**	**Food C**
Calcium	30	10	20
Iron	10	10	20
Vitamin A	10	30	20

(A) How many ounces of each food must be used to meet the diet requirements?*

(B) How is the diet in part (A) affected if food *C* is not used?*

(C) How is the diet in part (A) affected if the vitamin A requirement is dropped?*

84. Nutrition. Repeat Problem 83 if the diet is to include exactly 400 units of calcium, 160 units of iron, and 240 units of vitamin A.*

85. Plant food. A farmer can buy four types of plant food. Each barrel of mix *A* contains 30 pounds of phosphoric acid, 50 pounds of nitrogen, and 30 pounds of potash; each barrel of mix *B* contains 30 pounds of phosphoric acid, 75 pounds of nitrogen, and 20 pounds of potash; each barrel of mix *C* contains 30 pounds of phosphoric acid, 25 pounds of nitrogen, and 20 pounds of potash; and each barrel of mix *D* contains 60 pounds of phosphoric acid, 25 pounds of nitrogen, and 50 pounds of potash. Soil tests indicate that a particular field needs 900 pounds of phosphoric acid, 750 pounds of nitrogen, and 700 pounds of potash. How many barrels of each type of food should the farmer mix together to supply the necessary nutrients for the field?*

86. Animal feed. In a laboratory experiment, rats are to be fed 5 packets of food containing a total of 80 units of vitamin E. There are four different brands of food packets that can be used. A packet of brand *A* contains 5 units of vitamin E, a packet of brand *B* contains 10 units of vitamin E, a packet of brand *C* contains 15 units of vitamin E, and a packet of brand *D* contains 20 units of vitamin E. How many packets of each brand should be mixed and fed to the rats?*

87. Plant food. Refer to Problem 85. The costs of the four mixes are Mix *A*, $46; Mix *B*, $72; Mix *C*, $57; and Mix *D*, $63. Which of the solutions to Problem 85 would minimize the cost of the plant food?*

88. Animal feed. Refer to Problem 86. The costs of the four brands are Brand *A*, $1.50; Brand *B*, $3.00; Brand *C*, $3.75; and Brand *D*, $2.25. Which of the solutions to Problem 86 would minimize the cost of the rat food?*

89. Population growth. The U.S. population was approximately 75 million in 1900, 150 million in 1950, and 275 million in 2000. Construct a model for this data by finding a quadratic equation whose graph passes through the points $(0, 75)$, $(50, 150)$, and $(100, 275)$. Use this model to estimate the population in 2050. $y = 0.01x^2 + x + 75$, 450 million

90. Population growth. The population of California was approximately 24 million in 1980, 30 million in 1990, and 34 million in 2000. Construct a model for this data by finding a quadratic equation whose graph passes through the points $(0, 24)$, $(10, 30)$, and $(20, 34)$. Use this model to estimate the population in 2020. Do you think the estimate is plausible? Explain. $y = -0.01x^2 + 0.7x + 24$, 36 million

91. Female life expectancy. The life expectancy for females born during 1980–1985 was approximately 77.6 years. This grew to 78 years during 1985–1990 and to 78.6 years during 1990–1995. Construct a model for this data by finding a quadratic equation whose graph passes through the points $(0, 77.6)$, $(5, 78)$, and $(10, 78.6)$. Use this model to estimate the life expectancy for females born between 1995 and 2000 and for those born between 2000 and 2005.*

92. **Male life expectancy.** The life expectancy for males born during 1980–1985 was approximately 70.7 years. This grew to 71.1 years during 1985–1990 and to 71.8 years during 1990–1995. Construct a model for this data by finding a quadratic equation whose graph passes through the points $(0, 70.7)$, $(5, 71.1)$, and $(10, 71.8)$. Use this model to estimate the life expectancy for males born between 1995 and 2000 and for those born between 2000 and 2005.*

93. **Female life expectancy.** Refer to Problem 91. Subsequent data indicated that life expectancy grew to 79.1 years for females born during 1995–2000 and to 79.7 years for females born during 2000–2005. Add the points $(15, 79.1)$ and $(20, 79.7)$ to the data set in Problem 91. Use a graphing calculator to find a quadratic regression model for all five data points. Graph the data and the model in the same viewing window.*

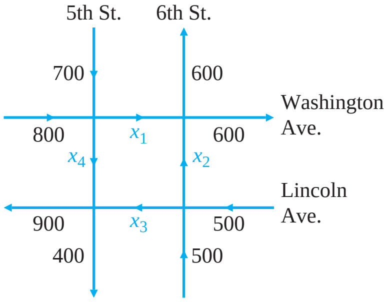
94. **Male life expectancy.** Refer to Problem 92. Subsequent data indicated that life expectancy grew to 73.2 years for males born during 1995–2000 and to 74.3 years for males born during 2000–2005. Add the points $(15, 73.2)$ and $(20, 74.3)$ to the data set in Problem 92. Use a graphing calculator to find a quadratic regression model for all five data points. Graph the data and the model in the same viewing window.*

95. **Sociology.** Two sociologists have grant money to study school busing in a particular city. They wish to conduct an opinion survey using 600 telephone contacts and 400 house contacts. Survey company *A* has personnel to do 30 telephone and 10 house contacts per hour; survey company *B* can handle 20 telephone and 20 house contacts per hour. How many hours should be scheduled for each firm to produce exactly the number of contacts needed?*

96. **Sociology.** Repeat Problem 95 if 650 telephone contacts and 350 house contacts are needed.*

97. **Traffic flow.** The rush-hour traffic flow for a network of four one-way streets in a city is shown in the figure. The numbers next to each street indicate the number of vehicles per hour that enter and leave the network on that street. The variables x_1, x_2, x_3, and x_4 represent the flow of traffic between the four intersections in the network.*

(A) For a smooth traffic flow, the number of vehicles entering each intersection should always equal the number leaving. For example, since 1,500 vehicles enter the intersection of 5th Street and Washington Avenue each hour and $x_1 + x_4$ vehicles leave this intersection, we see that $x_1 + x_4 = 1{,}500$. Find the equations determined by the traffic flow at each of the other three intersections.*

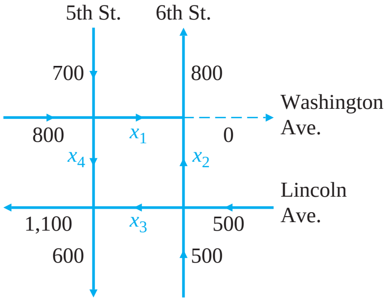


(B) Find the solution to the system in part (A).*

(C) What is the maximum number of vehicles that can travel from Washington Avenue to Lincoln Avenue on 5th Street? What is the minimum number? 1,300; 300

(D) If traffic lights are adjusted so that 1,000 vehicles per hour travel from Washington Avenue to Lincoln Avenue on 5th Street, determine the flow around the rest of the network.*

98. **Traffic flow.** Refer to Problem 97. Closing Washington Avenue east of 6th Street for construction changes the traffic flow for the network as indicated in the figure. Repeat parts (A)–(D) of Problem 97 for this traffic flow.*



Answers to Matched Problems

1. (A) Condition 2 is violated: The 3 in row 2 and column 2 should be a 1. Perform the operation $\frac{1}{3}R_2 \rightarrow R_2$ to obtain

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -2 \end{array}\right]$$

(B) Condition 3 is violated: The 5 in row 1 and column 2 should be a 0. Perform the operation $(-5)R_2 + R_1 \rightarrow R_1$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & -6 & 8 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(C) Condition 4 is violated. The leftmost 1 in the second row is not to the right of the leftmost 1 in the first row. Perform the operation $R_1 \leftrightarrow R_2$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

(D) Condition 1 is violated: The all-zero second row should be at the bottom. Perform the operation $R_2 \leftrightarrow R_3$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2. $x_1 = 1, x_2 = -1, x_3 = 0$

3. Inconsistent; no solution

4. $x_1 = 5t + 4, x_2 = 3t + 5, x_3 = t$, t any real number

5. $x_1 = s + 7, x_2 = s, x_3 = t - 2, x_4 = -3t - 1, x_5 = t$, s and t any real numbers

6. $t - 8$ cargo vans, $-2t + 24$ 15-foot trucks, and t 24-foot trucks, where $t = 8, 9, 10, 11$, or 12

4.4 Matrices: Basic Operations

- Addition and Subtraction
- Product of a Number k and a Matrix M
- Matrix Product

In the two preceding sections we introduced the important idea of matrices. In this and following sections, we develop this concept further. Matrices are both an ancient and a current mathematical concept. References to matrices and systems of equations can be found in Chinese manuscripts dating back to about 200 B.C. More recently, computers have made matrices a useful tool for a wide variety of applications. Most graphing calculators and computers are capable of performing calculations with matrices.

As we will see, matrix addition and multiplication are similar to real number addition and multiplication in many respects, but there are some important differences. A brief review of Appendix A, Section A.1, where real number operations are discussed, will help you understand the similarities and the differences.

Addition and Subtraction

Before we can discuss arithmetic operations for matrices, we have to define equality for matrices. Two matrices are **equal** if they have the same size and their corresponding elements are equal. For example,

$$\overset{2 \times 3}{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}} = \overset{2 \times 3}{\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}} \quad \text{if and only if} \quad \begin{matrix} a = u & b = v & c = w \\ d = x & e = y & f = z \end{matrix}$$

The **sum of two matrices of the same size** is the matrix with elements that are the sum of the corresponding elements of the two given matrices. Addition is not defined for matrices of different sizes.

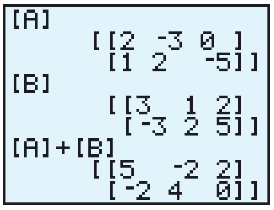
EXAMPLE 1 Matrix Addition

(A) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} (a + w) & (b + x) \\ (c + y) & (d + z) \end{bmatrix}$

(B) $\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ -3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 2 \\ -2 & 4 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 5 & 0 & -2 \\ 1 & -3 & 8 \end{bmatrix} + \begin{bmatrix} -1 & 7 \\ 0 & 6 \\ -2 & 8 \end{bmatrix}$ Not defined

Matched Problem 1 Add: $\begin{bmatrix} 3 & 2 \\ -1 & -1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$



```
[A]
     [[2 -3 0 ]
      [1 2  -5]]
[B]
      [[3  1 2]
       [-3 2 5]]
[A]+[B]
     [[5  -2 2]
      [-2 4  0]]
```

Figure 1 **Addition on a graphing calculator**

Graphing calculators can be used to solve problems involving matrix operations. Figure 1 illustrates the solution to Example 1B on a TI-84.

Because we add two matrices by adding their corresponding elements, it follows from the properties of real numbers that matrices of the same size are commutative and associative relative to addition. That is, if A, B, and C are matrices of the same size, then

Commutative: $A + B = B + A$

Associative: $(A + B) + C = A + (B + C)$

A matrix with elements that are all zeros is called a **zero matrix**. For example,

$$[0 \quad 0 \quad 0] \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are zero matrices of different sizes. [*Note:* The simpler notation “0” is often used to denote the zero matrix of an arbitrary size.] The **negative of a matrix *M***, denoted by $-M$, is a matrix with elements that are the negatives of the elements in M. Thus, if

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad -M = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

Note that $M + (-M) = 0$ (a zero matrix).

If A and B are matrices of the same size, we define **subtraction** as follows:

$$A - B = A + (-B)$$

So to subtract matrix B from matrix A, we simply add the negative of B to A.

EXAMPLE 2 Matrix Subtraction

$$\begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 2 & -4 \end{bmatrix}$$

Matched Problem 2 Subtract: $[2 \quad -3 \quad 5] - [3 \quad -2 \quad 1]$

EXAMPLE 3 Matrix Equations Find a, b, c, and d so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$

SOLUTION

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$

Subtract the matrices on the left side.

$$\begin{bmatrix} a - 2 & b - (-1) \\ c - (-5) & d - 6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$

Remove parentheses.

$$\begin{bmatrix} a - 2 & b + 1 \\ c + 5 & d - 6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$

Use the definition of *equality* to change this matrix equation into four real number equations.

$$\begin{aligned} a - 2 &= 4 & b + 1 &= 3 & c + 5 &= -2 & d - 6 &= 4 \\ a &= 6 & b &= 2 & c &= -7 & d &= 10 \end{aligned}$$

Matched Problem 3 Find a, b, c, and d so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 8 & 2 \end{bmatrix}$$

Product of a Number k and a Matrix M

The **product of a number *k* and a matrix *M***, denoted by ***kM***, is a matrix formed by multiplying each element of M by k.

EXAMPLE 4 Multiplication of a Matrix by a Number

$$-2\begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -6 & 2 & 0 \\ 4 & -2 & -6 \\ 0 & 2 & 4 \end{bmatrix}$$

Matched Problem 4 Find: $10\begin{bmatrix} 1.3 \\ 0.2 \\ 3.5 \end{bmatrix}$

The next example illustrates the use of matrix operations in an applied setting.

EXAMPLE 5 Sales Commissions Ms. Smith and Mr. Jones are salespeople in a new-car agency that sells only two models. August was the last month for this year's models, and next year's models were introduced in September. Gross dollar sales for each month are given in the following matrices:

$$\begin{array}{rcc} & \text{August sales} & & \text{September sales} \\ & \begin{array}{cc}\text{Compact} & \text{Luxury}\end{array} & & \begin{array}{cc}\text{Compact} & \text{Luxury}\end{array} \\ \begin{array}{r}\text{Ms. Smith}\\ \text{Mr. Jones}\end{array} & \left[\begin{array}{rr}\$54{,}000 & \$88{,}000\\ \$126{,}000 & 0\end{array}\right] = A & & \left[\begin{array}{rr}\$228{,}000 & \$368{,}000\\ \$304{,}000 & \$322{,}000\end{array}\right] = B \end{array}$$

For example, Ms. Smith had $54,000 in compact sales in August, and Mr. Jones had $322,000 in luxury car sales in September.

(A) What were the combined dollar sales in August and September for each salesperson and each model?

(B) What was the increase in dollar sales from August to September?

(C) If both salespeople receive 5% commissions on gross dollar sales, compute the commission for each person for each model sold in September.

SOLUTION

$$\begin{array}{rcl} & \begin{array}{cc}\text{Compact} & \text{Luxury}\end{array} & \\ \text{(A)}\quad A + B = & \left[\begin{array}{rr}\$282{,}000 & \$456{,}000\\ \$430{,}000 & \$322{,}000\end{array}\right] & \begin{array}{l}\text{Ms. Smith}\\ \text{Mr. Jones}\end{array}\end{array}$$

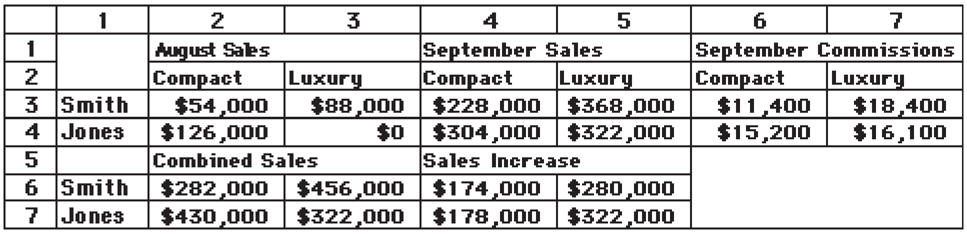
$$\begin{array}{rcl} & \begin{array}{cc}\text{Compact} & \text{Luxury}\end{array} & \\ \text{(B)}\quad B - A = & \left[\begin{array}{rr}\$174{,}000 & \$280{,}000\\ \$178{,}000 & \$322{,}000\end{array}\right] & \begin{array}{l}\text{Ms. Smith}\\ \text{Mr. Jones}\end{array}\end{array}$$

$$\begin{aligned}\text{(C)}\quad 0.05B &= \left[\begin{array}{cc}(0.05)(\$228{,}000) & (0.05)(\$368{,}000)\\ (0.05)(\$304{,}000) & (0.05)(\$322{,}000)\end{array}\right]\\ &= \left[\begin{array}{rr}\$11{,}400 & \$18{,}400\\ \$15{,}200 & \$16{,}100\end{array}\right]\begin{array}{l}\text{Ms. Smith}\\ \text{Mr. Jones}\end{array}\end{aligned}$$

Matched Problem 5 Repeat Example 5 with

$$A = \left[\begin{array}{rr}\$45{,}000 & \$77{,}000\\ \$106{,}000 & \$22{,}000\end{array}\right] \quad\text{and}\quad B = \left[\begin{array}{rr}\$190{,}000 & \$345{,}000\\ \$266{,}000 & \$276{,}000\end{array}\right]$$

Figure 2 illustrates a solution for Example 5 on a spreadsheet.



	1	2	3	4	5	6	7
1		August Sales		September Sales		September Commissions	
2		Compact	Luxury	Compact	Luxury	Compact	Luxury
3	Smith	$54,000	$88,000	$228,000	$368,000	$11,400	$18,400
4	Jones	$126,000	$0	$304,000	$322,000	$15,200	$16,100
5		Combined Sales		Sales Increase			
6	Smith	$282,000	$456,000	$174,000	$280,000		
7	Jones	$430,000	$322,000	$178,000	$322,000		

Figure 2

Matrix Product

Matrix multiplication was introduced by the English mathematician Arthur Cayley (1821–1895) in studies of systems of linear equations and linear transformations. Although this multiplication may seem strange at first, it is extremely useful in many practical problems.

We start by defining the product of two special matrices, a row matrix and a column matrix.

DEFINITION Product of a Row Matrix and a Column Matrix

The **product** of a $1 \times n$ row matrix and an $n \times 1$ column matrix is a 1×1 matrix given by

$$\overset{1 \times n}{[a_1\ a_2\ \dots\ a_n]}\ \overset{n \times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}} = [a_1b_1 + a_2b_2 + \cdots + a_nb_n]$$

Note that the number of elements in the row matrix and in the column matrix must be the same for the product to be defined.

EXAMPLE 6 Product of a Row Matrix and a Column Matrix

$$[2 \quad -3 \quad 0]\begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix} = [(2)(-5) + (-3)(2) + (0)(-2)]$$

$$= [-10 - 6 + 0] = [-16]$$

Matched Problem 6 $[-1 \quad 0 \quad 3 \quad 2]\begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix} = ?$

Refer to Example 6. The distinction between the real number -16 and the 1×1 matrix $[-16]$ is a technical one, and it is common to see 1×1 matrices written as real numbers without brackets. In the work that follows, we will frequently refer to 1×1 matrices as real numbers and omit the brackets whenever it is convenient to do so.

EXAMPLE 7 Labor Costs A factory produces a slalom water ski that requires 3 labor-hours in the assembly department and 1 labor-hour in the finishing department. Assembly personnel receive \$9 per hour and finishing personnel receive \$6 per hour. Total labor cost per ski is given by the product:

$$[3 \quad 1]\begin{bmatrix} 9 \\ 6 \end{bmatrix} = [(3)(9) + (1)(6)] = [27 + 6] = [33] \quad \text{or} \quad \$33 \text{ per ski}$$

Matched Problem 7 If the factory in Example 7 also produces a trick water ski that requires 5 labor-hours in the assembly department and 1.5 labor-hours in the finishing department, write a product between appropriate row and column matrices that will give the total labor cost for this ski. Compute the cost.

We now use the product of a $1 \times n$ row matrix and an $n \times 1$ column matrix to extend the definition of matrix product to more general matrices.

DEFINITION Matrix Product

If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then the **matrix product** of A and B, denoted AB, is an $m \times n$ matrix whose element in the ith row and jth column is the real number obtained from the product of the ith row of A and the jth column of B. If the number of columns in A does not equal the number of rows in B, the matrix product AB is **not defined**.

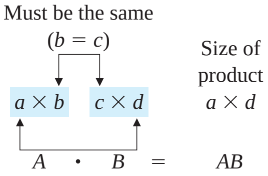


Figure 3

It is important to check sizes before starting the multiplication process. If A is an $a \times b$ matrix and B is a $c \times d$ matrix, then if $b = c$, the product AB will exist and will be an $a \times d$ matrix (see Fig. 3). If $b \neq c$, the product AB does not exist. The definition is not as complicated as it might first seem. An example should help clarify the process.

For

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

A is 2×3 and B is 3×2, so AB is 2×2. To find the first row of AB, we take the product of the first row of A with every column of B and write each result as a real number, not as a 1×1 matrix. The second row of AB is computed in the same manner. The four products of row and column matrices used to produce the four elements in AB are shown in the following dashed box. These products are usually calculated mentally or with the aid of a calculator, and need not be written out. The shaded portions highlight the steps involved in computing the element in the first row and second column of AB.

$$\overset{2\times 3}{\begin{bmatrix} 2 & 3 & -1 \\ -2 & 1 & 2 \end{bmatrix}} \overset{3\times 2}{\begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}} = \begin{bmatrix} [2 \quad 3 \quad -1]\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} & [2 \quad 3 \quad -1]\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \\ [-2 \quad 1 \quad 2]\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} & [-2 \quad 1 \quad 2]\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (2)(1) + (3)(2) + (-1)(-1) & (2)(3) + (3)(0) + (-1)(2) \\ (-2)(1) + (1)(2) + (2)(-1) & (-2)(3) + (1)(0) + (2)(2) \end{bmatrix} = \overset{2\times 2}{\begin{bmatrix} 9 & 4 \\ -2 & -2 \end{bmatrix}}$$

EXAMPLE 8 Matrix Multiplication Find the indicated matrix product, if it exists, where:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad E = \begin{bmatrix} 2 & -3 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix}$$

(A) $$AB = \overset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}} \overset{2\times 4}{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} (2)(1)+(1)(2) & (2)(-1)+(1)(1) & (2)(0)+(1)(2) & (2)(1)+(1)(0) \\ (1)(1)+(0)(2) & (1)(-1)+(0)(1) & (1)(0)+(0)(2) & (1)(1)+(0)(0) \\ (-1)(1)+(2)(2) & (-1)(-1)+(2)(1) & (-1)(0)+(2)(2) & (-1)(1)+(2)(0) \end{bmatrix}$$

$$= \overset{3\times 4}{\begin{bmatrix} 4 & -1 & 2 & 2 \\ 1 & -1 & 0 & 1 \\ 3 & 3 & 4 & -1 \end{bmatrix}}$$

(B) $$BA = \overset{2 \times 4}{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}} \overset{3 \times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}} \quad \text{Not defined}$$

(C) $$CD = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} (2)(1)+(6)(3) & (2)(2)+(6)(6) \\ (-1)(1)+(-3)(3) & (-1)(2)+(-3)(6) \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 40 \\ -10 & -20 \end{bmatrix}$$

(D) $$DC = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} (1)(2)+(2)(-1) & (1)(6)+(2)(-3) \\ (3)(2)+(6)(-1) & (3)(6)+(6)(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(E) $$EF = \begin{bmatrix} 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix} = [(2)(-5)+(-3)(2)+(0)(-2)] = [-16]$$

(F) $$FE = \begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} (-5)(2) & (-5)(-3) & (-5)(0) \\ (2)(2) & (2)(-3) & (2)(0) \\ (-2)(2) & (-2)(-3) & (-2)(0) \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 15 & 0 \\ 4 & -6 & 0 \\ -4 & 6 & 0 \end{bmatrix}$$

(G) $$A^{2*} = AA = \overset{3 \times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}} \overset{3 \times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}} \quad \text{Not defined}$$

(H) $$C^2 = CC = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(2)+(6)(-1) & (2)(6)+(6)(-3) \\ (-1)(2)+(-3)(-1) & (-1)(6)+(-3)(-3) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix}$$

Matched Problem 8 Find each product, if it is defined:

(A) $$\begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$$

(B) $$\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$$

(C) $$\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$$

(D) $$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

(E) $$\begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

(F) $$\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$$

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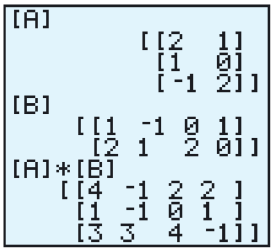
*Following standard algebraic notation, we write $A^2 = AA$, $A^3 = AAA$, and so on.



```
[A]
          [[2  1]
            [1  0]
            [-1 2]]
[B]
     [[1  -1 0 1]
       [2 1  2 0]]
[A]*[B]
    [[4  -1 2 2 ]
      [1  -1 0 1 ]
      [3 3  4  -1]]
```

Figure 4 **Multiplication on a graphing calculator**

Figure 4 illustrates a graphing calculator solution to Example 8A. What would you expect to happen if you tried to solve Example 8B on a graphing calculator?

CONCEPTUAL INSIGHT

In the arithmetic of real numbers, it does not matter in which order we multiply. For example, $5 \times 7 = 7 \times 5$. In matrix multiplication, however, it does make a difference. That is, AB does not always equal BA, even if both multiplications are defined and both products are the same size (see Examples 8C and 8D).

Matrix multiplication is not commutative.

The zero property of real numbers states that if the product of two real numbers is 0, then one of the numbers must be 0 (see Appendix A, Section A.1). This property is very important when solving equations. For example,

$$\begin{aligned} x^2 - 4x + 3 &= 0 \\ (x-1)(x-3) &= 0 \\ x - 1 = 0 \quad &\text{or} \quad x - 3 = 0 \\ x = 1 \quad & \quad\quad\;\; x = 3 \end{aligned}$$

For matrices, it is possible to find nonzero matrices A and B such that AB is a zero matrix (see Example 8D).

The zero property does not hold for matrix multiplication.

Explore and Discuss 1

In addition to the commutative and zero properties, there are other significant differences between real number multiplication and matrix multiplication.

(A) In real number multiplication, the only real number whose square is 0 is the real number 0 $(0^2 = 0)$. Find at least one 2×2 matrix A with all elements nonzero such that $A^2 = 0$, where 0 is the 2×2 zero matrix.

(B) In real number multiplication, the only nonzero real number that is equal to its square is the real number 1 $(1^2 = 1)$. Find at least one 2×2 matrix B with all elements nonzero such that $B^2 = B$.

(A) One answer is

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

(B) One answer is

$$B = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$$

EXAMPLE 9 Matrix Multiplication

Find a, b, c, and d so that

$$\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -6 & 17 \\ 7 & 4 \end{bmatrix}$$

SOLUTION The product of the matrices on the left side of the equation is

$$\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a - c & 2b - d \\ 5a + 3c & 5b + 3d \end{bmatrix}$$

Therefore,

$$\begin{aligned} 2a - c &= -6 & \qquad 2b - d &= 17 \\ 5a + 3c &= 7 & \qquad 5b + 3d &= 4 \end{aligned}$$

This gives a system of two equations in the variables a and c, and a second system of two equations in the variables b and d. Each system can be solved by substitution, or elimination by addition, or Gauss–Jordan elimination (the details are omitted). The solution of the first system is $a = -1$, $c = 4$, and the solution of the second system is $b = 5$, $d = -7$.

Matched Problem 9 Find a, b, c, and d so that

$$\begin{bmatrix} 6 & -5 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -16 & 64 \\ 24 & -6 \end{bmatrix}$$

Now we consider an application of matrix multiplication.

EXAMPLE 10 **Labor Costs** We can combine the time requirements for slalom and trick water skis discussed in Example 7 and Matched Problem 7 into one matrix:

$$\begin{array}{l|cc|} & \multicolumn{2}{c}{\text{Labor-hours per ski}} \\ & \text{Assembly department} & \text{Finishing department} \\ \text{Trick ski} & 5\text{ hr} & 1.5\text{ hr} \\ \text{Slalom ski} & 3\text{ hr} & 1\ \ \text{hr} \end{array} = L$$

Now suppose that the company has two manufacturing plants, one in California and the other in Maryland, and that their hourly rates for each department are given in the following matrix:

$$\begin{array}{l|cc|} & \multicolumn{2}{c}{\text{Hourly wages}} \\ & \text{California} & \text{Maryland} \\ \text{Assembly department} & \$12 & \$13 \\ \text{Finishing department} & \$7 & \$8 \end{array} = H$$

Since H and L are both 2×2 matrices, we can take the product of H and L in either order and the result will be a 2×2 matrix:

$$HL = \begin{bmatrix} 12 & 13 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 5 & 1.5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 99 & 31 \\ 59 & 18.5 \end{bmatrix}$$

$$LH = \begin{bmatrix} 5 & 1.5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 12 & 13 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 70.5 & 77 \\ 43 & 47 \end{bmatrix}$$

How can we interpret the elements in these products? Let's begin with the product HL. The element 99 in the first row and first column of HL is the product of the first row matrix of H and the first column matrix of L:

$$\begin{array}{c} \begin{array}{cc} \text{CA} & \text{MD} \end{array} \\ [12 \quad 13] \end{array}\begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{array}{l} \text{Trick} \\ \text{Slalom} \end{array} = 12(5) + 13(3) = 60 + 39 = 99$$

Notice that \$60 is the labor cost for assembling a trick ski at the California plant and \$39 is the labor cost for assembling a slalom ski at the Maryland plant. Although both numbers represent labor costs, it makes no sense to add them together. They do not pertain to the same type of ski or to the same plant. So even though the product HL happens to be defined mathematically, it has no useful interpretation in this problem.

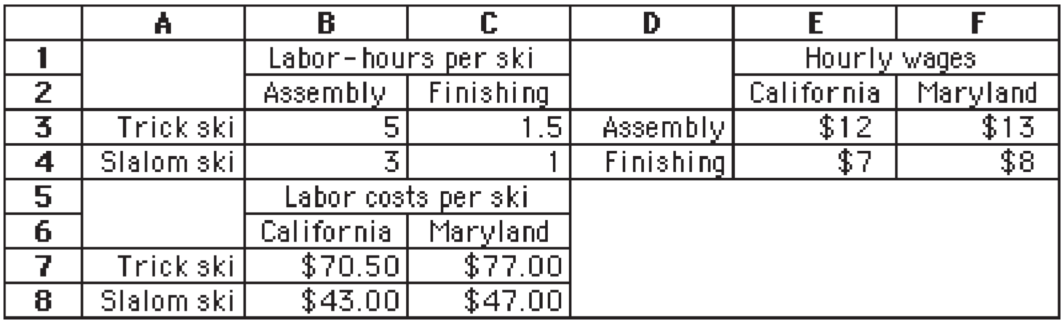
Now let's consider the product LH. The element 70.5 in the first row and first column of LH is given by the following product:

$$\begin{array}{c} \begin{array}{cc} \text{Assembly} & \text{Finishing} \end{array} \\ [5 \qquad\qquad 1.5] \end{array}\begin{bmatrix} 12 \\ 7 \end{bmatrix}\begin{array}{l} \text{Assembly} \\ \text{Finishing} \end{array} = 5(12) + 1.5(7) = 60 + 10.5 = 70.5$$

This time, \$60 is the labor cost for assembling a trick ski at the California plant and \$10.50 is the labor cost for finishing a trick ski at the California plant. So the sum is the total labor cost for producing a trick ski at the California plant. The other elements in LH also represent total labor costs, as indicated by the row and column labels shown below:

$$\begin{array}{c} \text{Labor costs per ski} \\ \begin{array}{cc} \text{CA} \qquad & \qquad \text{MD} \end{array} \\ LH = \left[\begin{array}{cc} \$70.50 & \$77 \\ \$43 & \$47 \end{array}\right] \begin{array}{l} \text{Trick} \\ \text{Slalom} \end{array} \end{array}$$

Figure 5 shows a solution to Example 9 on a spreadsheet.



	A	B	C	D	E	F
1		Labor-hours per ski			Hourly wages	
2		Assembly	Finishing		California	Maryland
3	Trick ski	5	1.5	Assembly	$12	$13
4	Slalom ski	3	1	Finishing	$7	$8
5		Labor costs per ski				
6		California	Maryland			
7	Trick ski	$70.50	$77.00			
8	Slalom ski	$43.00	$47.00			

Figure 5 **Matrix multiplication in a spreadsheet: The command MMULT(B3:C4, E3:F4) produces the matrix in B7:C8**

Matched Problem 10 Refer to Example 10. The company wants to know how many hours to schedule in each department in order to produce 2,000 trick skis and 1,000 slalom skis. These production requirements can be represented by either of the following matrices:

$$P = \begin{array}{c} \begin{array}{cc} \text{Trick skis} & \text{Slalom skis} \end{array} \\ [2{,}000 \quad 1{,}000] \end{array} \qquad Q = \left[\begin{array}{c} 2{,}000 \\ 1{,}000 \end{array}\right] \begin{array}{l} \text{Trick skis} \\ \text{Slalom skis} \end{array}$$

Using the labor-hour matrix L from Example 10, find PL or LQ, whichever has a meaningful interpretation for this problem, and label the rows and columns accordingly.

CONCEPTUAL INSIGHT

Example 10 and Matched Problem 10 illustrate an important point about matrix multiplication. Even if you are using a graphing calculator to perform the calculations in a matrix product, it is still necessary for you to know the definition of matrix multiplication so that you can interpret the results correctly.

Exercises 4.4

Skills Warm-up Exercises

W *In Problems 1–14, perform the indicated operation, if possible. (If necessary, review the definitions at the beginning of Section 4.4.)*

1. $[1 \quad 5] + [3 \quad 10]$*

2. $\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ -4 \end{bmatrix}$ $\begin{bmatrix} -4 \\ 6 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 0 \\ -3 & 6 \end{bmatrix} - \begin{bmatrix} 0 & -4 \\ -1 & 0 \end{bmatrix}$*

4. $\begin{bmatrix} -9 & 2 \\ 8 & 0 \end{bmatrix} + \begin{bmatrix} 9 & 0 \\ 0 & 8 \end{bmatrix}$*

5. $\begin{bmatrix} 3 \\ 6 \end{bmatrix} + [-1 \quad 9]$ Not defined

6. $4\begin{bmatrix} 2 & -6 & 1 \\ 8 & 5 & -3 \end{bmatrix}$*

7. $7[3 \quad -5 \quad 9 \quad 4]$*

8. $[10 \quad 12] + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$*

A **9.** $\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$

10. $\begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ $\begin{bmatrix} -6 \\ 14 \end{bmatrix}$

*Answer located in Additional Instructor's Answers section.

11. $\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$*

12. $\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix}$*

13. $\begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$*

14. $\begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}$*

In Problems 15–22, find the matrix product. Note that each product can be found mentally, without the use of a calculator or pencil-and-paper calculations.

15. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$*

16. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$*

17. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$*

18. $\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$*

19. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$*

20. $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$*

21. $\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$*

22. $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$*

B *Find the products in Problems 23–30.*

23. $\begin{bmatrix} 5 & -2 \end{bmatrix}\begin{bmatrix} -3 \\ -4 \end{bmatrix}$ $[-7]$

24. $\begin{bmatrix} -4 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $[11]$

25. $\begin{bmatrix} -3 \\ -4 \end{bmatrix}\begin{bmatrix} 5 & -2 \end{bmatrix}$*

26. $\begin{bmatrix} -2 \\ 1 \end{bmatrix}\begin{bmatrix} -4 & 3 \end{bmatrix}$*

27. $\begin{bmatrix} 3 & -2 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ $[11]$

28. $\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ $[6]$

29. $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}\begin{bmatrix} 3 & -2 & -4 \end{bmatrix}$*

30. $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$

Problems 31–48 refer to the following matrices:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 1 \\ 2 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 & 2 \\ 4 & -3 & 1 \\ -2 & 3 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & -2 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}$$

Perform the indicated operations, if possible.

31. AC*

32. CA Not defined

33. AB Not defined

34. BA*

35. B^2*

36. C^2*

37. $B + AD$*

38. $C + DA$*

39. $(0.1)DB$

40. $(0.2)CD$*

41. $(3)BA + (4)AC$*

42. $(2)DB + (5)CD$*

43. $(-2)BA + (6)CD$

44. $(-1)AC + (3)DB$

45. ACD*

46. CDA*

47. DBA*

48. BAD*

49. If a and b are nonzero real numbers,*

$$A = \begin{bmatrix} a & a \\ b & b \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} a & a \\ -a & -a \end{bmatrix}$$

find AB and BA.

43. Not defined **44.** Not defined

50. If a and b are nonzero real numbers,*

$$A = \begin{bmatrix} a & b \\ -a & -b \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

find AB and BA.

51. If a and b are nonzero real numbers and*

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

find A^2.

52. If a and b are nonzero real numbers and*

$$A = \begin{bmatrix} ab & b - ab^2 \\ a & 1 - ab \end{bmatrix}$$

find A^2.

In Problems 53 and 54, use a graphing calculator to calculate $B, B^2, B^3, \ldots$ and $AB, AB^2, AB^3, \ldots$. Describe any patterns you observe in each sequence of matrices.

53. $A = \begin{bmatrix} 0.3 & 0.7 \end{bmatrix}$ and $B = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix}$*

54. $A = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$ and $B = \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix}$*

C **55.** Find a, b, c, and d so that*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

56. Find w, x, y, and z so that*

$$\begin{bmatrix} 4 & -2 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix}$$

57. Find a, b, c, and d so that*

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

58. Find a, b, c, and d so that

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 7 & -7 \end{bmatrix}$$

$a = 3, b = 1, c = 1, d = -2$

In Problems 59–62, determine whether the statement is true or false.

59. There exist two 1×1 matrices A and B such that $AB \neq BA$. False

60. There exist two 2×2 matrices A and B such that $AB \neq BA$. True

61. There exist two nonzero 2×2 matrices A and B such that AB is the 2×2 zero matrix. True

62. There exist two nonzero 1×1 matrices A and B such that AB is the 1×1 zero matrix. False

63. A square matrix is a **diagonal matrix** if all elements not on the principal diagonal are zero. So a 2×2 diagonal matrix has the form

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

where a and d are real numbers. Discuss the validity of each of the following statements. If the statement is always true, explain why. If not, give examples.

(A) If A and B are 2×2 diagonal matrices, then $A + B$ is a 2×2 diagonal matrix. True

(B) If A and B are 2×2 diagonal matrices, then $A + B = B + A$. True

(C) If A and B are 2×2 diagonal matrices, then AB is a 2×2 diagonal matrix. True

(D) If A and B are 2×2 diagonal matrices, then $AB = BA$. True

64. A square matrix is an **upper triangular matrix** if all elements below the principal diagonal are zero. So a 2×2 upper triangular matrix has the form

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

where a, b, and d are real numbers. Discuss the validity of each of the following statements. If the statement is always true, explain why. If not, give examples.

(A) If A and B are 2×2 upper triangular matrices, then $A + B$ is a 2×2 upper triangular matrix. True

(B) If A and B are 2×2 upper triangular matrices, then $A + B = B + A$. True

(C) If A and B are 2×2 upper triangular matrices, then AB is a 2×2 upper triangular matrix. True

(D) If A and B are 2×2 upper triangular matrices, then $AB = BA$. False

Applications

65. Cost analysis. A company with two different plants manufactures guitars and banjos. Its production costs for each instrument are given in the following matrices:*

$$\begin{array}{rc} & \text{Plant X} \\ & \begin{array}{cc}\text{Guitar} & \text{Banjo}\end{array} \\ \begin{array}{r}\text{Materials}\\ \text{Labor}\end{array} & \begin{bmatrix} \$47 & \$39 \\ \$90 & \$125 \end{bmatrix} \end{array} = A \qquad \begin{array}{c} \text{Plant Y} \\ \begin{array}{cc}\text{Guitar} & \text{Banjo}\end{array} \\ \begin{bmatrix} \$56 & \$42 \\ \$84 & \$115 \end{bmatrix} \end{array} = B$$

Find $\frac{1}{2}(A + B)$, the average cost of production for the two plants.

66. Cost analysis. If both labor and materials at plant X in Problem 65 are increased by 20%, find $\frac{1}{2}(1.2A + B)$, the new average cost of production for the two plants.*

67. Markup. An import car dealer sells three models of a car. The retail prices and the current dealer invoice prices (costs) for the basic models and options indicated are given in the following two matrices (where "Air" means air-conditioning):*

$$\begin{array}{rc} & \text{Retail price} \\ & \begin{array}{cccc}\text{Basic Car} & \text{Air} & \text{AM/FM radio} & \text{Cruise control}\end{array} \\ \begin{array}{r}\text{Model A}\\ \text{Model B}\\ \text{Model C}\end{array} & \begin{bmatrix} \$35{,}075 & \$2{,}560 & \$1{,}070 & \$640 \\ \$39{,}045 & \$1{,}840 & \$770 & \$460 \\ \$45{,}535 & \$3{,}400 & \$1{,}415 & \$850 \end{bmatrix} \end{array} = M$$

$$\begin{array}{rc} & \text{Dealer invoice price} \\ & \begin{array}{cccc}\text{Basic Car} & \text{Air} & \text{AM/FM radio} & \text{Cruise control}\end{array} \\ \begin{array}{r}\text{Model A}\\ \text{Model B}\\ \text{Model C}\end{array} & \begin{bmatrix} \$30{,}996 & \$2{,}050 & \$850 & \$510 \\ \$34{,}857 & \$1{,}585 & \$660 & \$395 \\ \$41{,}667 & \$2{,}890 & \$1{,}200 & \$725 \end{bmatrix} \end{array} = N$$

We define the markup matrix to be $M - N$ (**markup** is the difference between the retail price and the dealer invoice price). Suppose that the value of the dollar has had a sharp decline and the dealer invoice price is to have an across-the-board 15% increase next year. To stay competitive with domestic cars, the dealer increases the retail prices 10%. Calculate a markup matrix for next year's models and the options indicated. (Compute results to the nearest dollar.)*

68. Markup. Referring to Problem 67, what is the markup matrix resulting from a 20% increase in dealer invoice prices and an increase in retail prices of 15%? (Compute results to the nearest dollar.) *

69. Labor costs. A company with manufacturing plants located in Massachusetts (MA) and Virginia (VA) has labor-hour and wage requirements for the manufacture of three types of inflatable boats as given in the following two matrices:*

$$M = \begin{array}{cl} \begin{array}{ccc} & \text{Labor-hours per boat} & \\ \text{Cutting department} & \text{Assembly department} & \text{Packaging department}\end{array} & \\ \begin{bmatrix} 0.6\text{ hr} & 0.6\text{ hr} & 0.2\text{ hr} \\ 1.0\text{ hr} & 0.9\text{ hr} & 0.3\text{ hr} \\ 1.5\text{ hr} & 1.2\text{ hr} & 0.4\text{ hr} \end{bmatrix} & \begin{array}{l}\text{One-person boat}\\ \text{Two-person boat}\\ \text{Four-person boat}\end{array} \end{array}$$

$$N = \begin{array}{cl} \text{Hourly wages} & \\ \begin{array}{cc}\text{MA} & \text{VA}\end{array} & \\ \begin{bmatrix} \$17.30 & \$14.65 \\ \$12.22 & \$10.29 \\ \$10.63 & \$9.66 \end{bmatrix} & \begin{array}{l}\text{Cutting department}\\ \text{Assembly department}\\ \text{Packaging department}\end{array} \end{array}$$

(A) Find the labor costs for a one-person boat manufactured at the Massachusetts plant. \$19.84

(B) Find the labor costs for a four-person boat manufactured at the Virginia plant. \$38.19

(C) Discuss possible interpretations of the elements in the matrix products MN and NM. MN gives the labor costs at each plant.

(D) If either of the products MN or NM has a meaningful interpretation, find the product and label its rows and columns.*

70. Inventory value. A personal computer retail company sells five different computer models through three stores. The inventory of each model on hand in each store is summarized in matrix M. Wholesale (W) and retail (R) values of each model computer are summarized in matrix N.*

$$M = \begin{array}{cl} \text{Model} & \\ \begin{array}{ccccc}\text{A} & \text{B} & \text{C} & \text{D} & \text{E}\end{array} & \\ \begin{bmatrix} 4 & 2 & 3 & 7 & 1 \\ 2 & 3 & 5 & 0 & 6 \\ 10 & 4 & 3 & 4 & 3 \end{bmatrix} & \begin{array}{l}\text{Store 1}\\ \text{Store 2}\\ \text{Store 3}\end{array} \end{array}$$

$$N = \begin{array}{cll} \begin{array}{cc}\text{W} & \text{R}\end{array} & & \\ \begin{bmatrix} \$700 & \$840 \\ \$1{,}400 & \$1{,}800 \\ \$1{,}800 & \$2{,}400 \\ \$2{,}700 & \$3{,}300 \\ \$3{,}500 & \$4{,}900 \end{bmatrix} & \begin{array}{l}\text{A}\\ \text{B}\\ \text{C}\\ \text{D}\\ \text{E}\end{array} & \text{Model} \end{array}$$

(A) What is the retail value of the inventory at store 2? $48,480

(B) What is the wholesale value of the inventory at store 3? $39.300

(C) Discuss possible interpretations of the elements in the matrix products *MN* and *NM*. *MN* gives the wholesale and retail values of the inventories at each store.

(D) If either product *MN* or *NM* has a meaningful interpretation, find the product and label its rows and columns.

(E) Discuss methods of matrix multiplication that can be used to find the total inventory of each model on hand at all three stores. State the matrices that can be used and perform the necessary operations.

(F) Discuss methods of matrix multiplication that can be used to find the total inventory of all five models at each store. State the matrices that can be used and perform the necessary operations.

71. **Cereal.** A nutritionist for a cereal company blends two cereals in three different mixes. The amounts of protein, carbohydrate, and fat (in grams per ounce) in each cereal are given by matrix M. The amounts of each cereal used in the three mixes are given by matrix N.

$$M = \begin{array}{c} \begin{array}{cc} \text{Cereal A} & \text{Cereal B} \end{array} \\ \left[\begin{array}{cc} 4\text{ g/oz} & 2\text{ g/oz} \\ 20\text{ g/oz} & 16\text{ g/oz} \\ 3\text{ g/oz} & 1\text{ g/oz} \end{array}\right] \end{array} \begin{array}{l} \\ \text{Protein} \\ \text{Carbohydrate} \\ \text{Fat} \end{array}$$

$$N = \begin{array}{c} \begin{array}{ccc} \text{Mix X} & \text{Mix Y} & \text{Mix Z} \end{array} \\ \left[\begin{array}{ccc} 15\text{ oz} & 10\text{ oz} & 5\text{ oz} \\ 5\text{ oz} & 10\text{ oz} & 15\text{ oz} \end{array}\right] \end{array} \begin{array}{l} \\ \text{Cereal A} \\ \text{Cereal B} \end{array}$$

(A) Find the amount of protein in mix X. 70 g

(B) Find the amount of fat in mix Z. 30 g

(C) Discuss possible interpretations of the elements in the matrix products *MN* and *NM*. *MN* gives the amount (in grams) of protein, carbohydrate, and fat in 20 oz of each mix.

(D) If either of the products *MN* or *NM* has a meaningful interpretation, find the product and label its rows and columns.

72. **Heredity.** Gregor Mendel (1822–1884) made discoveries that revolutionized the science of genetics. In one experiment, he crossed dihybrid yellow round peas (yellow and round are dominant characteristics; the peas also contained genes for the recessive characteristics green and wrinkled) and obtained peas of the types indicated in the matrix:

$$\begin{array}{l} \\ \text{Yellow} \\ \text{Green} \end{array} \begin{array}{c} \begin{array}{cc} \text{Round} & \text{Wrinkled} \end{array} \\ \left[\begin{array}{cc} 315 & 101 \\ 108 & 32 \end{array}\right] \end{array} = M$$

Suppose he carried out a second experiment of the same type and obtained peas of the types indicated in this matrix:

$$\begin{array}{l} \\ \text{Yellow} \\ \text{Green} \end{array} \begin{array}{c} \begin{array}{cc} \text{Round} & \text{Wrinkled} \end{array} \\ \left[\begin{array}{cc} 370 & 128 \\ 110 & 36 \end{array}\right] \end{array} = N$$

If the results of the two experiments are combined, discuss matrix multiplication methods that can be used to find the following quantities. State the matrices that can be used and perform the necessary operations.

(A) The total number of peas in each category *

(B) The total number of peas in all four categories*

(C) The percentage of peas in each category *

73. **Politics.** In a local California election, a public relations firm promoted its candidate in three ways: telephone calls, house calls, and letters. The cost per contact is given in matrix M, and the number of contacts of each type made in two adjacent cities is given in matrix N. *

$$M = \begin{array}{c} \text{Cost per contact} \\ \left[\begin{array}{c} \$1.20 \\ \$3.00 \\ \$1.45 \end{array}\right] \end{array} \begin{array}{l} \\ \text{Telephone call} \\ \text{House call} \\ \text{Letter} \end{array}$$

$$N = \begin{array}{c} \begin{array}{ccc} \text{Telephone call} & \text{House call} & \text{Letter} \end{array} \\ \left[\begin{array}{ccc} 1{,}000 & 500 & 5{,}000 \\ 2{,}000 & 800 & 8{,}000 \end{array}\right] \end{array} \begin{array}{l} \\ \text{Berkeley} \\ \text{Oakland} \end{array}$$

(A) Find the total amount spent in Berkeley. $9,950

(B) Find the total amount spent in Oakland. $16,400

(C) Discuss possible interpretations of the elements in the matrix products *MN* and *NM*. *NM* gives the total cost per town.

(D) If either product *MN* or *NM* has a meaningful interpretation, find the product and label its rows and columns. *

(E) Discuss methods of matrix multiplication that can be used to find the total number of telephone calls, house calls, and letters. State the matrices that can be used and perform the necessary operations. *

(F) Discuss methods of matrix multiplication that can be used to find the total number of contacts in Berkeley and in Oakland. State the matrices that can be used and perform the necessary operations. *

74. **Test averages.** A teacher has given four tests to a class of five students and stored the results in the following matrix:

$$\begin{array}{l} \\ \\ \text{Ann} \\ \text{Bob} \\ \text{Carol} \\ \text{Dan} \\ \text{Eric} \end{array} \begin{array}{c} \text{Tests} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 78 & 84 & 81 & 86 \\ 91 & 65 & 84 & 92 \\ 95 & 90 & 92 & 91 \\ 75 & 82 & 87 & 91 \\ 83 & 88 & 81 & 76 \end{array}\right] \end{array} = M$$

Discuss methods of matrix multiplication that the teacher can use to obtain the information indicated below. In each case, state the matrices to be used and then perform the necessary operations.

(A) The average on all four tests for each student, assuming that all four tests are given equal weight *

(B) The average on all four tests for each student, assuming that the first three tests are given equal weight and the fourth is given twice this weight *

(C) The class average on each of the four tests *

Answers to Matched Problems

1. $\begin{bmatrix} 1 & 5 \\ 0 & -2 \\ 2 & 1 \end{bmatrix}$

2. $[-1 \quad -1 \quad 4]$

3. $a = -6$
$b = 7$
$c = 9$
$d = -1$

4. $\begin{bmatrix} 13 \\ 2 \\ 35 \end{bmatrix}$

5. (A) $\begin{bmatrix} \$235{,}000 & \$422{,}000 \\ \$372{,}000 & \$298{,}000 \end{bmatrix}$

(B) $\begin{bmatrix} \$145{,}000 & \$268{,}000 \\ \$160{,}000 & \$254{,}000 \end{bmatrix}$

(C) $\begin{bmatrix} \$9{,}500 & \$17{,}250 \\ \$13{,}300 & \$13{,}800 \end{bmatrix}$

6. $[8]$

7. $[5 \quad 1.5]\begin{bmatrix} 9 \\ 6 \end{bmatrix} = [54]$, or \$54

8. (A) Not defined

(B) $\begin{bmatrix} 2 & 2 & -1 & 2 \\ 1 & 6 & 12 & -4 \\ -1 & 0 & 3 & -2 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} -6 & -12 \\ 3 & 6 \end{bmatrix}$

(E) $[11]$

(F) $\begin{bmatrix} 12 & -8 & 4 \\ 6 & -4 & 2 \\ 9 & -6 & 3 \end{bmatrix}$

9. $a = 4, c = 8, b = 9, d = -2$

10. $PL = \begin{array}{c} \begin{array}{cc} \text{Assembly} & \text{Finishing} \\ [13{,}000 & 4{,}000] \end{array} \end{array}$ Labor-hours

4.5 Inverse of a Square Matrix

- Identity Matrix for Multiplication
- Inverse of a Square Matrix
- Application: Cryptography

Identity Matrix for Multiplication

Does the set of all matrices of a given size have an identity element for multiplication? That is, if M is an arbitrary $m \times n$ matrix, does there exist an identity element I such that $IM = MI = M$? The answer, in general, is no. However, the set of all **square matrices of order *n*** (matrices with n rows and n columns) does have an identity element.

DEFINITION Identity Matrix

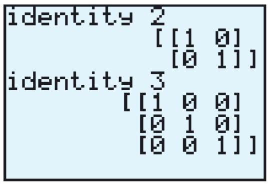
The **identity element for multiplication** for the set of all square matrices of order n is the square matrix of order n, denoted by I, with 1's along the principal diagonal (from the upper left corner to the lower right) and 0's elsewhere.

For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the identity matrices for all square matrices of order 2 and 3, respectively.

Most graphing calculators have a built-in command for generating the identity matrix of a given order (see Fig. 1).



```
identity 2
         [[1 0]
          [0 1]]
identity 3
       [[1 0 0]
        [0 1 0]
        [0 0 1]]
```

Figure 1 **Identity matrices**

EXAMPLE 1 Identity Matrix Multiplication

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 5 \\ 0 & 2 & -3 \\ -1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 2 & -3 \\ -1 & 4 & -2 \end{bmatrix}$

(B) $\begin{bmatrix} 3 & -2 & 5 \\ 0 & 2 & -3 \\ -1 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 2 & -3 \\ -1 & 4 & -2 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ -2 & 0 & 4 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & -1 & 3 \\ -2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ -2 & 0 & 4 \end{bmatrix}$

Matched Problem 1 Multiply:

(A) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 3 & -5 \\ 6 & 8 \end{bmatrix}$ and $\begin{bmatrix} 4 & 2 \\ 3 & -5 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

In general, we can show that if M is a square matrix of order n and I is the identity matrix of order n, then

$$IM = MI = M$$

If M is an $m \times n$ matrix that is not square $(m \neq n)$, it is still possible to multiply M on the left and on the right by an identity matrix, but not with the same size identity matrix (see Example 1C and D). To avoid the complications involved with associating two different identity matrices with each nonsquare matrix, we restrict our attention in this section to square matrices.

Explore and Discuss 1

(C) One answer is $\begin{bmatrix} 5 & 4 \\ -6 & -5 \end{bmatrix}$

The only real number solutions to the equation $x^2 = 1$ are $x = 1$ and $x = -1$.

(A) Show that $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ satisfies $A^2 = I$, where I is the 2×2 identity.

(B) Show that $B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ satisfies $B^2 = I$.

(C) Find a 2×2 matrix with all elements nonzero whose square is the 2×2 identity matrix.

Inverse of a Square Matrix

If r is an arbitary real number, then its **additive inverse** is the solution x to the equation $r + x = 0$. So the additive inverse of 3 is -3, and the additive inverse of -7 is 7. Similarly, if M is an arbitary $m \times n$ matrix, then M has an additive inverse $-M$, whose elements are just the additive inverses of the elements of M.

The situation is more complicated for *multiplicative* inverses. The **multiplicative inverse** of an arbitrary real number r is the solution x to the equation $r \cdot x = 1$. So the multiplicative inverse of 3 is $\frac{1}{3}$, and the multiplicative inverse of $\frac{-15}{4}$ is $\frac{-4}{15}$. Every real number has a multiplicative inverse except for 0. Because the equation $0 \cdot x = 1$ has no real solution, 0 does not have a multiplicative inverse.

Can we extend the multiplicative inverse concept to matrices? That is, given a matrix M, can we find another matrix N such that $MN = NM = I$, the matrix identity for multiplication? To begin, we consider the size of these matrices. Let M be an $n \times m$ matrix and N a $p \times q$ matrix. If both MN and NM are defined, then $m = p$ and $q = n$ (Fig. 2). If $MN = NM$, then $n = p$ and $q = m$ (Fig. 3). Thus, we have $m = p = n = q$. In other words, M and N must be square matrices of the same order. Later we will see that not all square matrices have inverses.

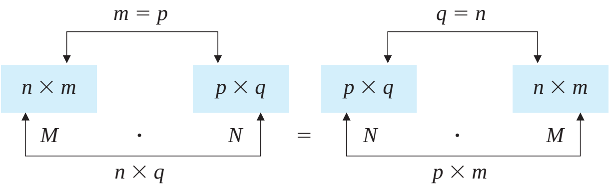


Figure 2

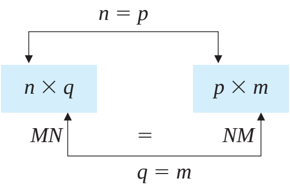


Figure 3

DEFINITION Inverse of a Square Matrix

Let M be a square matrix of order n and I be the identity matrix of order n. If there exists a matrix M^{-1} (read "M inverse") such that

$$M^{-1}M = MM^{-1} = I$$

then M^{-1} is called the **multiplicative inverse of *M*** or, more simply, the **inverse of *M***. If no such matrix exists, then M is said to be a **singular matrix**.

Let us use the definition above to find M^{-1} for

$$M = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

We are looking for

$$M^{-1} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

such that

$$MM^{-1} = M^{-1}M = I$$

So we write

$$\overset{M}{\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}}\overset{M^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

and try to find a, b, c, and d so that the product of M and M^{-1} is the identity matrix I. Multiplying M and M^{-1} on the left side, we obtain

$$\begin{bmatrix} (2a + 3b) & (2c + 3d) \\ (a + 2b) & (c + 2d) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is true only if

$$\begin{aligned} 2a + 3b &= 1 \\ a + 2b &= 0 \end{aligned} \qquad \begin{aligned} 2c + 3d &= 0 \\ c + 2d &= 1 \end{aligned}$$

Use Gauss–Jordan elimination to solve each system.

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 2 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_2 \qquad \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 1 & 2 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 3 & 1 \end{array}\right] \quad (-2)R_1 + R_2 \to R_2 \qquad \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 3 & 0 \end{array}\right] \quad (-2)R_1 + R_2 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 1 \end{array}\right] \quad (-1)R_2 \to R_2 \qquad \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & -2 \end{array}\right] \quad (-1)R_2 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1 \end{array}\right] \quad (-2)R_2 + R_1 \to R_1 \qquad \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \end{array}\right] \quad (-2)R_2 + R_1 \to R_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right]$$

$a = 2, b = -1$ $\qquad$ $c = -3, d = 2$

$$M^{-1} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

CHECK

$$\underset{M}{\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}} \underset{M^{-1}}{\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{M^{-1}}{\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}} \underset{M}{\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}}$$

Unlike nonzero real numbers, inverses do not always exist for square matrices. For example, if

$$N = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

then, using the previous process, we are led to the systems

$$\begin{aligned} 2a + b &= 1 \\ 4a + 2b &= 0 \end{aligned} \qquad \begin{aligned} 2c + d &= 0 \\ 4c + 2d &= 1 \end{aligned}$$

Use Gauss–Jordan elimination to solve each system.

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 2 & 0 \end{array}\right] \quad (-2)R_1 + R_2 \to R_2 \qquad \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 1 \end{array}\right] \quad (-2)R_1 + R_2 \to R_2$$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & -2 \end{array}\right] \qquad \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

The last row of each augmented matrix contains a contradiction. So each system is inconsistent and has no solution. We conclude that N^{-1} does not exist and N is a singular matrix.

Being able to find inverses, when they exist, leads to direct and simple solutions to many practical problems. In the next section, we show how inverses can be used to solve systems of linear equations.

The method outlined previously for finding M^{-1}, if it exists, gets very involved for matrices of order larger than 2. Now that we know what we are looking for, we can use augmented matrices (see Sections 4.2 and 4.3) to make the process more efficient.

EXAMPLE 2 Finding the Inverse of a Matrix Find the inverse, if it exists, of the matrix

$$M = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

SOLUTION We start as before and write

$$\begin{array}{cccc} M & M^{-1} & & I \\ \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix} & \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} & = & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

which is true only if

$$\begin{aligned} a - b + c &= 1 \\ 2b - c &= 0 \\ 2a + 3b \phantom{{}-c} &= 0 \end{aligned} \qquad \begin{aligned} d - e + f &= 0 \\ 2e - f &= 1 \\ 2d + 3e \phantom{{}-f} &= 0 \end{aligned} \qquad \begin{aligned} g - h + i &= 0 \\ 2h - i &= 0 \\ 2g + 3h \phantom{{}-i} &= 1 \end{aligned}$$

Now we write augmented matrices for each of the three systems:

$$\begin{array}{ccc} \text{First} & \text{Second} & \text{Third} \\ \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 2 & 3 & 0 & 0 \end{array}\right] & \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 2 & 3 & 0 & 0 \end{array}\right] & \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \end{array}$$

Since each matrix to the left of the vertical bar is the same, exactly the same row operations can be used on each augmented matrix to transform it into a reduced form. We can speed up the process substantially by combining all three augmented matrices into the single augmented matrix form below:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] = [M \mid I] \tag{1}$$

We now try to perform row operations on matrix (1) until we obtain a row-equivalent matrix of the form

$$\begin{array}{cc} I & B \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a & d & g \\ 0 & 1 & 0 & b & e & h \\ 0 & 0 & 1 & c & f & i \end{array}\right] \end{array} = [I \mid B] \tag{2}$$

If this can be done, the new matrix B to the right of the vertical bar will be M^{-1}. Now let's try to transform matrix (1) into a form like matrix (2). We follow the same sequence of steps as we did in the solution of linear systems by Gauss–Jordan elimination (see Section 4.3).

$$\begin{array}{c} \begin{array}{ccc|ccc} & M & & & I & \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \quad (-2)R_1 + R_3 \to R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right] \quad \tfrac{1}{2}R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 + R_1 \to R_1 \\ (-5)R_2 + R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -2 & -\frac{5}{2} & 1 \end{array}\right] \quad 2R_3 \to R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & -5 & 2 \end{array}\right] \quad \begin{array}{l} (-\frac{1}{2})R_3 + R_1 \to R_1 \\ \frac{1}{2}R_3 + R_2 \to R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & -1 \\ 0 & 1 & 0 & -2 & -2 & 1 \\ 0 & 0 & 1 & -4 & -5 & 2 \end{array}\right] = [I \mid B]$$

Converting back to systems of equations equivalent to our three original systems, we have

$$\begin{array}{lll} a = 3 & d = 3 & g = -1 \\ b = -2 & e = -2 & h = 1 \\ c = -4 & f = -5 & i = 2 \end{array}$$

And these are just the elements of M^{-1} that we are looking for!

$$M^{-1} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

Note that this is the matrix to the right of the vertical line in the last augmented matrix. That is, $M^{-1} = B$.

Since the definition of matrix inverse requires that

$$M^{-1}M = I \quad \text{and} \quad MM^{-1} = I \tag{3}$$

it appears that we must compute both $M^{-1}M$ and MM^{-1} to check our work. However, it can be shown that if one of the equations in (3) is satisfied, the other is also satisfied. So to check our answer it is sufficient to compute either $M^{-1}M$ or MM^{-1}; we do not need to do both.

CHECK

$$M^{-1}M = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Matched Problem 2 Let $M = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(A) Form the augmented matrix $[M \mid I]$.

(B) Use row operations to transform $[M \mid I]$ into $[I \mid B]$.

(C) Verify by multiplication that $B = M^{-1}$ (that is, show that $BM = I$).

The procedure shown in Example 2 can be used to find the inverse of any square matrix, if the inverse exists, and will also indicate when the inverse does not exist. These ideas are summarized in Theorem 1.

THEOREM 1 Inverse of a Square Matrix M

If $[M \mid I]$ is transformed by row operations into $[I \mid B]$, then the resulting matrix B is M^{-1}. However, if we obtain all 0's in one or more rows to the left of the vertical line, then M^{-1} does not exist.

Explore and Discuss 2

(A) Suppose that the square matrix M has a row of all zeros. Explain why M has no inverse.

(B) Suppose that the square matrix M has a column of all zeros. Explain why M has no inverse.

(A) Any product MN would have a row of all zeros, so could not equal I.

(B) Any product NM would have a column of all zeros, so could not equal I.

EXAMPLE 3 Finding a Matrix Inverse

Find M^{-1}, given $M = \begin{bmatrix} 4 & -1 \\ -6 & 2 \end{bmatrix}$.

SOLUTION

$$\left[\begin{array}{rr|rr} 4 & -1 & 1 & 0 \\ -6 & 2 & 0 & 1 \end{array}\right] \quad \tfrac{1}{4}R_1 \to R_1$$

$$\sim \left[\begin{array}{rr|rr} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ -6 & 2 & 0 & 1 \end{array}\right] \quad 6R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{rr|rr} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & 1 \end{array}\right] \quad 2R_2 \to R_2$$

$$\sim \left[\begin{array}{rr|rr} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & 3 & 2 \end{array}\right] \quad \tfrac{1}{4}R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{rr|rr} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 3 & 2 \end{array}\right]$$

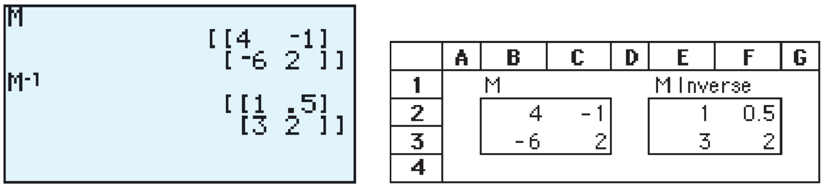
Therefore,

$$M^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{bmatrix}$$

Check by showing that $M^{-1}M = I$.

Matched Problem 3 Find M^{-1}, given $M = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$.

Most graphing calculators and spreadsheets can compute matrix inverses, as illustrated in Figure 4 for the solution to Example 3.



```
M
                [[4  -1]
                 [-6  2 ]]
M-1
                [[1  .5]
                 [3  2 ]]
```

	A	B	C	D	E	F	G
1		M			M Inverse		
2		4	-1		1	0.5	
3		-6	2		3	2	
4							

(A) The command M^{-1} produces the inverse on this graphing calculator

(B) The command MINVERSE (B2:C3) produces the inverse in this spreadsheet

Figure 4 **Finding a matrix inverse**

Explore and Discuss 3

(A) A^{-1} does not exist.

(B) $M^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{bmatrix}$

The inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$A^{-1} = \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad D = ad - bc$$

provided that $D \neq 0$.

(A) Use matrix multiplication to verify this formula. What can you conclude about A^{-1} if $D = 0$?

(B) Use this formula to find the inverse of matrix M in Example 3.

EXAMPLE 4 Finding a Matrix Inverse Find M^{-1}, given $M = \begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}$.

SOLUTION

$$\left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ -3 & 6 & 0 & 1 \end{array}\right] \quad \tfrac{1}{2}R_1 \rightarrow R_1$$

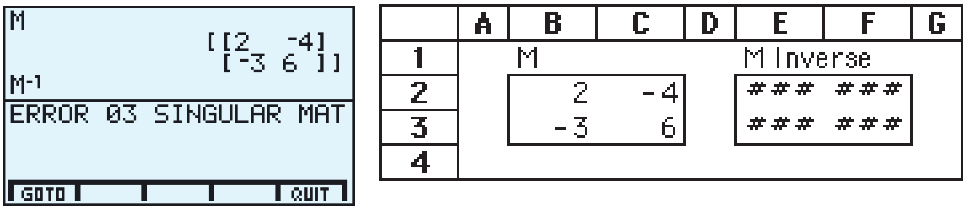
$$\sim \left[\begin{array}{cc|cc} 1 & -2 & \frac{1}{2} & 0 \\ -3 & 6 & 0 & 1 \end{array}\right] \quad 3R_1 + R_2 \rightarrow R_2$$

$$\sim \left[\begin{array}{cc|cc} 1 & -2 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} & 1 \end{array}\right]$$

We have all 0's in the second row to the left of the vertical bar; therefore, the inverse does not exist.

Matched Problem 4 Find N^{-1}, given $N = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$.

Square matrices that do not have inverses are called singular matrices. Graphing calculators and spreadsheets recognize singular matrices and generally respond with some type of error message, as illustrated in Figure 5 for the solution to Example 4.



```
M
                [[2  -4]
                 [-3 6 ]]
M-1
ERROR 03 SINGULAR MAT
```

	A	B	C	D	E	F	G
1		M			M Inverse		
2		2	-4		###	###	
3		-3	6		###	###	
4							

(A) A graphing calculator displays a clear error message

(B) A spreadsheet displays a more cryptic error message

Figure 5

Application: Cryptography

Matrix inverses can provide a simple and effective procedure for encoding and decoding messages. To begin, assign the numbers 1–26 to the letters in the alphabet, as shown below. Also assign the number 0 to a blank to provide for space between words. (A more sophisticated code could include both capital and lowercase letters and punctuation symbols.)

Blank	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

The message "SECRET CODE" corresponds to the sequence

$$19 \quad 5 \quad 3 \quad 18 \quad 5 \quad 20 \quad 0 \quad 3 \quad 15 \quad 4 \quad 5$$

Any matrix whose elements are positive integers and whose inverse exists can be used as an **encoding matrix**. For example, to use the 2×2 matrix

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

to encode the preceding message, first we divide the numbers in the sequence into groups of 2 and use these groups as the columns of a matrix B with 2 rows:

$$B = \begin{bmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{bmatrix}$$

Proceed down the columns, not across the rows.

Notice that we added an extra blank at the end of the message to make the columns come out even. Then we multiply this matrix on the left by A:

$$\begin{aligned} AB &= \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 91 & 66 & 80 & 9 & 72 & 20 \\ 24 & 21 & 25 & 3 & 19 & 5 \end{bmatrix} \end{aligned}$$

The coded message is

$$91 \quad 24 \quad 66 \quad 21 \quad 80 \quad 25 \quad 9 \quad 3 \quad 72 \quad 19 \quad 20 \quad 5$$

This message can be decoded simply by putting it back into matrix form and multiplying on the left by the **decoding matrix** A^{-1}. Since A^{-1} is easily determined if A is known, the encoding matrix A is the only key needed to decode messages that are encoded in this manner.

EXAMPLE 5 Cryptography The message

$$46 \quad 84 \quad 85 \quad 28 \quad 47 \quad 46 \quad 4 \quad 5 \quad 10 \quad 30 \quad 48 \quad 72 \quad 29 \quad 57 \quad 38 \quad 38 \quad 57 \quad 95$$

was encoded with the matrix A shown next. Decode this message.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

SOLUTION Since the encoding matrix A is 3×3, we begin by entering the coded message in the columns of a matrix C with three rows:

$$C = \begin{bmatrix} 46 & 28 & 4 & 30 & 29 & 38 \\ 84 & 47 & 5 & 48 & 57 & 57 \\ 85 & 46 & 10 & 72 & 38 & 95 \end{bmatrix}$$

If B is the matrix containing the uncoded message, then B and C are related by $C = AB$. To recover B, we find A^{-1} (details omitted) and multiply both sides of the equation $C = AB$ by A^{-1}:

$$
\begin{aligned}
B &= A^{-1}C \\
&= \begin{bmatrix} -5 & 2 & 1 \\ 2 & -1 & 0 \\ 4 & -1 & -1 \end{bmatrix}\begin{bmatrix} 46 & 28 & 4 & 30 & 29 & 38 \\ 84 & 47 & 5 & 48 & 57 & 57 \\ 85 & 46 & 10 & 72 & 38 & 95 \end{bmatrix} \\
&= \begin{bmatrix} 23 & 0 & 0 & 18 & 7 & 19 \\ 8 & 9 & 3 & 12 & 1 & 19 \\ 15 & 19 & 1 & 0 & 21 & 0 \end{bmatrix}
\end{aligned}
$$

Writing the numbers in the columns of this matrix in sequence and using the correspondence between numbers and letters noted earlier produces the decoded message:

23	8	15	0	9	19	0	3	1	18	12	0	7	1	21	19	19	0
W	H	O		I	S		C	A	R	L		G	A	U	S	S	

The answer to this question can be found earlier in this chapter.

Matched Problem 5 The message below was also encoded with the matrix A in Example 5. Decode this message:

46 84 85 28 47 46 32 41 78 25 42 53 25 37 63 43 71 83 19 37 25

Exercises 4.5

Skills Warm-up Exercises

W *In Problems 1–4, find the additive inverse and the multiplicative inverse, if defined, of each real number. (If necessary, review Section A.1).*

1. (A) 4 −4; 1/4 (B) −3 3; −1/3 (C) 0

2. (A) −7 7; −1/7 (B) 2 −2; 1/2 (C) −1 1; −1

3. (A) $\frac{2}{3}$ −2/3; 3/2 (B) $\frac{-1}{7}$ 1/7; −7 (C) 1.6

4. (A) $\frac{4}{5}$ −4/5; 5/4 (B) $\frac{12}{7}$ −12/7; 7/12 (C) −2.5

1. (C) 0; not defined **3. (C)** −1.6; 0.625 **4. (C)** 2.5; −0.4

In Problems 5–8, does the given matrix have a multiplicative inverse? Explain your answer.

5. $\begin{bmatrix} 2 & 5 \end{bmatrix}$ No

6. $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$ No

7. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ No

8. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Yes

A *In Problems 9–18, find the matrix products. Note that each product can be found mentally, without the use of a calculator or pencil-and-paper calculations.*

9. (A) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ * (B) $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *

10. (A) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix}$ * (B) $\begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *

*Answer located in Additional Instructor's Answers section.

11. (A) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ * (B) $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *

12. (A) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix}$ * (B) $\begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *

13. (A) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ * (B) $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *

14. (A) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix}$ * (B) $\begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *

15. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 5 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 5 & 1 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -4 & 0 \\ 1 & 2 & -5 \\ 6 & -3 & -1 \end{bmatrix}$ $\begin{bmatrix} 3 & -4 & 0 \\ 1 & 2 & -5 \\ 6 & -3 & -1 \end{bmatrix}$

17. $\begin{bmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 5 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 5 & 1 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 3 & -4 & 0 \\ 1 & 2 & -5 \\ 6 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 3 & -4 & 0 \\ 1 & 2 & -5 \\ 6 & -3 & -1 \end{bmatrix}$

In Problems 19–28, examine the product of the two matrices to determine if each is the inverse of the other.

19. $\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$; $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ Yes

20. $\begin{bmatrix} -2 & -1 \\ -4 & 2 \end{bmatrix}$; $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ No

21. $\begin{bmatrix}2&2\\-1&-1\end{bmatrix}; \begin{bmatrix}1&1\\-1&-1\end{bmatrix}$ No

22. $\begin{bmatrix}5&-7\\-2&3\end{bmatrix}; \begin{bmatrix}3&7\\2&5\end{bmatrix}$ Yes

23. $\begin{bmatrix}-5&2\\-8&3\end{bmatrix}; \begin{bmatrix}3&-2\\8&-5\end{bmatrix}$ Yes

24. $\begin{bmatrix}7&4\\-5&-3\end{bmatrix}; \begin{bmatrix}3&4\\-5&-7\end{bmatrix}$ Yes

25. $\begin{bmatrix}1&2&0\\0&1&0\\-1&-1&1\end{bmatrix}; \begin{bmatrix}1&-2&0\\0&1&0\\1&-1&0\end{bmatrix}$ No

26. $\begin{bmatrix}1&0&1\\-3&1&-2\\0&0&1\end{bmatrix}; \begin{bmatrix}1&0&-1\\3&1&-1\\0&0&1\end{bmatrix}$ Yes

27. $\begin{bmatrix}1&-1&1\\0&2&-1\\2&3&0\end{bmatrix}; \begin{bmatrix}3&3&-1\\-2&-2&1\\-4&-5&2\end{bmatrix}$ Yes

28. $\begin{bmatrix}1&0&-1\\3&1&-1\\0&0&0\end{bmatrix}; \begin{bmatrix}1&0&-1\\-3&1&-2\\0&0&1\end{bmatrix}$ No

Without performing any row operations, explain why each of the matrices in Problems 29–38 does not have an inverse.

29. $\begin{bmatrix}1&2&0\\-3&2&-1\end{bmatrix}$

30. $\begin{bmatrix}-2&3&-1\\4&0&1\end{bmatrix}$

31. $\begin{bmatrix}1&-2\\3&0\\2&-1\end{bmatrix}$

32. $\begin{bmatrix}0&-1\\2&-2\\1&-3\end{bmatrix}$

33. $\begin{bmatrix}-1&2\\0&0\end{bmatrix}$

34. $\begin{bmatrix}0&0\\1&2\end{bmatrix}$

35. $\begin{bmatrix}0&2\\0&-1\end{bmatrix}$

36. $\begin{bmatrix}1&0\\3&0\end{bmatrix}$

37. $\begin{bmatrix}1&2\\3&6\end{bmatrix}$

38. $\begin{bmatrix}-2&-3\\4&6\end{bmatrix}$

B *Given M in Problems 39–48, find M^{-1} and show that $M^{-1}M = I$.*

39. $\begin{bmatrix}-1&0\\-3&1\end{bmatrix}$ $\begin{bmatrix}-1&0\\-3&1\end{bmatrix}$

40. $\begin{bmatrix}1&-5\\0&-1\end{bmatrix}$ $\begin{bmatrix}1&-5\\0&-1\end{bmatrix}$

41. $\begin{bmatrix}1&2\\1&3\end{bmatrix}$ $\begin{bmatrix}3&-2\\-1&1\end{bmatrix}$

42. $\begin{bmatrix}2&1\\5&3\end{bmatrix}$ $\begin{bmatrix}3&-1\\-5&2\end{bmatrix}$

43. $\begin{bmatrix}1&3\\2&7\end{bmatrix}$ $\begin{bmatrix}7&-3\\-2&1\end{bmatrix}$

44. $\begin{bmatrix}2&1\\1&1\end{bmatrix}$ $\begin{bmatrix}1&-1\\-1&2\end{bmatrix}$

45. $\begin{bmatrix}1&-3&0\\0&1&1\\2&-1&4\end{bmatrix}$ *

46. $\begin{bmatrix}2&3&0\\1&2&3\\0&-1&-5\end{bmatrix}$ *

47. $\begin{bmatrix}1&1&0\\2&3&-1\\1&0&2\end{bmatrix}$ *

48. $\begin{bmatrix}1&0&-1\\2&-1&0\\1&1&-2\end{bmatrix}$ *

Find the inverse of each matrix in Problems 49–54, if it exists.

49. $\begin{bmatrix}4&3\\-3&-2\end{bmatrix}$ $\begin{bmatrix}-2&-3\\3&4\end{bmatrix}$

50. $\begin{bmatrix}-4&3\\-5&4\end{bmatrix}$ $\begin{bmatrix}-4&3\\-5&4\end{bmatrix}$

51. $\begin{bmatrix}2&6\\3&9\end{bmatrix}$ Does not exist

52. $\begin{bmatrix}2&-4\\-3&6\end{bmatrix}$ Does not exist

53. $\begin{bmatrix}2&1\\4&3\end{bmatrix}$ $\begin{bmatrix}1.5&-0.5\\-2&1\end{bmatrix}$

54. $\begin{bmatrix}-5&3\\2&-2\end{bmatrix}$ $\begin{bmatrix}-0.5&-0.75\\-0.5&-1.25\end{bmatrix}$

In Problems 55–60, find the inverse. Note that each inverse can be found mentally, without the use of a calculator or pencil-and-paper calculations.

55. $\begin{bmatrix}2&0\\0&2\end{bmatrix}$ $\begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{bmatrix}$

56. $\begin{bmatrix}4&0\\0&4\end{bmatrix}$ $\begin{bmatrix}\frac{1}{4}&0\\0&\frac{1}{4}\end{bmatrix}$

57. $\begin{bmatrix}3&0\\0&-5\end{bmatrix}$ $\begin{bmatrix}\frac{1}{3}&0\\0&-\frac{1}{5}\end{bmatrix}$

58. $\begin{bmatrix}-2&0\\0&\frac{1}{2}\end{bmatrix}$ $\begin{bmatrix}-\frac{1}{2}&0\\0&2\end{bmatrix}$

59. $\begin{bmatrix}4&0&0\\0&2&0\\0&0&-8\end{bmatrix}$ *

60. $\begin{bmatrix}3&0&0\\0&-6&0\\0&0&-5\end{bmatrix}$ *

C *Find the inverse of each matrix in Problems 61–68, if it exists.*

61. $\begin{bmatrix}-5&-2&-2\\2&1&0\\1&0&1\end{bmatrix}$ *

62. $\begin{bmatrix}2&-2&4\\1&1&1\\1&0&1\end{bmatrix}$ *

63. $\begin{bmatrix}2&1&1\\1&1&0\\-1&-1&0\end{bmatrix}$ *

64. $\begin{bmatrix}1&-1&0\\2&-1&1\\0&1&1\end{bmatrix}$ *

65. $\begin{bmatrix}-1&-2&2\\4&3&0\\4&0&4\end{bmatrix}$ *

66. $\begin{bmatrix}4&2&2\\4&2&0\\5&0&5\end{bmatrix}$ *

67. $\begin{bmatrix}2&-1&-2\\-4&2&8\\6&-2&-1\end{bmatrix}$ *

68. $\begin{bmatrix}-1&-1&4\\3&3&-22\\-2&-1&19\end{bmatrix}$ *

69. Show that $(A^{-1})^{-1} = A$ for: $A = \begin{bmatrix}4&3\\3&2\end{bmatrix}$

70. Show that $(AB)^{-1} = B^{-1}A^{-1}$ for

$$A = \begin{bmatrix}4&3\\3&2\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}2&5\\3&7\end{bmatrix}$$

71. Discuss the existence of M^{-1} for 2 × 2 diagonal matrices of the form

$$M = \begin{bmatrix}a&0\\0&d\end{bmatrix}$$

Generalize your conclusions to $n \times n$ diagonal matrices.*

72. Discuss the existence of M^{-1} for 2 × 2 upper triangular matrices of the form

$$M = \begin{bmatrix}a&b\\0&d\end{bmatrix}$$

Generalize your conclusions to $n \times n$ upper triangular matrices.*

In Problems 73–75, find A^{-1} and A^2.

73. $A = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$ **74.** $A = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$

75. $A = \begin{bmatrix} 4 & 3 \\ -5 & -4 \end{bmatrix}$ $A^{-1} = A;\ A^2 = I$

76. Based on your observations in Problems 73–75, if $A = A^{-1}$ for a square matrix A, what is A^2? Give a mathematical argument to support your conclusion. $A^2 = I$

73. $A^{-1} = A;\ A^2 = I$ **74.** $A^{-1} = A;\ A^2 = I$

Applications

Problems 77–80 refer to the encoding matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

77. Cryptography. Encode the message "WINGARDIUM LEVIOSA" using matrix A. *

78. Cryptography. Encode the message "FINITE INCANTATEM" using matrix A. *

79. Cryptography. The following message was encoded with matrix A. Decode this message:

52 70 17 21 5 5 29 43 4 4 52 70 25
35 29 33 15 18 5 5 PRIDE AND PREJUDICE

80. Cryptography. The following message was encoded with matrix A. Decode this message:

36 44 5 5 38 56 55 75 18 23 56 75
22 33 37 55 27 40 53 79 59 81
THE BROTHERS KARAMAZOV

Problems 81–84 require the use of a graphing calculator or computer. Use the 4 × 4 encoding matrix B given below. Form a matrix with 4 rows and as many columns as necessary to accommodate the message.

$$B = \begin{bmatrix} 2 & 2 & 1 & 3 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix}$$

81. Cryptography. Encode the message "DEPART ISTANBUL ORIENT EXPRESS" using matrix B. *

82. Cryptography. Encode the message "SAIL FROM LISBON IN MORNING" using matrix B. *

83. Cryptography. The following message was encoded with matrix B. Decode this message: *

85 74 27 109 31 27 13 40 139 73 58 154
61 70 18 93 69 59 23 87 18 13 9 22

84. Cryptography. The following message was encoded with matrix B. Decode this message: *

75 61 28 94 35 22 13 40 49 21 16 52
42 45 19 64 38 55 10 65 69 75 24 102
67 49 19 82 10 5 5 10

Problems 85–88 require the use of a graphing calculator or a computer. Use the 5 × 5 encoding matrix C given below. Form a matrix with 5 rows and as many columns as necessary to accommodate the message.

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 & 1 \end{bmatrix}$$

85. Cryptography. Encode the message "THE EAGLE HAS LANDED" using matrix C. *

86. Cryptography. Encode the message "ONE IF BY LAND AND TWO IF BY SEA" using matrix C. *

87. Cryptography. The following message was encoded with matrix C. Decode this message: *

37 72 58 45 56 30 67 50 46 60 27 77
41 45 39 28 24 52 14 37 32 58 70 36
76 22 38 70 12 67

88. Cryptography. The following message was encoded with matrix C. Decode this message: *

25 75 55 35 50 43 83 54 60 53 25 13
59 9 53 15 35 40 15 45 33 60 60 36
51 15 7 37 0 22

Answers to Matched Problems

1. (A) $\begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$ (B) $\begin{bmatrix} 4 & 2 \\ 3 & -5 \\ 6 & 8 \end{bmatrix}$

2. (A) $\left[\begin{array}{rrr|rrr} 3 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

(B) $\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$

(C) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$

4. Does not exist

5. WHO IS WILHELM JORDAN

4.6 Matrix Equations and Systems of Linear Equations

- Matrix Equations
- Matrix Equations and Systems of Linear Equations
- Application

The identity matrix and inverse matrix discussed in the preceding section can be put to immediate use in the solution of certain simple matrix equations. Being able to solve a matrix equation gives us another important method of solving systems of equations, provided that the system is independent and has the same number of variables as equations. If the system is dependent or if it has either fewer or more variables than equations, we must return to the Gauss–Jordan method of elimination.

Matrix Equations

Solving simple matrix equations is similar to solving real number equations but with two important differences:

1. there is *no* operation of division for matrices, and
2. matrix multiplication is *not* commutative.

Compare the real number equation $4x = 9$ and the matrix equation $AX = B$. The real number equation can be solved by dividing both sides of the equation by 4. However, that approach cannot be used for $AX = B$, because there is no operation of division for matrices. Instead, we note that $4x = 9$ can be solved by multiplying both sides of the equation by $\frac{1}{4}$, the multiplicative inverse of 4. So we solve $AX = B$ by multiplying both sides of the equation, *on the left*, by A^{-1}, the inverse of A. Because matrix multiplication is not commutative, multiplying both sides of an equation on the left by A^{-1} is different from multiplying both sides of an equation on the right by A^{-1}. In the case of $AX = B$, it is multiplication on the left that is required. The details are presented in Example 1.

In solving matrix equations, we will be guided by the properties of matrices summarized in Theorem 1.

THEOREM 1 Basic Properties of Matrices

Assuming that all products and sums are defined for the indicated matrices A, B, C, I, and 0, then

Addition Properties

Associative: $(A + B) + C = A + (B + C)$

Commutative: $A + B = B + A$

Additive identity: $A + 0 = 0 + A = A$

Additive inverse: $A + (-A) = (-A) + A = 0$

Multiplication Properties

Associative property: $A(BC) = (AB)C$

Multiplicative identity: $AI = IA = A$

Multiplicative inverse: If A is a square matrix and A^{-1} exists, then $AA^{-1} = A^{-1}A = I$.

Combined Properties

Left distributive: $A(B + C) = AB + AC$

Right distributive: $(B + C)A = BA + CA$

Equality

Addition: If $A = B$, then $A + C = B + C$.

Left multiplication: If $A = B$, then $CA = CB$.

Right multiplication: If $A = B$, then $AC = BC$.

EXAMPLE 1 **Solving a Matrix Equation** Given an $n \times n$ matrix A and $n \times 1$ column matrices B and X, solve $AX = B$ for X. Assume that all necessary inverses exist.

SOLUTION We are interested in finding a column matrix X that satisfies the matrix equation $AX = B$. To solve this equation, we multiply both sides on the left by A^{-1} to isolate X on the left side.

$$\begin{aligned} AX &= B && \text{Use the left multiplication property.} \\ A^{-1}(AX) &= A^{-1}B && \text{Use the associative property.} \\ (A^{-1}A)X &= A^{-1}B && A^{-1}A = I \\ IX &= A^{-1}B && IX = X \\ X &= A^{-1}B \end{aligned}$$

Matched Problem 1 Given an $n \times n$ matrix A and $n \times 1$ column matrices B, C, and X, solve $AX + C = B$ for X. Assume that all necessary inverses exist.

⚠ **CAUTION** Do not mix the left multiplication property and the right multiplication property. If $AX = B$, then

$$A^{-1}(AX) \neq BA^{-1}$$

▲

Matrix Equations and Systems of Linear Equations

Now we show how independent systems of linear equations with the same number of variables as equations can be solved. First, convert the system into a matrix equation of the form $AX = B$, and then use $X = A^{-1}B$ as obtained in Example 1.

EXAMPLE 2 **Using Inverses to Solve Systems of Equations** Use matrix inverse methods to solve the system:

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ 2x_2 - x_3 &= 1 \\ 2x_1 + 3x_2 \phantom{{}+x_3} &= 1 \end{aligned} \qquad (1)$$

SOLUTION The inverse of the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

provides an efficient method for solving this system. To see how, we convert system (1) into a matrix equation:

$$\overset{A}{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}} \overset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{B}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \qquad (2)$$

Check that matrix equation (2) is equivalent to system (1) by finding the product of the left side and then equating corresponding elements on the left with those on the right.

We are interested in finding a column matrix X that satisfies the matrix equation $AX = B$. In Example 1 we found that if A^{-1} exists, then

$$X = A^{-1}B$$

The inverse of A was found in Example 2, Section 4.5, to be

$$A^{-1} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

Therefore,

$$\overset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{A^{-1}}{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 5 \\ -3 \\ -7 \end{bmatrix}$$

and we can conclude that $x_1 = 5, x_2 = -3$, and $x_3 = -7$. Check this result in system (1).

Matched Problem 2 Use matrix inverse methods to solve the system:

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_2 &= 3 \\ x_1 + x_3 &= 2 \end{aligned}$$

[*Note:* The inverse of the coefficient matrix was found in Matched Problem 2, Section 4.5.]

At first glance, using matrix inverse methods seems to require the same amount of effort as using Gauss–Jordan elimination. In either case, row operations must be applied to an augmented matrix involving the coefficients of the system. The advantage of the inverse matrix method becomes readily apparent when solving a number of systems with a common coefficient matrix and different constant terms.

EXAMPLE 3 Using Inverses to Solve Systems of Equations Use matrix inverse methods to solve each of the following systems:

(A)
$$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 2x_2 - x_3 &= 1 \\ 2x_1 + 3x_2 &= 4 \end{aligned}$$

(B)
$$\begin{aligned} x_1 - x_2 + x_3 &= -5 \\ 2x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 &= -3 \end{aligned}$$

SOLUTION Notice that both systems have the same coefficient matrix A as system (1) in Example 2. Only the constant terms have changed. We can use A^{-1} to solve these systems just as we did in Example 2.

(A)

$$\overset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{A^{-1}}{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}} \overset{B}{\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}} = \begin{bmatrix} 8 \\ -4 \\ -9 \end{bmatrix}$$

$x_1 = 8, x_2 = -4$, and $x_3 = -9$.

(B)

$$\overset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{A^{-1}}{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}} \overset{B}{\begin{bmatrix} -5 \\ 2 \\ -3 \end{bmatrix}} = \begin{bmatrix} -6 \\ 3 \\ 4 \end{bmatrix}$$

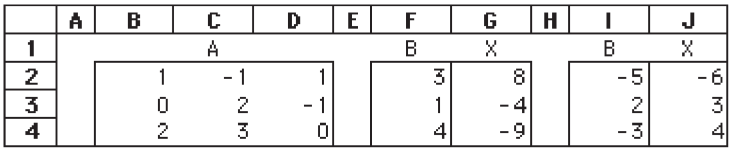
$x_1 = -6, x_2 = 3$, and $x_3 = 4$.

Matched Problem 3 Use matrix inverse methods to solve each of the following systems (see Matched Problem 2):

(A)
$$\begin{aligned} 3x_1 - x_2 + x_3 &= 3 \\ -x_1 + x_2 &= -3 \\ x_1 + x_3 &= 2 \end{aligned}$$

(B)
$$\begin{aligned} 3x_1 - x_2 + x_3 &= -5 \\ -x_1 + x_2 &= 1 \\ x_1 + x_3 &= -4 \end{aligned}$$

As Examples 2 and 3 illustrate, inverse methods are very convenient for hand calculations because once the inverse is found, it can be used to solve any new system formed by changing only the constant terms. Since most graphing calculators and computers can compute the inverse of a matrix, this method also adapts readily to graphing calculator and spreadsheet solutions (Fig. 1). However, if your graphing calculator (or spreadsheet) also has a built-in procedure for finding the reduced form of an augmented matrix, it is just as convenient to use Gauss–Jordan elimination. Furthermore, Gauss–Jordan elimination can be used in all cases and, as noted below, matrix inverse methods cannot always be used.



	A	B	C	D	E	F	G	H	I	J
1			A			B	X		B	X
2		1	-1	1		3	8		-5	-6
3		0	2	-1		1	-4		2	3
4		2	3	0		4	-9		-3	4

Figure 1 **Using inverse methods on a spreadsheet: The values in G2:G4 are produced by the command MMULT (MINVERSE(B2:D4),F2:F4)**

SUMMARY Using Inverse Methods to Solve Systems of Equations

If the number of equations in a system equals the number of variables and the coefficient matrix has an inverse, then the system will always have a unique solution that can be found by using the inverse of the coefficient matrix to solve the corresponding matrix equation.

Matrix equation	Solution
$AX = B$	$X = A^{-1}B$

CONCEPTUAL INSIGHT

There are two cases where inverse methods will not work:

Case 1. The coefficient matrix is singular.
Case 2. The number of variables is not the same as the number of equations.

In either case, use Gauss–Jordan elimination.

Application

The following application illustrates the usefulness of the inverse matrix method for solving systems of equations.

EXAMPLE 4 Investment Analysis An investment advisor currently has two types of investments available for clients: a conservative investment A that pays 10% per year and a higher risk investment B that pays 20% per year. Clients may divide their investments between the two to achieve any total return desired between 10% and 20%. However, the higher the desired return, the higher the risk. How should each client invest to achieve the indicated return?

	Client			
	1	**2**	**3**	***k***
Total investment	\$20,000	\$50,000	\$10,000	k_1
Annual return desired	\$ 2,400	\$ 7,500	\$ 1,300	k_2
	(12%)	(15%)	(13%)	

SOLUTION The answer to this problem involves six quantities, two for each client. Utilizing inverse matrices provides an efficient way to find these quantities. We will solve the problem for an arbitrary client k with unspecified amounts k_1 for the total investment and k_2 for the annual return. (Do not confuse k_1 and k_2 with variables. Their values are known—they just differ for each client.)

Let x_1 = amount invested in A by a given client
x_2 = amount invested in B by a given client

Then we have the following mathematical model:

$$\begin{aligned} x_1 + \quad x_2 &= k_1 && \text{Total invested} \\ 0.1x_1 + 0.2x_2 &= k_2 && \text{Total annual return desired} \end{aligned}$$

Write as a matrix equation:

$$\begin{array}{ccccc} A & & X & & B \\ \begin{bmatrix} 1 & 1 \\ 0.1 & 0.2 \end{bmatrix} & & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = & \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \end{array}$$

If A^{-1} exists, then

$$X = A^{-1}B$$

We now find A^{-1} by starting with the augmented matrix $[A \mid I]$ and proceeding as discussed in Section 4.5:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0.1 & 0.2 & 0 & 1 \end{array}\right] \quad 10R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 10 \end{array}\right] \quad (-1)R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 10 \end{array}\right] \quad (-1)R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -10 \\ 0 & 1 & -1 & 10 \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} 2 & -10 \\ -1 & 10 \end{bmatrix} \qquad \text{Check: } \begin{array}{ccccc} A^{-1} & & A & & I \\ \begin{bmatrix} 2 & -10 \\ -1 & 10 \end{bmatrix} & & \begin{bmatrix} 1 & 1 \\ 0.1 & 0.2 \end{bmatrix} & = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

and

$$\begin{array}{ccccc} X & & A^{-1} & & B \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = & \begin{bmatrix} 2 & -10 \\ -1 & 10 \end{bmatrix} & & \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \end{array}$$

To solve each client's investment problem, we replace k_1 and k_2 with appropriate values from the table and multiply by A^{-1}:

Client 1

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -10 \\ -1 & 10 \end{bmatrix} \begin{bmatrix} 20{,}000 \\ 2{,}400 \end{bmatrix} = \begin{bmatrix} 16{,}000 \\ 4{,}000 \end{bmatrix}$$

Solution: x_1 = \$16,000 in investment A, x_2 = \$4,000 in investment B

Client 2

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -10 \\ -1 & 10 \end{bmatrix} \begin{bmatrix} 50{,}000 \\ 7{,}500 \end{bmatrix} = \begin{bmatrix} 25{,}000 \\ 25{,}000 \end{bmatrix}$$

Solution: x_1 = \$25,000 in investment A, x_2 = \$25,000 in investment B

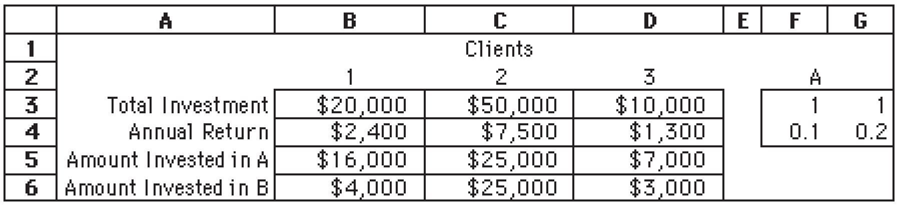
Client 3

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -10 \\ -1 & 10 \end{bmatrix} \begin{bmatrix} 10{,}000 \\ 1{,}300 \end{bmatrix} = \begin{bmatrix} 7{,}000 \\ 3{,}000 \end{bmatrix}$$

Solution: $x_1 = \$7{,}000$ in investment A, $x_2 = \$3{,}000$ in investment B

Matched Problem 4 Repeat Example 4 with investment A paying 8% and investment B paying 24%.

Figure 2 illustrates a solution to Example 4 on a spreadsheet.



	A	B	C	D	E	F	G
1			Clients				
2		1	2	3		A	
3	Total Investment	$20,000	$50,000	$10,000		1	1
4	Annual Return	$2,400	$7,500	$1,300		0.1	0.2
5	Amount Invested in A	$16,000	$25,000	$7,000			
6	Amount Invested in B	$4,000	$25,000	$3,000			

Figure 2

Explore and Discuss 1 Refer to the mathematical model in Example 4:

$$\begin{array}{ccccc} A & & X & & B \\ \begin{bmatrix} 1 & 1 \\ 0.1 & 0.2 \end{bmatrix} & & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = & \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \end{array} \qquad (3)$$

(A) Does matrix equation (3) always have a solution for any constant matrix B?

(B) Do all these solutions make sense for the original problem? If not, give examples.

(C) If the total investment is $k_1 = \$10{,}000$, describe all possible annual returns k_2.

(A) Yes
(B) No
(C) $\$1{,}000 \le k_2 \le \$2{,}000$

Exercises 4.6

W

Skills Warm-up Exercises

In Problems 1–8, solve each equation for x, where x represents a real number. (If necessary, review Section 1.1).

1. $5x = -3$ $-3/5$

2. $4x = 9$ $9/4$

3. $4x = 8x + 7$ $-7/4$

4. $6x = -3x + 14$ $14/9$

5. $6x + 8 = -2x + 17$ $9/8$

6. $-4x + 3 = 5x + 12$ -1

7. $10 - 3x = 7x + 9$ $1/10$

8. $2x + 7x + 1 = 8x + 3 - x$ 1

A *Write Problems 9–12 as systems of linear equations without matrices.*

9. $\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ $\begin{aligned} 3x_1 + x_2 &= 5 \\ 2x_1 - x_2 &= -4 \end{aligned}$

10. $\begin{bmatrix} -2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ $\begin{aligned} -2x_1 + x_2 &= -5 \\ -3x_1 + 4x_2 &= 7 \end{aligned}$

11. $\begin{bmatrix} -3 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$ $\begin{aligned} -3x_1 + x_2 &= 3 \\ 2x_1 + x_3 &= -4 \\ -x_1 + 3x_2 - 2x_3 &= 2 \end{aligned}$

12. $\begin{bmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 4 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix}$ $\begin{aligned} 2x_1 - x_2 &= 6 \\ -2x_1 + 3x_2 - x_3 &= -4 \\ 4x_1 + 3x_3 &= 7 \end{aligned}$

Write each system in Problems 13–16 as a matrix equation of the form AX = B.

13. $\begin{aligned} 3x_1 - 4x_2 &= 1 \\ 2x_1 + x_2 &= 5 \end{aligned}$ *

14. $\begin{aligned} 2x_1 + x_2 &= 8 \\ -5x_1 + 3x_2 &= -4 \end{aligned}$ *

15. $\begin{aligned} x_1 - 3x_2 + 2x_3 &= -3 \\ -2x_1 + 3x_2 &= 1 \\ x_1 + x_2 + 4x_3 &= -2 \end{aligned}$ *

16. $\begin{aligned} 3x_1 + 2x_3 &= 9 \\ -x_1 + 4x_2 + x_3 &= -7 \\ -2x_1 + 3x_2 &= 6 \end{aligned}$ *

Find x_1 and x_2 in Problems 17–20.

17. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

18. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

19. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

20. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

17. $x_1 = -8, x_2 = 2$ **18.** $x_1 = -8, x_2 = -7$
19. $x_1 = 0, x_2 = 4$ **20.** $x_1 = -7, x_2 = 2$

*Answer located in Additional Instructor's Answers section.

In Problems 21–24, find x_1 and x_2.

21. $\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ $x_1 = 3, x_2 = -2$

22. $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$ $x_1 = 18, x_2 = -3$

23. $\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$ $x_1 = 11, x_2 = 4$

24. $\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$ $x_1 = 8, x_2 = 2$

In Problems 25–30, solve for x_1 and x_2.

25. $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \end{bmatrix}$ $x_1 = 4, x_2 = 1$

26. $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$ $x_1 = -4, x_2 = 15$

27. $\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $x_1 = 9.5, x_2 = -6$

28. $\begin{bmatrix} 3 & -4 \\ -6 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ No solution

29. $\begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ No solution

30. $\begin{bmatrix} 3 & -1 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ $x_1 = 0, x_2 = 0$

B *In Problems 31–38, write each system as a matrix equation and solve using inverses. [Note: The inverses were found in Problems 41–48, Exercises 4.5.]*

31. $\begin{aligned} x_1 + 2x_2 &= k_1 \\ x_1 + 3x_2 &= k_2 \end{aligned}$

(A) $k_1 = 1, k_2 = 3$ $x_1 = -3, x_2 = 2$

(B) $k_1 = 3, k_2 = 5$ $x_1 = -1, x_2 = 2$

(C) $k_1 = -2, k_2 = 1$ $x_1 = -8, x_2 = 3$

32. $\begin{aligned} 2x_1 + x_2 &= k_1 \\ 5x_1 + 3x_2 &= k_2 \end{aligned}$

(A) $k_1 = 2, k_2 = 13$ $x_1 = -7, x_2 = 16$

(B) $k_1 = 2, k_2 = 4$ $x_1 = 2, x_2 = -2$

(C) $k_1 = 1, k_2 = -3$ $x_1 = 6, x_2 = -11$

33. $\begin{aligned} x_1 + 3x_2 &= k_1 \\ 2x_1 + 7x_2 &= k_2 \end{aligned}$

(A) $k_1 = 2, k_2 = -1$ $x_1 = 17, x_2 = -5$

(B) $k_1 = 1, k_2 = 0$ $x_1 = 7, x_2 = -2$

(C) $k_1 = 3, k_2 = -1$ $x_1 = 24, x_2 = -7$

34. $\begin{aligned} 2x_1 + x_2 &= k_1 \\ x_1 + x_2 &= k_2 \end{aligned}$

(A) $k_1 = -1, k_2 = -2$ $x_1 = 1, x_2 = -3$

(B) $k_1 = 2, k_2 = 3$ $x_1 = -1, x_2 = 4$

(C) $k_1 = 2, k_2 = 0$ $x_1 = 2, x_2 = -2$

35. $\begin{aligned} x_1 - 3x_2 \phantom{{}+ 4x_3} &= k_1 \\ x_2 + x_3 &= k_2 \\ 2x_1 - x_2 + 4x_3 &= k_3 \end{aligned}$

(A) $k_1 = 1, k_2 = 0, k_3 = 2$ $x_1 = 1, x_2 = 0, x_3 = 0$

(B) $k_1 = -1, k_2 = 1, k_3 = 0$ $x_1 = -7, x_2 = -2, x_3 = 3$

(C) $k_1 = 2, k_2 = -2, k_3 = 1$ $x_1 = 17, x_2 = 5, x_3 = -7$

36. $\begin{aligned} 2x_1 + 3x_2 \phantom{{}+ 3x_3} &= k_1 \\ x_1 + 2x_2 + 3x_3 &= k_2 \\ -x_2 - 5x_3 &= k_3 \end{aligned}$

(A) $k_1 = 0, k_2 = 2, k_3 = 1$ $x_1 = 39, x_2 = -26, x_3 = 5$

(B) $k_1 = -2, k_2 = 0, k_3 = 1$ $x_1 = 23, x_2 = -16, x_3 = 3$

(C) $k_1 = 3, k_2 = 1, k_3 = 0$ $x_1 = -6, x_2 = 5, x_3 = -1$

37. $\begin{aligned} x_1 + x_2 \phantom{{}- x_3} &= k_1 \\ 2x_1 + 3x_2 - x_3 &= k_2 \\ x_1 \phantom{{}+ 3x_2} + 2x_3 &= k_3 \end{aligned}$

(A) $k_1 = 2, k_2 = 0, k_3 = 4$ $x_1 = 8, x_2 = -6, x_3 = -2$

(B) $k_1 = 0, k_2 = 4, k_3 = -2$ $x_1 = -6, x_2 = 6, x_3 = 2$

(C) $k_1 = 4, k_2 = 2, k_3 = 0$ $x_1 = 20, x_2 = -16, x_3 = -10$

38. $\begin{aligned} x_1 \phantom{{}- x_2} - x_3 &= k_1 \\ 2x_1 - x_2 \phantom{{}- 2x_3} &= k_2 \\ x_1 + x_2 - 2x_3 &= k_3 \end{aligned}$

(A) $k_1 = 4, k_2 = 8, k_3 = 0$ $x_1 = 0, x_2 = -8, x_3 = -4$

(B) $k_1 = 4, k_2 = 0, k_3 = -4$ $x_1 = -12, x_2 = -24, x_3 = -16$

(C) $k_1 = 0, k_2 = 8, k_3 = -8$ $x_1 = 0, x_2 = -8, x_3 = 0$

In Problems 39–44, the matrix equation is not solved correctly. Explain the mistake and find the correct solution. Assume that the indicated inverses exist.

39. $AX = B, X = \dfrac{B}{A}$ $X = A^{-1}B$

40. $XA = B, X = \dfrac{B}{A}$ $X = BA^{-1}$

41. $XA = B, X = A^{-1}B$ $X = BA^{-1}$

42. $AX = B, X = BA^{-1}$ $X = A^{-1}B$

43. $AX = BA, X = A^{-1}BA, X = B$ $X = A^{-1}BA$

44. $XA = AB, X = ABA^{-1}, X = B$ $X = ABA^{-1}$

In Problems 45–50, explain why the system cannot be solved by matrix inverse methods. Discuss methods that could be used and then solve the system.

45. $\begin{aligned} -2x_1 + 4x_2 &= -5 \\ 6x_1 - 12x_2 &= 15 \end{aligned}$ $x_1 = 2t + 2.5, x_2 = t$, t any real number

46. $\begin{aligned} -2x_1 + 4x_2 &= 5 \\ 6x_1 - 12x_2 &= 15 \end{aligned}$ No solution

47. $\begin{aligned} x_1 - 3x_2 - 2x_3 &= -1 \\ -2x_1 + 6x_2 + 4x_3 &= 3 \end{aligned}$ No solution

48. $\begin{aligned} x_1 - 3x_2 - 2x_3 &= -1 \\ -2x_1 + 7x_2 + 3x_3 &= 3 \end{aligned}$ $x_1 = 5t + 2, x_2 = t + 1, x_3 = t$ for t any real number

49. $\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ 2x_1 - 3x_2 - 2x_3 &= 3 \\ x_1 - x_2 - 5x_3 &= 2 \end{aligned}$ $x_1 = 13t + 3, x_2 = 8t + 1, x_3 = t$, t any real number

50. $\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ 2x_1 - 3x_2 - 2x_3 &= 3 \\ x_1 - x_2 - 5x_3 &= 4 \end{aligned}$ No solution

C *For $n \times n$ matrices A and B, and $n \times 1$ column matrices C, D, and X, solve each matrix equation in Problems 51–56 for X. Assume that all necessary inverses exist.*

51. $AX - BX = C$*

52. $AX + BX = C$*

53. $AX + X = C$*

54. $AX - X = C$*

55. $AX - C = D - BX$*

56. $AX + C = BX + D$*

57. Use matrix inverse methods to solve the following system for the indicated values of k_1 and k_2.

$$\begin{aligned} x_1 + 2.001x_2 &= k_1 \\ x_1 + 2x_2 &= k_2 \end{aligned}$$

(A) $k_1 = 1, k_2 = 1$ $x_1 = 1, x_2 = 0$

(B) $k_1 = 1, k_2 = 0$ $x_1 = -2{,}000, x_2 = 1{,}000$

(C) $k_1 = 0, k_2 = 1$ $x_1 = 2{,}001, x_2 = -1{,}000$

Discuss the effect of small changes in the constant terms on the solution set of this system.

58. Repeat Problem 57 for the following system:*

$$\begin{aligned} x_1 - 3.001x_2 &= k_1 \\ x_1 - 3x_2 &= k_2 \end{aligned}$$

In Problems 59–62, write each system as a matrix equation and solve using the inverse coefficient matrix. Use a graphing calculator or computer to perform the necessary calculations.

59.

$$\begin{aligned} x_1 + 8x_2 + 7x_3 &= 135 \\ 6x_1 + 6x_2 + 8x_3 &= 155 \\ 3x_1 + 4x_2 + 6x_3 &= 75 \end{aligned}$$

$x_1 = 18.2, x_2 = 27.9, x_3 = -15.2$

60.

$$\begin{aligned} 5x_1 + 3x_2 - 2x_3 &= 112 \\ 7x_1 + 5x_2 &= 70 \\ 3x_1 + x_2 - 9x_3 &= 96 \end{aligned}$$

*

61.

$$\begin{aligned} 6x_1 + 9x_2 + 7x_3 + 5x_4 &= 250 \\ 6x_1 + 4x_2 + 7x_3 + 3x_4 &= 195 \\ 4x_1 + 5x_2 + 3x_3 + 2x_4 &= 145 \\ 4x_1 + 3x_2 + 8x_3 + 2x_4 &= 125 \end{aligned}$$

*

62.

$$\begin{aligned} 3x_1 + 3x_2 + 6x_3 + 5x_4 &= 10 \\ 4x_1 + 5x_2 + 8x_3 + 2x_4 &= 15 \\ 3x_1 + 6x_2 + 7x_3 + 4x_4 &= 30 \\ 4x_1 + x_2 + 6x_3 + 3x_4 &= 25 \end{aligned}$$

*

Applications

Construct a mathematical model for each of the following problems. (The answers in the back of the book include both the mathematical model and the interpretation of its solution.) Use matrix inverse methods to solve the model and then interpret the solution.

63. Concert tickets. A concert hall has 10,000 seats and two categories of ticket prices, \$25 and \$35. Assume that all seats in each category can be sold.

	Concert		
	1	2	3
Tickets sold	10,000	10,000	10,000
Return required	\$275,000	\$300,000	\$325,000

(A) How many tickets of each category should be sold to bring in each of the returns indicated in the table? *

(B) Is it possible to bring in a return of \$200,000? Of \$400,000? Explain. No

(C) Describe all the possible returns.
$\$250{,}000 + 10t, 0 \le t \le 10{,}000$

64. Parking receipts. Parking fees at a zoo are \$5.00 for local residents and \$7.50 for all others. At the end of each day, the total number of vehicles parked that day and the gross receipts for the day are recorded, but the number of vehicles in each category is not. The following table contains the relevant information for a recent 4-day period:

	Day			
	1	2	3	4
Vehicles parked	1,200	1,550	1,740	1,400
Gross receipts	\$7,125	\$9,825	\$11,100	\$8,650

(A) How many vehicles in each category used the zoo's parking facilities each day?*

(B) If 1,200 vehicles are parked in one day, is it possible to take in gross receipts of \$5,000? Of \$10,000? Explain.*

(C) Describe all possible gross receipts on a day when 1,200 vehicles are parked.*

65. Production scheduling. A supplier manufactures car and truck frames at two different plants. The production rates (in frames per hour) for each plant are given in the table:*

Plant	Car Frames	Truck Frames
A	10	5
B	8	8

How many hours should each plant be scheduled to operate to exactly fill each of the orders in the following table?

	Orders		
	1	2	3
Car frames	3,000	2,800	2,600
Truck frames	1,600	2,000	2,200

66. Production scheduling. Labor and material costs for manufacturing two guitar models are given in the table:*

Guitar Model	Labor Cost	Material Cost
A	\$30	\$20
B	\$40	\$30

(A) If a total of \$3,000 a week is allowed for labor and material, how many of each model should be produced each week to use exactly each of the allocations of the \$3,000 indicated in the following table?*

	Weekly Allocation		
	1	2	3
Labor	\$1,800	\$1,750	\$1,720
Material	\$1,200	\$1,250	\$1,280

(B) Is it possible to use an allocation of \$1,600 for labor and \$1,400 for material? Of \$2,000 for labor and \$1,000 for material? Explain. No

67. Incentive plan. A small company provides an incentive plan for its top executives. Each executive receives as a bonus a percentage of the portion of the annual profit that remains after the bonuses for the other executives have been deducted (see the table). If the company has an annual profit of $2 million, find the bonus for each executive. Round each bonus to the nearest hundred dollars. *

Officer	Bonus
President	3%
Executive vice-president	2.5%
Associate vice-president	2%
Assistant vice-president	1.5%

68. Incentive plan. Repeat Problem 67 if the company decides to include a 1% bonus for the sales manager in the incentive plan. *

69. Diets. A biologist has available two commercial food mixes containing the percentage of protein and fat given in the table.

Mix	Protein(%)	Fat(%)
A	20	4
B	14	3

(A) How many ounces of each mix should be used to prepare each of the diets listed in the following table? *

	Diet		
	1	2	3
Protein	80 oz	90 oz	100 oz
Fat	17 oz	18 oz	21 oz

(B) Is it possible to prepare a diet consisting of 100 ounces of protein and 22 ounces of fat? Of 80 ounces of protein and 15 ounces of fat? Explain. No

70. Education. A state university system is planning to hire new faculty at the rank of lecturer or instructor for several of its two-year community colleges. The number of sections taught and the annual salary (in thousands of dollars) for each rank are given in the table. *

	Rank	
	Lecturer	Instructor
Sections taught	3	4
Annual salary (thousand $)	20	25

The number of sections taught by new faculty and the amount budgeted for salaries (in thousands of dollars) at each of the colleges are given in the following table. How many faculty of each rank should be hired at each college to exactly meet the demand for sections and completely exhaust the salary budget?

	Community College		
	1	2	3
Demand for sections	30	33	35
Salary budget (thousand $)	200	210	220

Answers to Matched Problems

1.
$$\begin{aligned}
AX + C &= B\\
(AX + C) - C &= B - C\\
AX + (C - C) &= B - C\\
AX + 0 &= B - C\\
AX &= B - C\\
A^{-1}(AX) &= A^{-1}(B - C)\\
(A^{-1}A)X &= A^{-1}(B - C)\\
IX &= A^{-1}(B - C)\\
X &= A^{-1}(B - C)
\end{aligned}$$

2. $x_1 = 2, x_2 = 5, x_3 = 0$

3. (A) $x_1 = -2, x_2 = -5, x_3 = 4$

(B) $x_1 = 0, x_2 = 1, x_3 = -4$

4. $A^{-1} = \begin{bmatrix} 1.5 & -6.25 \\ -0.5 & 6.25 \end{bmatrix}$; client 1: $15,000 in *A* and $5,000 in *B*; client 2: $28,125 in *A* and $21,875 in *B*; client 3: $6,875 in *A* and $3,125 in *B*

4.7 Leontief Input–Output Analysis

- Two-Industry Model
- Three-Industry Model

An important application of matrices and their inverses is **input–output analysis**. Wassily Leontief (1905–1999), the primary force behind this subject, was awarded the Nobel Prize in economics in 1973 because of the significant impact his work had on economic planning for industrialized countries. Among other things, he conducted a comprehensive study of how 500 sectors of the U.S. economy interacted with each other. Of course, large-scale computers played a crucial role in this analysis.

Our investigation will be more modest. In fact, we start with an economy comprised of only two industries. From these humble beginnings, ideas and definitions will evolve that can be readily generalized for more realistic economies. Input–output analysis attempts to establish equilibrium conditions under which industries in an economy have just enough output to satisfy each other's demands in addition to final (outside) demands.

Two-Industry Model

We start with an economy comprised of only two industries, electric company E and water company W. Output for both companies is measured in dollars. The electric company uses both electricity and water (inputs) in the production of electricity (output), and the water company uses both electricity and water (inputs) in the production of water (output). Suppose that the production of each dollar's worth of electricity requires \$0.30 worth of electricity and \$0.10 worth of water, and the production of each dollar's worth of water requires \$0.20 worth of electricity and \$0.40 worth of water. If the final demand (the demand from all other users of electricity and water) is

$$\begin{aligned} d_1 &= \$12 \text{ million for electricity} \\ d_2 &= \$8 \text{ million for water} \end{aligned}$$

how much electricity and water should be produced to meet this final demand?

To begin, suppose that the electric company produces \$12 million worth of electricity and the water company produces \$8 million worth of water. Then the production processes of the companies would require

$$\overset{\substack{\text{Electricity}\\ \text{required to}\\ \text{produce}\\ \text{electricity}}}{0.3(12)} + \overset{\substack{\text{Electricity}\\ \text{required to}\\ \text{produce}\\ \text{water}}}{0.2(8)} = \$5.2 \text{ million of electricity}$$

and

$$\overset{\substack{\text{Water}\\ \text{required to}\\ \text{produce}\\ \text{electricity}}}{0.1(12)} + \overset{\substack{\text{Water}\\ \text{required to}\\ \text{produce}\\ \text{water}}}{0.4(8)} = \$4.4 \text{ million of water}$$

leaving only \$6.8 million of electricity and \$3.6 million of water to satisfy the final demand. To meet the internal demands of both companies and to end up with enough electricity for the final outside demand, both companies must produce more than just the final demand. In fact, they must produce exactly enough to meet their own internal demands plus the final demand. To determine the total output that each company must produce, we set up a system of equations.

If

$$\begin{aligned} x_1 &= \text{total output from electric company} \\ x_2 &= \text{total output from water company} \end{aligned}$$

then, reasoning as before, the internal demands are

$$\begin{aligned} &0.3x_1 + 0.2x_2 \quad \text{Internal demand for electricity} \\ &0.1x_1 + 0.4x_2 \quad \text{Internal demand for water} \end{aligned}$$

Combining the internal demand with the final demand produces the following system of equations:

$$\begin{aligned} \overset{\substack{\text{Total}\\ \text{output}}}{x_1} &= \overset{\substack{\text{Internal}\\ \text{demand}}}{0.3x_1 + 0.2x_2} + \overset{\substack{\text{Final}\\ \text{demand}}}{d_1} \\ x_2 &= 0.1x_1 + 0.4x_2 + d_2 \end{aligned} \tag{1}$$

or, in matrix form,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

or

$$X = MX + D \tag{2}$$

where

$$D = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad \text{Final demand matrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{Output matrix}$$

$$M = \begin{array}{c|cc} & E & W \\ \hline E & 0.3 & 0.2 \\ W & 0.1 & 0.4 \end{array} \quad \text{Technology matrix}$$

The **technology matrix** is the heart of input–output analysis. The elements in the technology matrix are determined as follows (read from left to right and then up):

$$\begin{array}{ccc} & & \text{Output} \\ & & \begin{array}{cc} \overset{E}{\uparrow} \qquad\qquad\qquad & W \end{array} \\ \text{Input} & \begin{array}{c} E \longrightarrow \\ \\ \\ W \end{array} & \begin{bmatrix} \begin{pmatrix} \text{input from } E \\ \text{to produce } \$1 \\ \text{of electricity} \end{pmatrix} & \begin{pmatrix} \text{input from } E \\ \text{to produce } \$1 \\ \text{of water} \end{pmatrix} \\ \begin{pmatrix} \text{input from } W \\ \text{to produce } \$1 \\ \text{of electricity} \end{pmatrix} & \begin{pmatrix} \text{input from } W \\ \text{to produce } \$1 \\ \text{of water} \end{pmatrix} \end{bmatrix} = M \end{array}$$

CONCEPTUAL INSIGHT

Labeling the rows and columns of the technology matrix with the first letter of each industry is an important part of the process. The same order must be used for columns as for rows, and that same order must be used for the entries of D (the final demand matrix) and the entries of X (the output matrix). In this book we normally label the rows and columns in alphabetical order.

Now we solve equation (2) for X. We proceed as in Section 4.6:

$$\begin{aligned} X &= MX + D \\ X - MX &= D \\ IX - MX &= D & \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ (I - M)X &= D \\ X &= (I - M)^{-1}D & \quad \text{Assuming } I - M \text{ has an inverse} \end{aligned} \tag{3}$$

Omitting the details of the calculations, we find

$$I - M = \begin{bmatrix} 0.7 & -0.2 \\ -0.1 & 0.6 \end{bmatrix} \quad \text{and} \quad (I - M)^{-1} = \begin{bmatrix} 1.5 & 0.5 \\ 0.25 & 1.75 \end{bmatrix}$$

Then we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.5 & 0.5 \\ 0.25 & 1.75 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1.5 & 0.5 \\ 0.25 & 1.75 \end{bmatrix} \begin{bmatrix} 12 \\ 8 \end{bmatrix} = \begin{bmatrix} 22 \\ 17 \end{bmatrix} \tag{4}$$

Therefore, the electric company must produce an output of \$22 million and the water company must produce an output of \$17 million so that each company can meet both internal and final demands.

CHECK We use equation (2) to check our work:

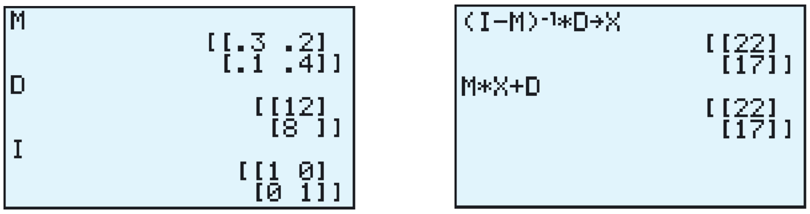
$$X = MX + D$$

$$\begin{bmatrix} 22 \\ 17 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}\begin{bmatrix} 22 \\ 17 \end{bmatrix} + \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 22 \\ 17 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 10 \\ 9 \end{bmatrix} + \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 22 \\ 17 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 22 \\ 17 \end{bmatrix}$$

To solve this input–output problem on a graphing calculator, simply store matrices M, D, and I in memory; then use equation (3) to find X and equation (2) to check your results. Figure 1 illustrates this process on a graphing calculator.



```
M
            [[.3 .2]
             [.1 .4]]
D
               [[12]
                [8 ]]
I
             [[1 0]
              [0 1]]
```

```
(I-M)⁻¹*D→X
               [[22]
                [17]]
M*X+D
               [[22]
                [17]]
```

(A) Store M, D, and I in the graphing calculator's memory

(B) Compute X and check in equation (2)

Figure 1

Actually, equation (4) solves the original problem for arbitrary final demands d_1 and d_2. This is very useful, since equation (4) gives a quick solution not only for the final demands stated in the original problem, but also for various other projected final demands. If we had solved system (1) by Gauss–Jordan elimination, then we would have to start over for each new set of final demands.

Suppose that in the original problem the projected final demands 5 years from now are $d_1 = 24$ and $d_2 = 16$. To determine each company's output for this projection, we simply substitute these values into equation (4) and multiply:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.5 & 0.5 \\ 0.25 & 1.75 \end{bmatrix}\begin{bmatrix} 24 \\ 16 \end{bmatrix} = \begin{bmatrix} 44 \\ 34 \end{bmatrix}$$

We summarize these results for convenient reference.

SUMMARY Solution to a Two-Industry Input–Output Problem

Given two industries, C_1 and C_2, with

$$M = \overset{\text{Technology matrix}}{\begin{array}{c|cc} & C_1 & C_2 \\ \hline C_1 & a_{11} & a_{12} \\ C_2 & a_{21} & a_{22} \end{array}} \qquad X = \overset{\text{Output matrix}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \qquad D = \overset{\text{Final demand matrix}}{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}}$$

where a_{ij} is the input required from C_i to produce a dollar's worth of output for C_j, the solution to the input–output matrix equation

$$\overset{\text{Total output}}{X} = \overset{\text{Internal demand}}{MX} + \overset{\text{Final demand}}{D}$$

is

$$X = (I - M)^{-1}D \tag{3}$$

assuming that $I - M$ has an inverse.

Three-Industry Model

Equations (2) and (3) in the solution to a two-industry input–output problem are the same for a three-industry economy, a four-industry economy, or an economy with n industries (where n is any natural number). The steps we took going from equation (2) to equation (3) hold for arbitrary matrices as long as the matrices have the correct sizes and $(I - M)^{-1}$ exists.

Explore and Discuss 1 If equations (2) and (3) are valid for an economy with n industries, discuss the size of all the matrices in each equation. $X, D: n \times 1 \quad M, I, (I - M)^{-1}: n \times n$

The next example illustrates the application of equations (2) and (3) to a three-industry economy.

EXAMPLE 1 **Input–Output Analysis** An economy is based on three sectors, agriculture (A), energy (E), and manufacturing (M). Production of a dollar's worth of agriculture requires an input of \$0.20 from the agriculture sector and \$0.40 from the energy sector. Production of a dollar's worth of energy requires an input of \$0.20 from the energy sector and \$0.40 from the manufacturing sector. Production of a dollar's worth of manufacturing requires an input of \$0.10 from the agriculture sector, \$0.10 from the energy sector, and \$0.30 from the manufacturing sector. Find the output from each sector that is needed to satisfy a final demand of \$20 billion for agriculture, \$10 billion for energy, and \$30 billion for manufacturing.

SOLUTION Since this is a three-industry problem, the technology matrix will be a 3×3 matrix, and the output and final demand matrices will be 3×1 column matrices.

To begin, we form a blank 3×3 technology matrix and label the rows and columns in alphabetical order.

Technology matrix

Output

$$\begin{array}{cc} & \begin{array}{ccc} A & E & M \end{array} \\ \text{Input} \begin{array}{c} A \\ E \\ M \end{array} & \left[\begin{array}{ccc} \quad & \quad & \quad \\ & & \\ & & \end{array}\right] \end{array} = M$$

Now we analyze the production information given in the problem, beginning with agriculture.

> **"Production of a dollar's worth of agriculture requires an input of \$0.20 from the agriculture sector and \$0.40 from the energy sector."**

We organize this information in a table and then insert it in the technology matrix. Since manufacturing is not mentioned in the agriculture production information, the input from manufacturing is \$0.

Agriculture

$$\begin{array}{ccc} \text{Input} & & \text{Output} \\ A & \xrightarrow{0.2} & A \\ E & \xrightarrow{0.4} & A \\ M & \xrightarrow{0} & A \end{array} \qquad \begin{array}{cc} & \begin{array}{ccc} A & E & M \end{array} \\ \begin{array}{c} A \\ E \\ M \end{array} & \left[\begin{array}{ccc} 0.2 & \quad & \quad \\ 0.4 & & \\ 0 & & \end{array}\right] \end{array}$$

"Production of a dollar's worth of energy requires an input of $0.20 from the energy sector and $0.40 from the manufacturing sector."

$$\begin{array}{c}\text{Energy}\\ \begin{array}{ccc}\text{Input} & & \text{Output}\\ A & \xrightarrow{0} & E\\ E & \xrightarrow{0.2} & E\\ M & \xrightarrow{0.4} & E\end{array}\end{array}\qquad \begin{array}{c|ccc|} & A & E & M\\ A & 0.2 & 0 & \\ E & 0.4 & 0.2 & \\ M & 0 & 0.4 & \end{array}$$

"Production of a dollar's worth of manufacturing requires an input of $0.10 from the agriculture sector, $0.10 from the energy sector and $0.30 from the manufacturing sector."

$$\begin{array}{c}\text{Manufacturing}\\ \begin{array}{ccc}\text{Input} & & \text{Output}\\ A & \xrightarrow{0.1} & M\\ E & \xrightarrow{0.1} & M\\ M & \xrightarrow{0.3} & M\end{array}\end{array}\qquad \begin{array}{c|ccc|} & A & E & M\\ A & 0.2 & 0 & 0.1\\ E & 0.4 & 0.2 & 0.1\\ M & 0 & 0.4 & 0.3\end{array}$$

Therefore,

$$M = \begin{array}{c}\text{Technology matrix}\\ \begin{array}{c|ccc|} & A & E & M\\ A & 0.2 & 0 & 0.1\\ E & 0.4 & 0.2 & 0.1\\ M & 0 & 0.4 & 0.3\end{array}\end{array}\qquad D = \begin{array}{c}\text{Final demand}\\ \text{matrix}\\ \begin{bmatrix}20\\10\\30\end{bmatrix}\end{array}\qquad X = \begin{array}{c}\text{Output}\\ \text{matrix}\\ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\end{array}$$

where M, X, and D satisfy the input–output equation $X = MX + D$. Since the solution to this equation is $X = (I - M)^{-1}D$, we must first find $I - M$ and then $(I - M)^{-1}$. Omitting the details of the calculations, we have

$$I - M = \begin{bmatrix}0.8 & 0 & -0.1\\ -0.4 & 0.8 & -0.1\\ 0 & -0.4 & 0.7\end{bmatrix}$$

and

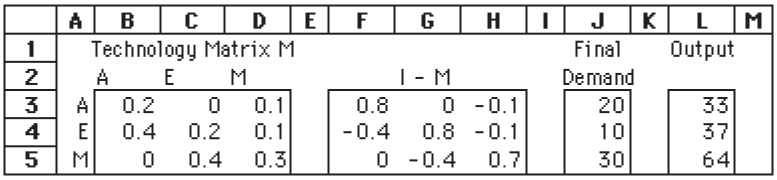
$$(I - M)^{-1} = \begin{bmatrix}1.3 & 0.1 & 0.2\\ 0.7 & 1.4 & 0.3\\ 0.4 & 0.8 & 1.6\end{bmatrix}$$

So the output matrix X is given by

$$\overset{X}{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}} = \overset{(I - M)^{-1}}{\begin{bmatrix}1.3 & 0.1 & 0.2\\ 0.7 & 1.4 & 0.3\\ 0.4 & 0.8 & 1.6\end{bmatrix}}\overset{D}{\begin{bmatrix}20\\10\\30\end{bmatrix}} = \begin{bmatrix}33\\37\\64\end{bmatrix}$$

An output of $33 billion for agriculture, $37 billion for energy, and $64 billion for manufacturing will meet the given final demands. You should check this result in equation (2).

Figure 2 illustrates a spreadsheet solution for Example 1.



	A	B	C	D	E	F	G	H	I	J	K	L	M
1		Technology Matrix M								Final		Output	
2		A	E	M			I - M			Demand			
3	A	0.2	0	0.1		0.8	0	-0.1		20		33	
4	E	0.4	0.2	0.1		-0.4	0.8	-0.1		10		37	
5	M	0	0.4	0.3		0	-0.4	0.7		30		64	

Figure 2 **The command MMULT(MINVERSE(F3:H5), J3:J5) produces the output in L3:L5**

Matched Problem 1 An economy is based on three sectors, coal, oil, and transportation. Production of a dollar's worth of coal requires an input of \$0.20 from the coal sector and \$0.40 from the transportation sector. Production of a dollar's worth of oil requires an input of \$0.10 from the oil sector and \$0.20 from the transportation sector. Production of a dollar's worth of transportation requires an input of \$0.40 from the coal sector, \$0.20 from the oil sector, and \$0.20 from the transportation sector.

(A) Find the technology matrix M.

(B) Find $(I - M)^{-1}$.

(C) Find the output from each sector that is needed to satisfy a final demand of \$30 billion for coal, \$10 billion for oil, and \$20 billion for transportation.

Exercises 4.7

W

Skills Warm-up Exercises

In Problems 1–8, solve each equation for x, where x represents a real number. (If necessary, review Section 1.1).

1. $x = 3x + 6$ -3

2. $x = 4x - 5$ $5/3$

3. $x = 0.9x + 10$ 100

4. $x = 0.6x + 84$ 210

5. $x = 0.2x + 3.2$ 4

6. $x = 0.3x + 4.2$ 6

7. $x = 0.68x + 2.56$ 8

8. $x = 0.98x + 8.24$ 412

A *Problems 9–14 pertain to the following input–output model: Assume that an economy is based on two industrial sectors, agriculture (A) and energy (E). The technology matrix M and final demand matrices (in billions of dollars) are*

$$\begin{array}{cc} & \begin{array}{cc} A & E \end{array} \\ \begin{array}{c} A \\ E \end{array} & \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.1 \end{bmatrix} \end{array} = M$$

$$D_1 = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \qquad D_2 = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \qquad D_3 = \begin{bmatrix} 12 \\ 9 \end{bmatrix}$$

9. How much input from A and E are required to produce a dollar's worth of output for A? 40¢ from A; 20¢ from E

10. How much input from A and E are required to produce a dollar's worth of output for E? 20¢ from A; 10¢ from E

11. Find $I - M$ and $(I - M)^{-1}$. $\begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.9 \end{bmatrix}; \begin{bmatrix} 1.8 & 0.4 \\ 0.4 & 1.2 \end{bmatrix}$

12. Find the output for each sector that is needed to satisfy the final demand D_1.*

13. Repeat Problem 12 for D_2.*

14. Repeat Problem 12 for D_3.*

B *Problems 15–20 pertain to the following input–output model: Assume that an economy is based on three industrial sectors: agriculture (A), building (B), and energy (E). The technology matrix M and final demand matrices (in billions of dollars) are*

$$\begin{array}{cc} & \begin{array}{ccc} A & B & E \end{array} \\ \begin{array}{c} A \\ B \\ E \end{array} & \begin{bmatrix} 0.3 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.1 \end{bmatrix} \end{array} = M$$

$$D_1 = \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} \qquad D_2 = \begin{bmatrix} 20 \\ 15 \\ 10 \end{bmatrix}$$

15. How much input from A, B, and E are required to produce a dollar's worth of output for B?*

16. How much of each of B's output dollars is required as input for each of the three sectors? 10¢ for each sector

17. Show that

$$I - M = \begin{bmatrix} 0.7 & -0.2 & -0.2 \\ -0.1 & 0.9 & -0.1 \\ -0.2 & -0.1 & 0.9 \end{bmatrix}$$

18. Given

$$(I - M)^{-1} = \begin{bmatrix} 1.6 & 0.4 & 0.4 \\ 0.22 & 1.18 & 0.18 \\ 0.38 & 0.22 & 1.22 \end{bmatrix}$$

show that $(I - M)^{-1}(I - M) = I$.

19. Use $(I - M)^{-1}$ in Problem 18 to find the output for each sector that is needed to satisfy the final demand D_1.*

20. Repeat Problem 19 for D_2.*

In Problems 21–26, find $(I - M)^{-1}$ and X.

21. $M = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix}; \quad D = \begin{bmatrix} 10 \\ 25 \end{bmatrix}$ $\begin{bmatrix} 1.4 & 0.4 \\ 0.6 & 1.6 \end{bmatrix}; \begin{bmatrix} 24 \\ 46 \end{bmatrix}$

22. $M = \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}; \quad D = \begin{bmatrix} 30 \\ 50 \end{bmatrix}$ I-M is singular; X does not exist.

23. $M = \begin{bmatrix} 0.7 & 0.8 \\ 0.3 & 0.2 \end{bmatrix}; \quad D = \begin{bmatrix} 25 \\ 75 \end{bmatrix}$ I-M is singular; X does not exist.

24. $M = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}; \quad D = \begin{bmatrix} 15 \\ 20 \end{bmatrix}$ $\begin{bmatrix} 1.75 & 0.25 \\ 0.5 & 1.5 \end{bmatrix}; \begin{bmatrix} 31.25 \\ 37.5 \end{bmatrix}$

C **25.** $M = \begin{bmatrix} 0.3 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}; \quad D = \begin{bmatrix} 20 \\ 5 \\ 10 \end{bmatrix}$ *

26. $M = \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 \end{bmatrix}; \quad D = \begin{bmatrix} 10 \\ 25 \\ 15 \end{bmatrix}$ *

---

*Answer located in Additional Instructor's Answers section.

27. The technology matrix for an economy based on agriculture (*A*) and manufacturing (*M*) is

$$M = \begin{array}{c} \\ A \\ M \end{array} \begin{array}{cc} A & M \\ \hline \end{array}$$

$$M = \begin{array}{cc} & \begin{array}{cc} A & M \end{array} \\ \begin{array}{c} A \\ M \end{array} & \begin{bmatrix} 0.3 & 0.25 \\ 0.1 & 0.25 \end{bmatrix} \end{array}$$

(A) Find the output for each sector that is needed to satisfy a final demand of \$40 million for agriculture and \$40 million for manufacturing.*

(B) Discuss the effect on the final demand if the agriculture output in part (A) is increased by \$20 million and manufacturing output remains unchanged.*

28. The technology matrix for an economy based on energy (*E*) and transportation (*T*) is

$$M = \begin{array}{cc} & \begin{array}{cc} E & T \end{array} \\ \begin{array}{c} E \\ T \end{array} & \begin{bmatrix} 0.25 & 0.25 \\ 0.4 & 0.2 \end{bmatrix} \end{array}$$

(A) Find the output for each sector that is needed to satisfy a final demand of \$50 million for energy and \$50 million for transportation.*

(B) Discuss the effect on the final demand if the transportation output in part (A) is increased by \$40 million and the energy output remains unchanged.*

29. Refer to Problem 27. Fill in the elements in the following technology matrix.

$$T = \begin{array}{cc} & \begin{array}{cc} M & A \end{array} \\ \begin{array}{c} M \\ A \end{array} & \begin{bmatrix} & \\ & \end{bmatrix} \end{array} \quad \begin{bmatrix} 0.25 & 0.1 \\ 0.25 & 0.3 \end{bmatrix}$$

Use this matrix to solve Problem 27. Discuss any differences in your calculations and in your answers.

30. Refer to Problem 28. Fill in the elements in the following technology matrix.

$$T = \begin{array}{cc} & \begin{array}{cc} T & E \end{array} \\ \begin{array}{c} T \\ E \end{array} & \begin{bmatrix} & \\ & \end{bmatrix} \end{array} \quad \begin{bmatrix} 0.2 & 0.4 \\ 0.25 & 0.25 \end{bmatrix}$$

Use this matrix to solve Problem 28. Discuss any differences in your calculations and in your answers.

31. The technology matrix for an economy based on energy (*E*) and mining (*M*) is

$$M = \begin{array}{cc} & \begin{array}{cc} E & M \end{array} \\ \begin{array}{c} E \\ M \end{array} & \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.3 \end{bmatrix} \end{array}$$

The management of these two sectors would like to set the total output level so that the final demand is always 40% of the total output. Discuss methods that could be used to accomplish this objective.

32. The technology matrix for an economy based on automobiles (*A*) and construction (*C*) is

$$M = \begin{array}{cc} & \begin{array}{cc} A & C \end{array} \\ \begin{array}{c} A \\ C \end{array} & \begin{bmatrix} 0.1 & 0.4 \\ 0.1 & 0.1 \end{bmatrix} \end{array}$$

The management of these two sectors would like to set the total output level so that the final demand is always 70% of the total output. Discuss methods that could be used to accomplish this objective.*

31. The total output of the energy sector should be 75% of the total output of the mining sector.

33. All the technology matrices in the text have elements between 0 and 1. Why is this the case? Would you ever expect to find an element in a technology matrix that is negative? That is equal to 0? That is equal to 1? That is greater than 1? Each element should be between 0 and 1, inclusive.

34. The sum of the elements in a column of any of the technology matrices in the text is less than 1. Why is this the case? Would you ever expect to find a column with a sum equal to 1? Greater than 1? How would you describe an economic system where the sum of the elements in every column of the technology matrix is 1?

Applications

35. Coal, steel. An economy is based on two industrial sectors, coal and steel. Production of a dollar's worth of coal requires an input of \$0.10 from the coal sector and \$0.20 from the steel sector. Production of a dollar's worth of steel requires an input of \$0.20 from the coal sector and \$0.40 from the steel sector. Find the output for each sector that is needed to satisfy a final demand of \$20 billion for coal and \$10 billion for steel. Coal: \$28 billion; steel: \$26 billion

36. Transportation, manufacturing. An economy is based on two sectors, transportation and manufacturing. Production of a dollar's worth of transportation requires an input of \$0.10 from each sector and production of a dollar's worth of manufacturing requires an input of \$0.40 from each sector. Find the output for each sector that is needed to satisfy a final demand of \$5 billion for transportation and \$20 billion for manufacturing.*

37. Agriculture, tourism. The economy of a small island nation is based on two sectors, agriculture and tourism. Production of a dollar's worth of agriculture requires an input of \$0.20 from agriculture and \$0.15 from tourism. Production of a dollar's worth of tourism requires an input of \$0.40 from agriculture and \$0.30 from tourism. Find the output from each sector that is needed to satisfy a final demand of \$60 million for agriculture and \$80 million for tourism.*

38. Agriculture, oil. The economy of a country is based on two sectors, agriculture and oil. Production of a dollar's worth of agriculture requires an input of \$0.40 from agriculture and \$0.35 from oil. Production of a dollar's worth of oil requires an input of \$0.20 from agriculture and \$0.05 from oil. Find the output from each sector that is needed to satisfy a final demand of \$40 million for agriculture and \$250 million for oil. Agriculture: \$176 million; oil: \$328 million

39. Agriculture, manufacturing, energy. An economy is based on three sectors, agriculture, manufacturing, and energy. Production of a dollar's worth of agriculture requires inputs of \$0.20 from agriculture, \$0.20 from manufacturing, and \$0.20 from energy. Production of a dollar's worth of manufacturing requires inputs of $0.40 from agriculture, $0.10 from manufacturing, and $0.10 from energy. Production of a dollar's worth of energy requires inputs of $0.30 from agriculture, $0.10 from manufacturing, and $0.10 from energy. Find the output for each sector that is needed to satisfy a final demand of $10 billion for agriculture, $15 billion for manufacturing, and $20 billion for energy.*

40. Electricity, natural gas, oil. A large energy company produces electricity, natural gas, and oil. The production of a dollar's worth of electricity requires inputs of $0.30 from electricity, $0.10 from natural gas, and $0.20 from oil. Production of a dollar's worth of natural gas requires inputs of $0.30 from electricity, $0.10 from natural gas, and $0.20 from oil. Production of a dollar's worth of oil requires inputs of $0.10 from each sector. Find the output for each sector that is needed to satisfy a final demand of $25 billion for electricity, $15 billion for natural gas, and $20 billion for oil.*

41. Four sectors. An economy is based on four sectors, agriculture (*A*), energy (*E*), labor (*L*), and manufacturing (*M*). The table gives the input requirements for a dollar's worth of output for each sector, along with the projected final demand (in billions of dollars) for a 3-year period. Find the output for each sector that is needed to satisfy each of these final demands. Round answers to the nearest billion dollars.*

		Output				**Final Demand**		
		A	***E***	***L***	***M***	**1**	**2**	**3**
Input	***A***	0.05	0.17	0.23	0.09	23	32	55
	E	0.07	0.12	0.15	0.19	41	48	62
	L	0.25	0.08	0.03	0.32	18	21	25
	M	0.11	0.19	0.28	0.16	31	33	35

42. Repeat Problem 41 with the following table:

		Output				**Final Demand**		
		A	***E***	***L***	***M***	**1**	**2**	**3**
Input	***A***	0.07	0.09	0.27	0.12	18	22	37
	E	0.14	0.07	0.21	0.24	26	31	42
	L	0.17	0.06	0.02	0.21	12	19	28
	M	0.15	0.13	0.31	0.19	41	45	49

Answers to Matched Problems

1. (A) $\begin{bmatrix} 0.2 & 0 & 0.4 \\ 0 & 0.1 & 0.2 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}$ (B) $\begin{bmatrix} 1.7 & 0.2 & 0.9 \\ 0.2 & 1.2 & 0.4 \\ 0.9 & 0.4 & 1.8 \end{bmatrix}$

(C) $71 billion for coal, $26 billion for oil, and $67 billion for transportation

Chapter 4 Summary and Review

Important Terms, Symbols, and Concepts

4.1 Review: Systems of Linear Equations in Two Variables

EXAMPLES

- The **solution** of a system is an ordered pair of real numbers that satisfies each equation in the system. Solution by **graphing** is one method that can be used to find a solution.
  Ex. 1, p. 174
  Ex. 2, p. 175
- A linear system is **consistent** and **independent** if it has a unique solution, **consistent** and **dependent** if it has more than one solution, and **inconsistent** if it has no solution. A linear system that is consistent and dependent actually has an infinite number of solutions.
- A **graphing calculator** provides accurate solutions to a linear system.
  Ex. 3, p. 177
- The **substitution** method can also be used to solve linear systems.
  Ex. 4, p. 177
- The **method of elimination by addition** is easily extended to larger systems.
  Ex. 5, p. 179

4.2 Systems of Linear Equations and Augmented Matrices

- A **matrix** is a rectangular array of real numbers. **Row operations** performed on an **augmented matrix** produce equivalent systems (Theorem 1, page 189).
  Ex. 1, p. 189
  Ex. 2, p. 191
- There are only three possible final forms for the augmented matrix for a linear system of two equations in two variables (p. 194).
  Ex. 3, p. 192
  Ex. 4, p. 193

4.3 Gauss–Jordan Elimination

- There are many possibilities for the final **reduced form** of the augmented matrix of a larger system of linear equations. Reduced form is defined on page 197.
  Ex. 1, p. 197
  Ex. 2, p. 198
- The **Gauss–Jordan elimination procedure** is described on pages 199 and 200.
  Ex. 3, p. 200
  Ex. 4, p. 201
  Ex. 5, p. 202

4.4 Matrices: Basic Operations

- Two matrices are **equal** if they are the same size and their corresponding elements are equal. The **sum** of two matrices of the same size is the matrix with elements that are the sum of the corresponding elements of the two given matrices. Ex. 1, p. 210
- The **negative of a matrix** is the matrix with elements that are the negatives of the given matrix. If A and B are matrices of the same size, then B can subtracted from A by adding the negative of B to A. Ex. 2, p. 211
- Matrix equations involving addition and subtraction are solved much like real number equations. Ex. 3, p. 211
- The product of a real number k and a matrix M is the matrix formed by multiplying each element of M by k. Ex. 4, p. 211
- The product of a row matrix and a column matrix is defined on page 213. Ex. 6, p. 213
- The matrix product of an $m \times p$ matrix with a $p \times n$ is defined on page 213. Ex. 8, p. 214

4.5 Inverse of a Square Matrix

- The **identity matrix** for multiplication is defined on page 222. Ex. 1, p. 223
- The **inverse** of a square matrix is defined on page 224. Ex. 2, p. 226

Ex. 3, p. 228

Ex. 4, p. 229

4.6 Matrix Equations and Systems of Linear Equations

- Basic properties of matrices are summarized in Theorem 1 on page 234. Ex. 1, p. 235
- **Matrix inverse methods** for solving systems of equations are described in the Summary on page 237. Ex. 2, p. 235

Ex. 3, p. 236

4.7 Leontief Input–Output Analysis

- Leontief **input–output** analysis is summarized on page 245. Ex. 1, p. 246

Review Exercises

Work through all the problems in this chapter review and check your answers in the back of the book. Answers to all problems are there along with section numbers in italics to indicate where each type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

A **1.** Solve the following system by graphing:*

$$\begin{aligned} 2x - y &= 4 \\ x - 2y &= -4 \end{aligned}$$

2. Solve the system in Problem 1 by substitution.*

3. If a matrix is in reduced form, say so. If not, explain why and state the row operation(s) necessary to transform the matrix into reduced form.

(A) $\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 3 \end{array}\right]$* (B) $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 3 & 3 \end{array}\right]$*

(C) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right]$* (D) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right]$*

4. Given matrices A and B,

$$A = \begin{bmatrix} 5 & 3 & -1 & 0 & 2 \\ -4 & 8 & 1 & 3 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 2 \\ 0 & 4 \\ -1 & 7 \end{bmatrix}$$

(A) What is the size of A? Of B? $2 \times 5, 3 \times 2$

(B) Find a_{24}, a_{15}, b_{31}, and b_{22}.
$a_{24} = 3, a_{15} = 2, b_{31} = -1, b_{22} = 4$

(C) Is AB defined? Is BA defined? AB is not defined; BA is defined *(4.2, 4.4)*

5. Find x_1 and x_2 :

(A) $\begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $x_1 = 8, x_2 = 2$

(B) $\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 25 \\ 14 \end{bmatrix} = \begin{bmatrix} 18 \\ 22 \end{bmatrix}$
$x_1 = -15.5, x_2 = 23.5$ *(4.6)*

In Problems 6–14, perform the operations that are defined, given the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad C = [2 \quad 3] \quad D = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

6. $A + B$* **7.** $B + D$ Not defined *(4.4)*

8. $A - 2B$* **9.** AB*

10. AC Not defined *(4.4)* **11.** AD*

12. DC* **13.** CD $[8]$ *(4.4)*

*Answer located in Additional Instructor's Answers section.

14. $C + D$ Not defined *(4.4)*

15. Find the inverse of the matrix A given below by appropriate row operations on $[A \mid I]$. Show that $A^{-1}A = I$.

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \quad \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \ (4.5)$$

16. Solve the following system using elimination by addition:

$$\begin{aligned} 4x_1 + 3x_2 &= 3 \\ 3x_1 + 2x_2 &= 5^* \end{aligned}$$

17. Solve the system in Problem 16 by performing appropriate row operations on the augmented matrix of the system. $x_1 = 9, x_2 = -11$ *(4.2)*

18. Solve the system in Problem 16 by writing the system as a matrix equation and using the inverse of the coefficient matrix (see Problem 15). Also, solve the system if the constants 3 and 5 are replaced by 7 and 10, respectively. By 4 and 2, respectively.

$x_1 = 9, x_2 = -11; x_1 = 16, x_2 = -19; x_1 = -2, x_2 = 4$ *(4.6)*

B *In Problems 19–24, perform the operations that are defined, given the following matrices:*

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \quad C = [2 \quad 1 \quad 3]$$

$$D = \begin{bmatrix} 3 & -2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 3 & -4 \\ -1 & 0 \end{bmatrix}$$

19. $A + D$ *

20. $E + DA$ *

21. $DA - 3E$ *

22. BC *

23. CB $[9]$ *(4.4)*

24. $AD - BC$ *

25. Find the inverse of the matrix A given below by appropriate row operations on $[A \mid I]$. Show that $A^{-1}A = I$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} -\frac{5}{2} & 2 & -\frac{1}{2} \\ 1 & -1 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \ (4.5)$$

26. Solve by Gauss–Jordan elimination:

(A)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 3x_2 + 4x_3 &= 3 \\ x_1 + 2x_2 + x_3 &= 3 \quad x_1 = 2, x_2 = 1, x_3 = -1 \end{aligned}$$

(B)
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= -3 \\ 3x_1 + 5x_2 &= -1 \ ^* \end{aligned}$$

(C)
$$\begin{aligned} x_1 + x_2 + x_3 &= 8 \\ 3x_1 + 2x_2 + 4x_3 &= 21 \ ^* \end{aligned}$$

27. Solve the system in Problem 26A by writing the system as a matrix equation and using the inverse of the coefficient matrix (see Problem 25). Also, solve the system if the constants 1, 3, and 3 are replaced by 0, 0, and −2, respectively. By −3, −4, and 1, respectively. *

28. Discuss the relationship between the number of solutions of the following system and the constant k.*

$$\begin{aligned} 2x_1 - 6x_2 &= 4 \\ -x_1 + kx_2 &= -2 \end{aligned}$$

29. An economy is based on two sectors, agriculture and energy. Given the technology matrix M and the final demand matrix D (in billions of dollars), find $(I - M)^{-1}$ and the output matrix X:*

$$M = \begin{array}{c} \\ A \\ E \end{array} \begin{array}{c} \begin{array}{cc} A & E \end{array} \\ \begin{bmatrix} 0.2 & 0.15 \\ 0.4 & 0.3 \end{bmatrix} \end{array} \quad D = \begin{array}{c} A \\ E \end{array} \begin{bmatrix} 30 \\ 20 \end{bmatrix}$$

30. Use the matrix M in Problem 29 to fill in the elements in the following technology matrix.

$$T = \begin{array}{c} \\ E \\ A \end{array} \begin{array}{c} \begin{array}{cc} E & A \end{array} \\ \begin{bmatrix} & \\ & \end{bmatrix} \end{array}$$

Use this matrix to solve Problem 29. Discuss any differences in your calculations and in your answers.*

31. An economy is based on two sectors, coal and steel. Given the technology matrix M and the final demand matrix D (in billions of dollars), find $(I - M)^{-1}$ and the output matrix X:*

$$M = \begin{array}{c} \\ C \\ S \end{array} \begin{array}{c} \begin{array}{cc} C & S \end{array} \\ \begin{bmatrix} 0.45 & 0.65 \\ 0.55 & 0.35 \end{bmatrix} \end{array} \quad D = \begin{array}{c} C \\ S \end{array} \begin{bmatrix} 40 \\ 10 \end{bmatrix}$$

32. Use graphical approximation techniques on a graphing calculator to find the solution of the following system to two decimal places:

$$\begin{aligned} x - 5y &= -5 \\ 2x + 3y &= 12 \quad x = 3.46, y = 1.69 \ (4.1) \end{aligned}$$

C **33.** Find the inverse of the matrix A given below. Show that $A^{-1}A = I$.

$$A = \begin{bmatrix} 4 & 5 & 6 \\ 4 & 5 & -4 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -0.9 & -0.1 & 5 \\ 0.8 & 0.2 & -4 \\ 0.1 & -0.1 & 0 \end{bmatrix} \ (4.5)$$

34. Solve the system

$$\begin{aligned} 0.04x_1 + 0.05x_2 + 0.06x_3 &= 360 \\ 0.04x_1 + 0.05x_2 - 0.04x_3 &= 120 \\ x_1 + x_2 + x_3 &= 7{,}000 \end{aligned}$$

by writing it as a matrix equation and using the inverse of the coefficient matrix. (Before starting, multiply the first two equations by 100 to eliminate decimals. Also, see Problem 33.) $x_1 = 1{,}400, x_2 = 3{,}200, x_3 = 2{,}400$ *(4.6)*

35. Solve Problem 34 by Gauss–Jordan elimination.
$x_1 = 1{,}400, x_2 = 3{,}200, x_3 = 2{,}400$ *(4.3)*

36. Given the technology matrix M and the final demand matrix D (in billions of dollars), find $(I - M)^{-1}$ and the output matrix X: *

$$M = \begin{bmatrix} 0.2 & 0 & 0.4 \\ 0.1 & 0.3 & 0.1 \\ 0 & 0.4 & 0.2 \end{bmatrix} \quad D = \begin{bmatrix} 40 \\ 20 \\ 30 \end{bmatrix}$$

37. Discuss the number of solutions for a system of n equations in n variables if the coefficient matrix

(A) Has an inverse.*

(B) Does not have an inverse.*

38. Discuss the number of solutions for the system corresponding to the reduced form shown below if

(A) $m \neq 0$ Unique solution

(B) $m = 0$ and $n \neq 0$ No solution

(C) $m = 0$ and $n = 0$ Infinite number of solutions *(4.3)*

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & m & n \end{array}\right]$$

39. One solution to the input–output equation $X = MX + D$ is given by $X = (I - M)^{-1}D$. Discuss the validity of each step in the following solutions of this equation. (Assume that all necessary inverses exist.) Are both solutions correct?

(A)
$$\begin{aligned} X &= MX + D \\ X - MX &= D \\ X(I - M) &= D \\ X &= D(I - M)^{-1} \end{aligned}$$

(B)
$$\begin{aligned} X &= MX + D \\ -D &= MX - X \\ -D &= (M - I)X \\ X &= (M - I)^{-1}(-D) \end{aligned}$$

(B) is the only correct solution. *(4.6)*

Applications

40. Break-even analysis. A cookware manufacturer is preparing to market a new pasta machine. The company's fixed costs for research, development, tooling, etc., are \$243,000 and the variable costs are \$22.45 per machine. The company sells the pasta machine for \$59.95.

(A) Find the cost and revenue equations.
$C = 243{,}000 + 22.45x$; $R = 59.95x$

(B) Find the break-even point.
$x = 6{,}480$ machines; $R = C = \$388{,}476$

(C) Graph both equations in the same coordinate system and show the break-even point. Use the graph to determine the production levels that will result in a profit and in a loss. *

41. Resource allocation. An international mining company has two mines in Voisey's Bay and Hawk Ridge. The composition of the ore from each field is given in the table. How many tons of ore from each mine should be used to obtain exactly 6 tons of nickel and 8 tons of copper? *

Mine	Nickel (%)	Copper (%)
Voisey's Bay	2	4
Hawk Ridge	3	2

42. Resource allocation.

(A) Set up Problem 41 as a matrix equation and solve using the inverse of the coefficient matrix. *

(B) Solve Problem 41 as in part (A) if 7.5 tons of nickel and 7 tons of copper are needed. *

43. Business leases. A grain company wants to lease a fleet of 20 covered hopper railcars with a combined capacity of 108,000 cubic feet. Hoppers with three different carrying capacities are available: 3,000 cubic feet, 4,500 cubic feet, and 6,000 cubic feet.

(A) How many of each type of hopper should they lease? *

(B) The monthly rates for leasing these hoppers are \$180 for 3,000 cubic feet, \$225 for 4,500 cubic feet, and \$325 for 6,000 cubic feet. Which of the solutions in part (A) would minimize the monthly leasing costs? *

44. Material costs. A manufacturer wishes to make two different bronze alloys in a metal foundry. The quantities of copper, tin, and zinc needed are indicated in matrix M. The costs for these materials (in dollars per pound) from two suppliers are summarized in matrix N. The company must choose one supplier or the other.

$$M = \begin{array}{cccc} \text{Copper} & \text{Tin} & \text{Zinc} & \\ 4{,}800\text{ lb} & 600\text{ lb} & 300\text{ lb} & \text{Alloy 1} \\ 6{,}000\text{ lb} & 1{,}400\text{ lb} & 700\text{ lb} & \text{Alloy 2} \end{array}$$

$$N = \begin{array}{ccc} \text{Supplier A} & \text{Supplier B} & \\ \$0.75 & \$0.70 & \text{Copper} \\ \$6.50 & \$6.70 & \text{Tin} \\ \$0.40 & \$0.50 & \text{Zinc} \end{array}$$

(A) Discuss possible interpretations of the elements in the matrix products MN and NM. Elements in MN give the cost of materials for each alloy from each supplier.

(B) If either product MN or NM has a meaningful interpretation, find the product and label its rows and columns. *

(C) Discuss methods of matrix multiplication that can be used to determine the supplier that will provide the necessary materials at the lowest cost. *

45. Labor costs. A company with manufacturing plants in California and Texas has labor-hour and wage requirements for the manufacture of two inexpensive calculators as given in matrices M and N below:

Labor-hours per calculator

$$M = \begin{array}{cccc} \text{Fabricating department} & \text{Assembly department} & \text{Packaging department} & \\ 0.15\text{ hr} & 0.10\text{ hr} & 0.05\text{ hr} & \text{Model A} \\ 0.25\text{ hr} & 0.20\text{ hr} & 0.05\text{ hr} & \text{Model B} \end{array}$$

Hourly wages

$$N = \begin{array}{ccc} \text{California plant} & \text{Texas plant} & \\ \$12 & \$10 & \text{Fabricating department} \\ \$15 & \$12 & \text{Assembly department} \\ \$\ 7 & \$\ 6 & \text{Packaging department} \end{array}$$

(A) Find the labor cost for producing one model B calculator at the California plant. \$6.35

(B) Discuss possible interpretations of the elements in the matrix products MN and NM. Elements in MN give the total labor costs for each calculator at each plant.

(C) If either product MN or NM has a meaningful interpretation, find the product and label its rows and columns. *

46. **Investment analysis.** A person has $5,000 to invest, part at 5% and the rest at 10%. How much should be invested at each rate to yield $400 per year? Solve using augmented matrix methods. *

47. **Investment analysis.** Solve Problem 46 by using a matrix equation and the inverse of the coefficient matrix. $2,000 at 5% and $3,000 at 10% *(4.6)*

48. **Investment analysis.** In Problem 46, is it possible to have an annual yield of $200? Of $600? Describe all possible annual yields.*

49. **Ticket prices.** An outdoor amphitheater has 25,000 seats. Ticket prices are $8, $12, and $20, and the number of tickets priced at $8 must equal the number priced at $20. How many tickets of each type should be sold (assuming that all seats can be sold) to bring in each of the returns indicated in the table? Solve using the inverse of the coefficient matrix. *

	Concert		
	1	**2**	**3**
Tickets sold	25,000	25,000	25,000
Return required	$320,000	$330,000	$340,000

50. **Ticket prices.** Discuss the effect on the solutions to Problem 49 if it is no longer required to have an equal number of $8 tickets and $20 tickets. *

51. **Input–output analysis.** An economy is based on two industrial sectors, agriculture and fabrication. Production of a dollar's worth of agriculture requires an input of $0.30 from the agriculture sector and $0.20 from the fabrication sector. Production of a dollar's worth of fabrication requires $0.10 from the agriculture sector and $0.40 from the fabrication sector.

(A) Find the output for each sector that is needed to satisfy a final demand of $50 billion for agriculture and $20 billion for fabrication. Agriculture: $80 billion; fabrication: $60 billion

(B) Find the output for each sector that is needed to satisfy a final demand of $80 billion for agriculture and $60 billion for fabrication. Agriculture: $135 billion; fabrication: $145 billion *(4.7)*

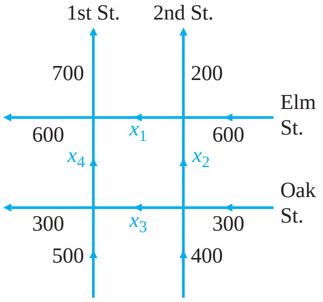
52. **Cryptography.** The following message was encoded with the matrix B shown below. Decode the message. *

7 25 30 19 6 24 20 8 28 5 14 14

9 23 28 15 6 21 13 1 14 21 26 29

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

53. **Traffic flow.** The rush-hour traffic flow (in vehicles per hour) for a network of four one-way streets is shown in the figure. *



(A) Write the system of equations determined by the flow of traffic through the four intersections. *

(B) Find the solution of the system in part (A). *

(C) What is the maximum number of vehicles per hour that can travel from Oak Street to Elm Street on 1st Street? What is the minimum number? 900; 200

(D) If traffic lights are adjusted so that 500 vehicles per hour travel from Oak Street to Elm Street on 1st Street, determine the flow around the rest of the network.

Elm St.: 800; 2nd St.: 400; Oak St.: 300 *(4.3)*